RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II

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Übungsblatt 8

Symmetric spaces

Exercise 1. Consider the symmetric space

$$S_n = \{M \in \mathcal{P} \mid \det(M) = 1\}$$

from Sheet 6, Exercise 2. Note that $S_n = SL(n, R)/SO(n)$.

(a) Compute the Cartan decomposition associated to the point

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in S_n,$$

where $\lambda_1 > \ldots > \lambda_n$ and $\lambda_1 \cdot \cdots \cdot \lambda_n = 1$.

(b) Denote by Θ^Λ (resp. Θ^{Id}) the associated Cartan involutions, and by $\mathfrak{p}^\Lambda$ (resp. $\mathfrak{p}^{\mathrm{Id}}$ the eigenspaces for -1. What is $\mathfrak{p}^{\mathrm{Id}} \cap \mathfrak{p}^\Theta$?

Exercise 2. (Orthogonal symmetric Lie algebras)

(a) Show that, if the OSLA $(\mathfrak{g}, \sigma)$ is of compact type, and comes from the symmetric space $\mathcal{X}$, then $\mathcal{X}$ is compact.

(b) Show that the symmetric space $SL(n, \mathbb{R})/SO(n)$ is non-compact. Does the associated orthogonal symmetric Lie algebra have a well defined type?

(c) Consider the symmetric space $\mathbb{R}^n$, with $G = O(n) \ltimes \mathbb{R}^n$. Show that $\mathfrak{g}$ is of Euclidean type.

Exercise 3. (Exponential maps) Let $M = G/K$ be a globally symmetric space and $p \in M$. Denote by $s_p : M \to M$ the geodesic symmetry about $p \in M$. Recall that $\sigma : G \ni g \to s_p g s_p \in G$, and $\pi : G \ni g \to gp \in M$. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated decomposition. Denote by $\exp : \mathfrak{g} \to G$ the exponential map in the sense of Lie groups and by $\mathrm{Exp}_p : T_pM \to M$ the exponential map in the sense of Riemannian manifolds. Recall from class that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{D_e\pi|_\mathfrak{p}} & T_pM \\ \downarrow & & \downarrow \\ G & \xrightarrow{\pi} & M \end{array}$$

where the arrow $\mathfrak{p} \to G$ is given by $\exp|_\mathfrak{p}$ and the arrow $T_pM \to M$ is given by Exp_p.

Make this explicit for $M = \mathbb{R}^2, \mathbb{S}^2, \mathbb{H}^2$.