RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II

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Übungsblatt 6

Examples of symmetric spaces

Exercise 1. Compute the killing form for $SL(n, \mathbb{R})$ and prove that $SL(n, \mathbb{R})$ is semisimple.

Exercise 2. Let $\mathrm{Sym}(n, \mathbb{R})$ denote the vector space of symmetric $n \times n$ matrices. Consider the open subset of positive definite matrices:

$$\mathcal{P} = \{M \in \mathrm{Sym}(n, \mathbb{R}) \mid M > 0\}$$

As usual for open subsets of vector spaces, we will identify the tangent space at every point with $\mathrm{Sym}(n, \mathbb{R})$ itself:

$$\forall M \in \mathcal{P}, \quad T_M\mathcal{P} = \mathrm{Sym}(n, \mathbb{R})$$

We will denote by $\mathrm{Id} \in \mathcal{P}$ the identity matrix.

(a) Prove that the formula

$$g_M(S, T) = \mathrm{tr}(M^{-1}SM^{-1}T)$$

intended as a bilinear form on the vector space $T_M\mathcal{P}$, for a $M \in \mathcal{P}$, defines a Riemannian metric on $\mathcal{P}$.

(b) Prove that the formula

$$GL(n, \mathbb{R}) \times \mathcal{P} \ni (A, M) \longrightarrow A^T M A \in \mathcal{P}$$

defines an action of $GL(n, \mathbb{R})$ on $\mathcal{P}$, given by base change of scalar products.

(c) Prove that the given action of $GL(n, \mathbb{R})$ is transitive.

(d) Prove that the given action of $GL(n, \mathbb{R})$ is by isometries for the Riemannian metric g.

(e) Compute the stabilizer of the point Id.

(f) Prove that $(\mathcal{P}, g)$ is a symmetric space.

(g) Consider the submanifold

$$\{M \in \mathcal{P} \mid \det(M) = 1\}$$

with the induced Riemannian metric. Prove that it is a symmetric space.

Exercise 3. Consider the space $\mathbb{R}^n$ with scalar product $\langle .,. \rangle$. Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace, and let $V^\perp$ be its orthogonal. We denote by $O(k) \times O(n-k)$ the subgroup of $O(n)$ that preserves the orthogonal decomposition $V \oplus V^\perp$, where the factor $O(k)$ is acting on V and the factor $O(n-k)$ is acting on $V^\perp$.

Consider the group $G = SO(n)$ and the subgroups $K_1 = SO(k) \times SO(n-k)$, $K_2 = S(O(k) \times O(n-k))$, where $S(H)$ denotes all the elements of the group H with determinant 1.

(a) Prove that the spaces G/K_1 and G/K_2 are symmetric spaces.

(b) Find a natural covering map $G/K_1 \to G/K_2$.

(c) Show that the group K_2/K_1 has a natural action on G/K_1, that makes it the group of deck transformations of the covering above.

(d) What are G/K_1 and G/K_2? Can you find a geometric meaning of G/K_1 and G/K_2 that relates them to objects coming from linear algebra?