RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
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ÜBUNGSBLATT 5

Compact symmetric spaces

Not to hand in.

Exercise 1. Orthogonal complex structures. Consider $\mathbb{R}^{2n}$ with the standard scalar product. A complex structure on $\mathbb{R}^{2n}$ is a multiplication by i, i.e. a linear map J such that $J^2 = -1$. A complex structure J is orthogonal if J is an orthogonal linear map.

1. Prove that the set S of orthogonal complex structures coincides with the set of all matrices that are orthogonal and antisymmetric.
2. Prove that the group $O(2n)$ acts by conjugation on S, that the action is transitive and that the stabilizer of any point is isomorphic to the unitary group $U(n)$.
3. Put a structure of symmetric space on S.

Exercise 2. Real structures. Consider the complex vector space $\mathbb{C}^n$, with the standard positive definite Hermitian form. It can be seen as a real vector space $\mathbb{R}^{2n}$, with a linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $J^2 = -1$. The real part of h gives a scalar product $<,>$ on $\mathbb{R}^{2n}$. A totally real subspace is a real vector subspace V of dimension n such that V is orthogonal to $J(V)$ w.r.t. $<,>$. A linear map A is complex antilinear if it anticommutes with J, i.e. if $A(J(v)) = -J(A(v))$.

1. Verify that, for the scalar product $<,>$ on $\mathbb{R}^{2n}$, the map J is orthogonal.
2. Prove that the map that associates to a complex anti-linear reflection its set of fixed points is a bijection between the set of complex anti-linear reflections and the set of totally real subspaces.
3. The set $TR(n)$ of totally real subspaces is a subset of the Grassmannian $P(n, 2n)$. Prove that it is a totally geodesic submanifold. (Hint: show that $TR(n)$ is the subset of fixed points of an isometry).
4. Verify that the real span of an unitary basis is a totally real subspace, and that an orthogonal basis of a totally real subspace is an unitary basis.
5. Describe the set $TR(n)$ as $U(n)/O(n)$, and give it the structure of a symmetric space.

Exercise 3. Fubini-Study metric. Consider the vector space $\mathbb{C}^{n+1}$ with the standard positive definite Hermitian form:

$$h(x, y) = \sum_{i=0}^{n} x_i \bar{y}_i$$

Denote by $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ the usual projection into the projective space. Recall also that you can identify the tangent space at every point of $\mathbb{C}^{n+1}$ with the vector space $\mathbb{C}^{n+1}$.

1. Show that, $d\pi_v$ identifies the orthogonal space at v for the form h with the tangent space at $x = \pi(v)$.
2. Define a Riemannian metric on $\mathbb{CP}^n$ using the real part of the restriction of h to these orthogonal subspaces. This metric is usually called the Fubini-Study metric.
3. Show that every element of $U(n+1)$ induces an isometry of this metric.
4. Show that the stabilizer of any point for the action of $U(n+1)$ is isomorphic to $U(1)\times U(n)$.
5. Show that, with this metric, the space $\mathbb{CP}^n$ is a symmetric space.