RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
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ÜBUNGSBLATT 11

Symmetric spaces

Exercise 1. (Totally geodesic subspaces) Consider $O(p,q) < SL(n,\mathbb{R})$.

(a) Choose a point $x \in SL(n,\mathbb{R})/O(n)$ such that the $O(p,q)$-orbit of x is a copy of the positive Grassmannian embedded as a totally geodesic subspace of $SL(n,\mathbb{R})/O(n)$.

(b) Compute the associated Lie triple system.

Exercise 2. (Totally geodesic submanifold)

(a) Show that if the symmetric space $\mathcal{X}$ splits as the Riemannian product $\mathcal{X}_1 \times \mathcal{X}_2$, for every Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g} = \mathrm{Lie}(\mathrm{Isom}^0(\mathcal{X}))$ associated to a point $x \in \mathcal{X}$, then there are Lie triple systems $\mathfrak{n}_1, \mathfrak{n}_2 < \mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.

(b) In $\mathfrak{sp}(2n,\mathbb{R})$ denote by $\mathfrak{n}_1$ (resp $\mathfrak{n}_2$) the Lie triple system

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \mid A \text{ symmetric} \right\}$$

$$\mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \mid A \text{ symmetric} \right\}$$

Show that there is a Cartan decomposition such that $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.

(c) Is the symmetric space associated to $\mathfrak{sp}(2n,\mathbb{R})$ irreducible?

Exercise 3. (Flat strip theorem) Show that the hypothesis that the geodesics are biinfinite is crucial in the flat strip theorem by exhibiting two geodesic rays in a Hadamard manifold that stay at bounded distance but do not bound a flat strip.