RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 3.07.2013

Exercise sheet 10

Hadamard manifolds

To hand in by Wednesday July 10, 13:00

Exercise 1. Consider the vector space $\mathbb{R}^{n+1}$, with the standard bilinear form of signature $(1, n)$:

$$< x, y >= -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$$

The hyperboloid model of the hyperbolic space is the set

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid < x, x >= -1, x_0 > 0\}$$

Consider the projection in the projective space $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$. The projective model of the hyperbolic space is the ellipsoid

$$\mathbb{B}^n = \{\pi(x) \in \mathbb{RP}^n \mid < x, x >< 0\}$$

The Riemannian metric of the hyperbolic space at the point x can be defined, in the hyperboloid model, by the restriction of $<, >$ to the orthogonal subspace $x^\perp$. In the projective model it is defined by the diffeomorphism $\pi|_{\mathbb{H}^n}$.

1. Prove that the group of isometries of the hyperbolic space is isomorphic to $O^+(1, n)$, namely the subgroup of $O(1, n)$ made of matrices that preserve $\mathbb{H}^n$, and that the stabilizer of a point is isomorphic to $O(n)$.

2. Prove that the geodesics in the hyperboloid model are intersections of $\mathbb{H}^n$ with vector subspaces of dimension 2. (Hint: use bases and the isometry group).

3. Prove that geodesics in the projective model are intersections of projective lines with $\mathbb{B}^n$.

4. (Bonus) Prove that the distance between two points $x, y \in \mathbb{H}^n$ is given by $\operatorname{arcosh}(< x, y >)$, and that the distance between two points $x, y \in \mathbb{B}^n$ is given by

$$\frac{1}{2} \log \frac{|x - y_0||y - x_0|}{|x - x_0||y - y_0|}$$

   where x_0, y_0 are the extremes of the segment intersection between the line containing x, y and $\mathbb{B}^n$, x_0 on the side of x, and y_0 on the side of y. (This exercise is a bonus exercise because it is not very interesting, just computations. But if you have time, you can try.)

5. Find a parametrization of a geodesic in $\mathbb{B}^n$ with unit speed (Hint: find a projective map from $\mathbb{RP}^1$ to your line that sends 0 in x_0 and ∞ in y_0, and use logarithms).

6. Show that two geodesic rays are asymptotics if and only if they intersect the boundary of $\mathbb{B}^n$ in the same point. This identifies the visual boundary of $\mathbb{H}^n$ with the boundary of $\mathbb{B}^n$.

7. Compute the orbit of a point in the visual boundary.

Exercise 2. Let (M, g) be a Hadamard manifold, and $x, y, z \in M$. Assume that the angles of the triangle $T = T_M(x, y, z)$ satisfy $\hat{x} + \hat{y} + \hat{z} = \pi$. Prove that the convex hull $\mathrm{Conv}(T)$ is isometric to the convex hull of the comparison triangle $T_0 = T(x_0, y_0, z_0)$.

(Hint: Define a function $\Phi : \mathrm{Conv}(T_0) \to \mathrm{Conv}(T)$ in the following way: if $a_0 \in \partial T_0$, for example $a_0 \in \gamma_{x_0 y_0}$, define $\Phi(a_0) = a$, where a is the point of γ_{xy} that has the same distance from x, y as a_0 from x_0, y_0. If $a_0 \in \mathrm{Conv}(T_0) \setminus \partial T$, there exists a unique point $a_0' \in \gamma_{x_0 y_0}$ such that $a_0 \in \gamma_{z_0 a_0'}$. Since $d(z_0, a_0') = d(z, a')$, you can define $\Phi(a_0)$ as the point $a \in \gamma_{z,a'}$ that has the same distance from z, a' as a_0 from z_0, a_0'. To prove that this map preserves distances, given $a_0, b_0 \in \mathrm{Conv}(T_0) \setminus \partial T$, you can prove that $T(z_0, a_0', b_0')$ is a comparison triangle for $T(z, a', b')$. To prove that this map is surjective, you can use the fact that a curve that realizes the distance between two points in a Riemannian manifold is a geodesic arc.)