RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Übungsblatt 9

Decompositions into products

Not to hand in.

Exercise 1. (Totally geodesic submanifold)

(a) Show that if the symmetric space $\mathcal{X}$ splits as the Riemannian product $\mathcal{X}_1 \times \mathcal{X}_2$, for every Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g} = \mathrm{Lie}(\mathrm{Isom}^0(\mathcal{X}))$ associated to a point $x \in \mathcal{X}$, then there are Lie triple systems $\mathfrak{n}_1, \mathfrak{n}_2 < \mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.

(b) In $\mathfrak{sp}(2n, \mathbb{R})$ denote by $\mathfrak{n}_1$ (resp $\mathfrak{n}_2$) the Lie triple system

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \mid A \text{ symmetric} \right\}$$

$$\mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \mid A \text{ symmetric} \right\}$$

Show that there is a Cartan decomposition such that $\mathfrak{p} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$.

(c) Is the symmetric space associated to $\mathfrak{sp}(2n, \mathbb{R})$ irreducible?

Exercise 2. (Iwasawa decomposition)

(a) Determine the factors K, A, N of the Iwasawa decomposition for the following Lie groups: $SL_n(\mathbb{R})$, $Sp(2n, \mathbb{R})$, $SO_0(p, q)$.

(b) Given a unimodular basis of $\mathbb{R}^n$, let M be the matrix of change of basis, and let M' be the matrix of change of basis of the orthogonal basis obtained from the Gram-Schmidt process. Verify that M' is the K-factor of the Iwasawa decomposition of M.

(c) If we start with a symplectic basis of $\mathbb{R}^{2n}$, does the orthogonal basis obtained from the Gram-Schmidt process assume a special form?