RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 7

Compact symmetric spaces

Exercise 1. (Riemannian symmetric pairs)

(a) Explicitely give two distinct involutive automorphisms $\sigma_i : SO(n) \to SO(n)$ such that the groups G_{σ_i} are non tivial and such that the quotients G/G_{σ_i} are isomorphic. What are the groups G_{σ_i} in your example?

(b) Find on Wikipedia the list of all simply connected symmetric spaces M having $\mathrm{Iso}^0(M) = SO(n)$. For each one of them, give one example of an involutive automorphism σ, such that $G/G_\sigma = M$.

Exercise 2. (Fubini-Study metric) Consider the vector space $\mathbb{C}^{n+1}$ with the standard positive definite Hermitian form:

$$h(x,y) = \sum_{i=0}^{n} \bar{x}_i y_i$$

Denote by $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ the usual projection into the projective space. Recall also that you can identify the tangent space at every point of $\mathbb{C}^{n+1}$ with the vector space $\mathbb{C}^{n+1}$.

1. Show that, $d\pi_v$ identifies the orthogonal space at v for the form h with the tangent space at $x = \pi(v)$.
2. Define a Riemannian metric on $\mathbb{CP}^n$ using the real part of the restriction of h to these orthogonal subspaces. This metric is usually called the Fubini-Study metric.
3. Show that every element of $U(n+1)$ induces an isometry of this metric.
4. Show that the stabilizer of any point for the action of $U(n+1)$ is isomorphic to $U(1)\times U(n)$.
5. Show that, with this metric, the space $\mathbb{CP}^n$ is a symmetric space.
6. Find the associated Riemannian symmetric.

Exercise 3. Let G be a compact Lie group.

(a) Show that G admits a bi-invariant Riemannian metric.

Hint: Find a bijection between bi-invariant Riemannian metrics and $\mathrm{Ad}(G)$-invariant scalar products on $\mathfrak{g}$. Then construct one such scalar product.

(b) Show that for any bi-invariant metric, the one parameter subgroups are geodesics.

Hint: Use Kozul's formula to show that $\nabla_X Y = \frac{1}{2}[X, Y]$, where X, Y are left invariant vector fields.

(c) How many bi-invariant metrics are there on the torus?