RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 3

Recap of differential geometry

Exercise 1. Consider the vector space $\mathbb{R}^{n+1}$ with the bilinear form

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

Consider the subset

$$C = \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle < 0\}.$$

The set C is an open cone, and it projects to an open ball $\mathbb{B} \subset \mathbb{RP}^n$. For every $p \in B$ and $v \in T_p\mathbb{B}$, let ℓ_v be the projective line passing through p with tangent v. ℓ_v intersects the boundary $\partial\mathbb{B}$ in two points, v^+, v^-. There is a unique projective map $\phi : \ell_v \to \mathbb{RP}^1$ such that $\phi(v^-) = 0$, $\phi(v^+) = \infty$ and $\phi(p) = 1$. Define the number

$$\|v\|_p = |d(\log \circ \phi)[v]|$$

where $\log : \mathbb{R}_{>0} \to \mathbb{R}$ and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}$.

(a) Prove that there exists a unique Riemannian metric g on $\mathbb{B}$ such that

$$\|v\|_p = \sqrt{g_p(v, v)}$$

(b) Prove that the group $PO(n, 1)$ acts on $\mathbb{RP}^n$ preserving $\mathbb{B}$, and it acts on $\mathbb{B}$ by isometries for g.

(c) Prove that g has constant sectional curvature.

(d) Find all the geodesics for g.

(e) Prove that g is complete.

(f) Prove that the sectional curvature is negative.

Exercise 2. (a) Find two Riemannian manifolds M, N and a local diffeomorphism $f : M \to N$ such that for every $p \in M$ and for every $v \in T_pM$,

$$|df_p(v)| \geq |v|$$

and f is not a covering map.

(b) Find two Riemannian manifolds M, N with M complete, and a local diffeomorphism $f : M \to N$ such that there exists $p \in M$ such that for every $v \in T_pM$,

$$|df_p(v)| \geq |v|$$

and f is not a covering map.

Exercise 3. Let (M, g) be a Riemannian manifold of dimension n with constant sectional curvature k.

(a) Show that for every vector fields X, Y, Z, T, we have the relation

$$g(R(X,Y)Z,T) = k(g(X,T)g(Y,Z) - g(X,Z)g(Y,T)).$$

(b) Let $\gamma : \mathbb{R} \to M$ be a geodesic parametrized by arc-length. Choose an orthonormal basis $e_1, \ldots, e_n$ of $T_{\gamma(0)}M$ such that $e_1 = \dot{\gamma}(0)$. Let $e_i(t)$ be the parallel transport of e_i along γ. Let J be a normal Jacobi field along γ. In the basis $e_i(t)$, it looks like:

$$J(t) = \sum_{i=2}^{n} J^i(t) e_i(t),$$

where the $J^i : \mathbb{R} \to \mathbb{R}$ are functions. Write the functions J^i explicitly, each of them depends on two parameters.