RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematatisches Institut

Vorlesung Differentialgeometrie I
Heidelberg, 26.10.2017

Übungsblatt 3

Symmetric spaces

Not to hand in

Exercise 1. Let (M, g_M), (N, g_N) be two Riemannian manifolds with parallel curvature tensors. Given $m \in M$ and $n \in N$, assume that there exists a linear isometry $\phi : T_m M \to T_n N$ preserving the Riemann curvature tensors, i.e. such that for all $u, v, w \in T_m M$,

$$R^N_n(\phi(u), \phi(v))\phi(w) = \phi(R^M_m(u, v)w).$$

Prove that there exist normal neighborhoods U, V of m and n respectively and an isometry $f : U \to V$ such that $f(m) = n$ and $D_m f = \phi$.

Exercise 2. Let $\mathrm{Sym}(n, \mathbb{R})$ denote the vector space of symmetric $n \times n$ matrices. Consider the open subset of positive definite matrices:

$$\mathcal{P} = \{M \in \mathrm{Sym}(n, \mathbb{R}) \mid M > 0\}$$

As usual for open subsets of vector spaces, we will identify the tangent space at every point with $\mathrm{Sym}(n, \mathbb{R})$ itself:

$$\forall M \in \mathcal{P}, \quad T_M\mathcal{P} = \mathrm{Sym}(n, \mathbb{R})$$

We will denote by $\mathrm{Id} \in \mathcal{P}$ the identity matrix.

(a) Prove that the formula
$$g_M(S, T) = \mathrm{tr}(M^{-1}SM^{-1}T)$$
intended as a bilinear form on the vector space $T_M\mathcal{P}$, for a $M \in \mathcal{P}$, defines a Riemannian metric on $\mathcal{P}$.

(b) Prove that the formula
$$GL(n, \mathbb{R}) \times \mathcal{P} \ni (A, M) \longrightarrow A^T M A \in \mathcal{P}$$
defines an action of $GL(n, \mathbb{R})$ on $\mathcal{P}$, given by base change of scalar products.

(c) Prove that the given action of $GL(n, \mathbb{R})$ is transitive.

(d) Prove that the given action of $GL(n, \mathbb{R})$ is by isometries for the Riemannian metric g.

(e) Compute the stabilizer of the point Id.

(f) Prove that $(\mathcal{P}, g)$ is a symmetric space.

(g) Consider the submanifold

$$\{M \in \mathcal{P} \mid \det(M) = 1\}$$

with the induced Riemannian metric. Prove that it is a symmetric space.

Exercise 3. Consider the action of the group $\mathbb{Z}^n$ on $\mathbb{R}^n$ by integral translations:

$$\mathbb{Z}^n \times \mathbb{R}^n \ni ((m_1, \dots, m_n), (x_1, \dots, x_n)) \longrightarrow (x_1 + m_1, \dots, x_n + m_n) \in \mathbb{R}^n.$$

Notice that the translations are isometries for the standard Riemannian metric of $\mathbb{R}^n$. The flat torus is the quotient $\mathbb{R}^n/\mathbb{Z}^n$, with its induced Riemannian metric. Prove that the flat torus is a symmetric space.