RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
Heidelberg, 7.12.2017

ÜBUNGSBLATT 6

Riemannian symmetric pairs

To hand in by December 20, 14:00.

Exercise 1. (Riemannian symmetric pairs)

(a) Explicitely give two distinct involutive automorphisms $\sigma_i : SO(n) \to SO(n)$ such that the groups G_{σ_i} are non tivial and such that the quotients G/G_{σ_i} are isomorphic. What are the groups G_{σ_i} in your example?

(b) Find on Wikipedia the list of all symmetric spaces for $SO(n)$, and for every one of them, give one example of an involutive automorphism $\sigma : SO(n) \to SO(n)$ that gives it as quotient.

Exercise 2. (Orthogonal symmetric Lie algebras)

(a) Show that, if the OSLA $(\mathfrak{g}, \sigma)$ is of compact type, and comes from the symmetric space $\mathcal{X}$, then $\mathcal{X}$ is compact.

(b) Show that the symmetric space $SL(n, \mathbb{R})/SO(n)$ is non-compact. Does the associated orthogonal symmetric Lie algebra have a well defined type?

(c) Consider the symmetric space $\mathbb{R}^n$, with $G = O(n) \ltimes \mathbb{R}^n$. Show that $\mathfrak{g}$ is of Euclidean type.

Exercise 3. (The Symmetric space asssociated to $SO_0(p, q)$)

(a) What is the Lie algebra of $SO_0(p, q)$?

(b) What is the Cartan involution and Cartan decomposition associated to a basepoint of the symmetric space $SO(p, q)/S(O(p)xO(q))$?

(c) Consider the open subset of the Grassmannian $P(p, p+q)$ consisting of all the p-planes such that the restriction of the symmetric bilinear form of signature (p, q) is positive definite. Identify this open subset with the symmetric space associated to $SO_0(p, q)$.

Exercise 4. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be subalgebras of $\mathfrak{g}_1$ an $\mathfrak{g}_2$ respectively, and set $\mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$. Prove that $\mathfrak{t}$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively.