RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II

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Übungsblatt 2

Recap of differential geometry

Exercise 1. The aim of this exercise is to prove the following proposition:

Proposition. Let B and C be manifolds, with B connected. Let $p : C \to B$ be a local diffeomorphism, satisfying the path lifting property. Then p is a covering map.

Recall that a map $p : B \to C$ is said to satisfy the path lifting property if for every path $\gamma : [0,1] \to B$ and for all $c \in p^{-1}(\gamma(0))$ there is a path $\bar{\gamma} : [0,1] \to C$ with $\bar{\gamma}(0) = c$ and $p \circ \bar{\gamma} = \gamma$.

Now, let B, C, p be as in the proposition. Prove the following:

(a) Squares can be lifted, i.e. if $\phi : [0,1] \times [0,1] \to B$ is a map and $c \in p^{-1}(\phi(0,0))$, then there is a unique $\bar{\phi} : [0,1] \times [0,1] \to C$ with $\bar{\phi}(0,0) = c$ and $p \circ \bar{\phi} = \phi$.

(b) Let $\gamma_1, \gamma_2 : [0,1] \to B$ be two paths with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Let $c \in p^{-1}(\gamma(0))$, and consider their lifts $\bar{\gamma_1}, \bar{\gamma_2}$ such that $\bar{\gamma_1}(0) = \bar{\gamma_2}(0) = c$. Prove that if γ_1 and γ_2 are homotopic relatively to the end-points, then $\bar{\gamma_1}(1) = \bar{\gamma_2}(1)$ and $\bar{\gamma_1}$ and $\bar{\gamma_2}$ are also homotopic relatively to the end-points.

(c) Let X be a simply connected manifold and $f : X \to B$ a smooth map. Choose $x \in X$ and $c \in p^{-1}(f(x))$. Then f can be lifted to some $\bar{f} : X \to C$ such that $\bar{f}(x) = c$ and $p \circ \bar{f} = f$.

(d) Assume that C is connected and B is simply connected. Prove that in this case p is a diffeomorphism.

(e) Prove the proposition.

Exercise 2. Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of latitude on $\mathbb{S}^2$ and v a tangent vector to $\mathbb{S}^2$ at a point of c. Describe geometrically the parallel transport of v along c.

Hint: Consider the cone C tangent to $\mathbb{S}^2$ along c, and show that the parallel transport of v along c is the same, whether taken relative to $\mathbb{S}^2$ or to C.

Exercise 3. Let $p : \bar{M} \to M$ be a covering map, and g be a Riemannian metric on M.

(a) Show that there exists a Riemannian metric $\bar{g}$ on $\bar{M}$ such that p is a local isometry. Such metric is called the *covering metric.*

(b) Show that g is complete if and only if $\bar{g}$ is complete.