RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
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ÜBUNGSBLATT 5

Symmetric and locally symmetric spaces

Exercise 1. Let M be a symmetric space and $G = \mathrm{Isom}_0(M)$. Let ω be a G-invariant differential form. Prove that ω is closed.

Exercise 2. Let M be one of the symmetric spaces $\mathbb{S}^n$ or $\mathbb{H}^n$. For $\mathbb{H}^n$, you can use the model $(\mathbb{B}, g)$ constructed in Sheet 3, Exercise 1, or any other model you know. Let $p \in M$ be a point.

(a) Write the geodesic symmetry at p.

(b) Find two transvections T, S with $T(p) = S(p)$ and $T \neq S$.

Exercise 3. For $X \in \mathrm{Mat}(n, n, \mathbb{C})$, consider the limit

$$\sum_{k=1}^{\infty} \frac{1}{k!} X^k = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k!} X^k$$

(a) Prove that for every X the limit exists in $\mathrm{Mat}(n, n, \mathbb{C})$. Denote

$$e^X = \sum_{k=1}^{\infty} \frac{1}{k!} X^k \in \mathrm{Mat}(n, n, \mathbb{C})$$

(Hint: choose a suitable norm on $\mathrm{Mat}(n, n, \mathbb{C})$, prove that $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k!} |X|^k$ exists in $\mathbb{R}$, and from this deduce the same result for matrices.)

(b) Prove that for X, Y commuting matrices, $e^{X+Y} = e^X e^Y$.

(c) Prove that for all $X \in \mathrm{Mat}(n, n, \mathbb{C})$, $e^X \in GL(n, \mathbb{C})$.

(d) Prove that for every $X \in \mathrm{Mat}(n, n, \mathbb{C})$, the map

$$\mathbb{R} \ni t \to e^{tX} \in GL(n, \mathbb{C})$$

is a one-parameter subgroup of $GL(n, \mathbb{C})$.

(e) Compute the derivative of the one-parameter subgroup above:

$$\frac{d}{dt} e^{tX} = X e^{tX} = e^{tX} X$$

(f) Prove that the map

$$\mathrm{Mat}(n, n, \mathbb{C}) \ni X \to e^X \in GL(n, \mathbb{C})$$

is locally invertible around 0.

(g) Prove that the above map is the exponential map of the Lie group $GL(n, \mathbb{C})$.