RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 4

Symmetric and locally symmetric spaces

Exercise 1. Let (M, g_M), (N, g_N) be two Riemannian manifolds with parallel curvature tensors. Given $m \in M$ and $n \in N$, assume that there exists a linear isometry $\phi : T_mM \to T_nN$ preserving the Riemann curvature tensors, i.e. such that for all $u, v, w \in T_mM$,

$$R^N_n(\phi(u), \phi(v))\phi(w) = \phi(R^M_m(u, v)w).$$

Prove that for every normal neighborhood U of m there exists a normal neighborhood V of n and a local isometry $f : U \to V$ such that $f(m) = n$ and $D_mf = \phi$.

Exercise 2. Consider the action of the group $\mathbb{Z}^n$ on $\mathbb{R}^n$ by integral translations:

$$\mathbb{Z}^n \times \mathbb{R}^n \ni ((m_1, \ldots, m_n), (x_1, \ldots, x_n)) \longrightarrow (x_1 + m_1, \ldots, x_n + m_n) \in \mathbb{R}^n.$$

Notice that the translations are isometries for the standard Riemannian metric of $\mathbb{R}^n$. The flat torus is the quotient $\mathbb{R}^n/\mathbb{Z}^n$, with its induced Riemannian metric. Prove that the flat torus is a symmetric space.

Exercise 3. Consider the open ball $\mathbb{B} \subset \mathbb{RP}^2$ with the Riemannian metric g constructed in Sheet 3, Exercise 1. The Riemannian manifold $(\mathbb{B}, g)$ will be called the *hyperbolic plane.* Consider the matrix

$$A = \begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Prove that $A \in PO(2,1)$. This implies that A acts on $(\mathbb{B}, g)$ by isometries.

(b) Prove that the action of A on $\mathbb{B}$ leaves a geodesic invariant, call it a.

(c) Prove that the quotient $\mathbb{B}/(A)$, with its induced Riemannian metric, is a complete Riemannian manifold, which is homeomorphic to a cylinder.

Hint: Given a point $p \in a$, consider the geodesics m_1, m_2 perpendicular to a and meeting a at p and $A(p)$ respectively, as in the picture. A maps m_1 to m_2.

(d) Prove that $\mathbb{B}/(A)$ is locally symmetric, but not symmetric.

Hint: Consider a point $q \in \mathbb{B}$, with $q \notin a$, and q between m_1 and m_2. From the picture, you can see that the geodesic symmetry around q cannot be globally well defined on $\mathbb{B}/A$.