RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 9

**Symmetric spaces**

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**Exercise 1.** Let $Q$ denote the algebra of quaternions:

$$Q = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

with the non-commutative product defined by the relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

The conjugate of a quaternion is defined as

$$\overline{a + bi + cj + dk} = a - bi - cj - dk,$$

and the norm is defined by

$$|q|^2 = q\overline{q}.$$

The norm is multiplicative, and it helps finding the multiplicative inverses:

$$|pq| = |p||q|, \qquad\qquad q^{-1} = \frac{\overline{q}}{q^2}.$$

Define the set of pure quaternions as

$$P = \{bi + cj + dk \mid b, c, d \in \mathbb{R}\}.$$

Let $SO(3)$ denote the connected component of the group of all $\mathbb{R}$-linear maps from $P$ to itself that preserve the norm $|\cdot|$. Similarly, let $SO(4)$ denote the connected component of the group of all $\mathbb{R}$-linear maps from $Q$ to itself that preserve the norm $|\cdot|$. Define the sphere group as

$$\mathbb{S} = \{q \in Q \mid |q| = 1\}$$

This is a multiplicative subgroup of $Q$. For every $x, y \in \mathbb{S}$, consider the linear map

$$T_{x,y} : Q \ni q \to xqy^{-1} \in Q$$

(a) Prove that for every $x \in \mathbb{S}$, $T_{x,x}$ preserves $P$.

(b) Show that this gives a surjective homomorphism $\mathbb{S} \to SO(3)$, find the kernel and describe the topology of $SO(3)$.

(c) Show that this gives a surjective homomorphism of $\mathbb{S} \times \mathbb{S} \to SO(4)$, and find the kernel.

(d) Show that $PO(4)$ is isomorphic to $SO(3) \times SO(3)$.

**Exercise 2.** Orthogonal complex structures. Consider $\mathbb{R}^{2n}$ with the standard scalar product. A complex structure on $\mathbb{R}^{2n}$ is a multiplication by $i$, i.e. a linear map $J$ such that $J^2 = -1$. A complex structure $J$ is orthogonal if $J$ is an orthogonal linear map.

1. Prove that the set $S$ of orthogonal complex structures coincides with the set of all matrices that are orthogonal and antisymmetric.
2. Prove that the group $O(2n)$ acts by conjugation on $S$, that the action is transitive and that the stabilizer of any point is isomorphic to the unitary group $U(n)$.
3. Put a structure of symmetric space on $S$.
4. Determine the non-compact dual of this symmetric space.

**Exercise 3.** (Restriction of the Killing form) Give an example of a simple Lie algebra $\mathfrak{g}$ with a subalgebra $\mathfrak{t}$ such that $B_{\mathfrak{g}}|_{\mathfrak{t}}$ is not a scalar multiple of $B_{\mathfrak{t}}$. (Hint: find a non-semisimple Lie algebra that is subalgebra of a compact one.)