RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
Heidelberg, 26.10.2017

Übungsblatt 2

Hyperbolic spaces and Jacobi fields

To hand in by November 9, at 14:00

Exercise 1. Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

(a) Given $a, b, c > 0$, consider points $[A], [B], [C] \in \mathbb{K}H^n$ such that:

$$d([A],[B]) = a + b, \quad d([B],[C]) = b + c, \quad d([C],[A]) = c + a.$$

Consider the point $[F]$ on the geodesic segment joining $[A]$ and $[B]$ with $d([A],[F]) = a$, and the point $[G]$ on the geodesic segment joining $[A]$ and $[C]$ with $d([A],[G]) = a$.

Prove that $d([F],[G])$ is bounded by a universal constant that does not depend on n, on a, b, c nor on $[A], [B], [C]$.

(Hint: choose representative vectors $A, B, C \in \mathbb{K}^{n+1}$. Compute the tangent vectors $u, v \in A^\perp$ such that u is tangent to the geodesic from $[A]$ to $[B]$ and v is tangent to the geodesic from $[A]$ to $[C]$. Write the geodesics and find representatives of the points $[F]$ and $[G]$. Express $\langle F, G \rangle$ as a function of a and $\langle u, v \rangle$, then find a bound. It can be useful to separate the analysis in two cases: $a \leq 1$ and $a \geq 1$).

(b) Prove that all the metric spaces $\mathbb{K}H^n$ are δ-hyperbolic for some δ, i.e. that all triangles are δ-thin.

Exercise 2. Consider the matrix

$$I_{n,1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Consider the following subsets of the space of matrices:

$$O(n,1) = \{A \in \mathrm{Mat}(n+1, n+1, \mathbb{R}) \mid A^T I_{n,1} A = I_{n,1}\}$$

$$U(n,1) = \{A \in \mathrm{Mat}(n+1, n+1, \mathbb{C}) \mid \bar{A}^T I_{n,1} A = I_{n,1}\}$$

$$Sp(n,1) = \{A \in \mathrm{Mat}(n+1, n+1, \mathbb{H}) \mid \bar{A}^T I_{n,1} A = I_{n,1}\}$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, we will denote by $O_{\mathbb{K}}(I_{n,1})$ the subspace $O(n,1), U(n,1)$ or $Sp(n,1)$ that is embedded in $\mathrm{Mat}(n+1, n+1, \mathbb{K})$.

(a) Prove that these subsets are groups for matrix multiplication.

(b) Prove that they are submanifolds of $\mathrm{Mat}(n+1, n+1, \mathbb{K})$, and compute their dimension.

(c) Prove that the induced action of $O_{\mathbb{K}}(I_{n,1})$ on $\mathbb{KP}^n$ preserves the subset $\mathbb{K}H^n$, and that they act on $\mathbb{K}H^n$ by isometries.

(d) Prove that the action of $O_{\mathbb{K}}(I_{n,1})$ on $\mathbb{K}H^n$ is transitive and isotropic (i.e. it is transitive on the tangent bundle).

(e) Compute the stabilizer of the point $(0, \ldots, 0, 1) \in \mathbb{K}H^n$, prove that it is isomorphic to $O(n) \times O(1)$, $U(n) \times U(1)$, $Sp(n) \times Sp(1)$.

(f) Describe the action of the second factor of the stabilizer.

(g) Prove that there exists an element of $O_{\mathbb{K}}(I_{n,1})$ that fixes $(0, \ldots, 0, 1) \in \mathbb{K}H^n$ and whose differential acts on the tangent space as $-\mathrm{Id}$. Conclude that $\mathbb{K}H^n$ is a symmetric space.

Exercise 3. Let M be a Riemannian manifold of constant sectional curvature K, and let $\gamma : [0, \ell] \to M$ be a geodesic parametrized by arc-lenght, let γ' be its derivative and let J be a Jacobi field along γ, normal to γ'.

Recall a fact from Differential Geometry 1: the Riemann tensor of M can be written explicitely as

$$\langle R(X,Y)W, Z\rangle = K\left(\langle X, W\rangle \langle Y, Z\rangle - \langle Y, W\rangle \langle X, Z\rangle\right)$$

(a) Prove that $R(\gamma', J)\gamma' = KJ$.

(b) Prove that the Jacobi equation can be written as

$$\frac{D^2 J}{dt^2} + KJ = 0$$

(c) Let $w(t)$ be a parallel field along γ with $\langle \gamma'(t), w(t)\rangle = 0$ and $|w(t)| = 1$. Prove that the solution of the Jacobi equation with initial conditions $J(0) = 0$, $J'(0) = w(0)$ are given by:

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t), & \text{if } K > 0 \\ t w(t), & \text{if } K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t), & \text{if } K < 0 \end{cases}$$

(d) Find all the pairs of conjugate points on the sphere $\mathbb{S}^n$.