RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
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Übungsblatt 1

Grassmannians

Not to hand in

Exercise 1. Consider the following subsets of the space of matrices:

$$O(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{R}) \mid A^T A = AA^T = I\}$$

$$U(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{C}) \mid \bar{A}^T A = A\bar{A}^T = I\}$$

$$Sp(n) = \{A \in \mathrm{Mat}(n, n, \mathbb{H}) \mid \bar{A}^T A = A\bar{A}^T = I\}$$

(a) Prove that these subsets are groups for matrix multiplication.

(b) Prove that they are submanifolds of $\mathrm{Mat}(n, n, \mathbb{R})$, $\mathrm{Mat}(n, n, \mathbb{C})$ or $\mathrm{Mat}(n, n, \mathbb{H})$, respectively, and compute their dimension.

(c) Prove that they are compact subsets.

(d) Show that some of them are connected and some are disconnected.

Exercise 2. Consider the vector space $\mathbb{R}^n$, and let b denote the standard scalar product: $b(v, w) = \langle v, w \rangle$. Consider the vector space $\mathrm{End}(\mathbb{R}^n)$ of linear maps from $\mathbb{R}^n$ to itself.

(a) By identifying $\mathrm{End}(\mathbb{R}^n)$ with $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, show that $b^{\mathrm{End}} = b^* \otimes b$ defines a scalar product on $\mathrm{End}(\mathbb{R}^n)$.

(b) For every element $A \in O(n)$, prove that the map:

$$\mathrm{End}(\mathbb{R}^n) \ni M \longrightarrow AMA^{-1} \in \mathrm{End}(\mathbb{R}^n)$$

is linear and it preserves the scalar product b^{End}.

Exercise 3. Define the subset

$$P(n) = \{M \in \mathrm{End}(\mathbb{R}^n) \mid M = M^T, M^2 = M\}$$

(a) Prove that every matrix in $P(n)$ has integer trace between 0 and n. Denote by $P(k,n)$ the set of elements of $P(n)$ with trace k.

(b) Choose a basis of $\mathbb{R}^n$, and write $M \in \mathrm{End}(\mathbb{R}^n)$ as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are submatrices. Write the conditions for $M \in P(n)$ in terms of A, B, C, D.

(c) With the notation above, assume that A is a $k \times k$ matrix of rank k. Prove that M has rank k if and only if $D = CA^{-1}B$. (Hint: multiply on the right by the matrix $\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}$)

(d) Prove that, if A is a $k \times k$ matrix of rank k, then $M \in P(k,n)$ if and only if

$$A = A^T, C = B^T, D = CA^{-1}B, A^2 + BC = A$$

(e) Prove that

$$\{(A,B) \in \mathrm{Mat}(k,k,\mathbb{R}) \times \mathrm{Mat}(k,n-k,\mathbb{R}) \mid A^2 + BB^T = A\}$$

is a manifold around the point $(I, 0)$.

(f) Prove that $P(k,n)$ is a manifold, and compute the dimension.

Exercise 4. Denote by $\mathrm{Gr}(k,n)$ the set of all vector subspaces of $\mathbb{R}^n$ of dimension k. This set is called the *Grassmannian* of k-planes.

(a) Show that the map that associates to every element of $P(k,n)$ its image is a bijection with $\mathrm{Gr}(k,n)$. This gives the Grassmannian a structure of smooth manifold.

(b) Consider the Riemannian metric on $P(k,n)$ induced by the scalar product b^{End}. Prove that the action of $O(n)$ on $\mathrm{End}(\mathbb{R}^n)$ preserves $P(k,n)$, and it acts on it by Riemannian isometries.

(c) Prove that the action of $O(n)$ on $P(k,n)$ is transitive.

(d) Choose a point $M \in P(k,n)$ and prove that the stabilizer of this point:

$$\mathrm{Stab}_{O(n)}(M) = \{A \in O(n) \mid A(M) = M\}$$

is isomorphic to $O(k) \times O(n-k)$.

(e) Prove that $P(k,n)$ is compact.