RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 21.12.2017

Übungsblatt 7

Christmas exercise sheet

Not to hand in.
Optionally, if handed in by January 10, 14:00, every exercise is worth up to 3 bonus points.

Exercise 1. (Restriction of the Killing form) Give an example of a simple Lie algebra $\mathfrak{g}$ with a subalgebra $\mathfrak{t}$ such that $B_{\mathfrak{g}}|_{\mathfrak{t}}$ is not a scalar multiple of $B_{\mathfrak{t}}$. (Hint: find a non-semisimple Lie algebra that is subalgebra of a compact one.)

Exercise 2. (Exponential maps) Let $M = G/K$ be a globally symmetric space and $p \in M$. Denote by $s_p : M \to M$ the geodesic symmetry about $p \in M$. Recall that $\sigma : G \ni g \to s_p g s_p \in G$, and $\pi : G \ni g \to gp \in M$. Denote by $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ the associated decomposition. Denote by $\exp : \mathfrak{g} \to G$ the exponential map in the sense of Lie groups and by $\mathrm{Exp}_p : T_pM \to M$ the exponential map in the sense of Riemannian manifolds. Recall from class that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{D_e\pi|_{\mathfrak{p}}} & T_pM \\ \downarrow & & \downarrow \\ G & \xrightarrow{\pi} & M \end{array}$$

where the arrow $\mathfrak{p} \to G$ is given by $\exp|_{\mathfrak{p}}$ and the arrow $T_pM \to M$ is given by Exp_p.

Make this explicit for $M = \mathbb{R}^2, \mathbb{S}^2, \mathbb{H}^2$.

Exercise 3. (Irreducible OSLAs) Show that $(\mathfrak{sl}(n, \mathbb{R}), \Theta)$ is an irreducible orthogonal symmetric Lie algebra.

Exercise 4. (Totally geodesic subspaces) Consider $O(p, q) < SL(n, \mathbb{R})$. Choose a point $p \in SL(n, \mathbb{R})/O(n)$ such that the $O(p, q)$-orbit of p is a copy of the positive Grassmannian embedded as a totally geodesic subspace of $SL(n, \mathbb{R})/O(n)$.

Exercise 5. (Spin group) In the 4-dimensional vector space $\mathbb{H}$ of the quaternions, consider the 3-dimensional subspace V of all totally imaginary quaternions. Consider the sphere of the quaternions

$$\mathbb{S}^3 = \{q \in \mathbb{H} \mid |q| = 1\}$$

Prove that $\mathbb{S}^3$, is a Lie group with reference to the quaternion multiplication. Consider the action of $\mathbb{S}^3$ on $\mathbb{H}$ by conjugation:

$$\mathbb{S}^3 \times \mathbb{H} \ni (q, p) \to qpq^{-1} \in \mathbb{H}$$

Prove that this action preserves V, and use this action on V to construct a group homomorphism

$$\mathbb{S}^3 \longrightarrow SO(3)$$

which is a covering. Compute the kernel of this homomorphism and deduce the fundamental group of $SO(3)$. This show that $\mathbb{S}^3$ is the universal covering of $SO(3)$. This group is usually called spin group and denoted by Spin(3).