RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
Heidelberg, 26.06.2013

EXERCISE SHEET 9
Abelian subalgebras
To hand in by Wednesday July 3, 13:00

Exercise 1. Consider the Lie algebra

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid tr(A) = 0\}$$

Consider the involution $\sigma(A) = -A^T$, where A^T is the transpose of A.

1. Compute the decomposition $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{k} + \mathfrak{p}$.
2. Prove that the set

$$\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \sum \lambda_i = 0 \right\}$$

   is a maximal abelian subalgebra of $\mathfrak{p}$.
3. Find the roots and the root spaces.
4. Verify that $\mathfrak{g}_0$ (the root space associated with the root 0) is equal to $\mathfrak{a}$.
5. Compute a Weil chamber.

Exercise 2. On the vector space $\mathbb{R}^{2n}$, consider the standard symplectic form

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. The simplectic Lie algebra is the set of matrices

$$\mathfrak{sp}(2n,\mathbb{R}) = \{A \in \mathfrak{gl}(2n,\mathbb{R}) \mid \Omega A + A^T \Omega = 0\}$$

Consider the involution $\sigma(A) = -A^T$.

1. Compute the decomposition $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{k} + \mathfrak{p}$.

2. Prove that the set

$$\left\{ \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix} \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}$$

is a maximal abelian subalgebra of $\mathfrak{p}$.

3. Find the roots and the root spaces.

4. Verify that $\mathfrak{g}_0$ (the root space associated with the root 0) is equal to $\mathfrak{a}$.

5. Compute a Weil chamber.

Exercise 3. On the vector space $\mathbb{R}^{k+n}$, with $n > k+1$, consider the following bilnear form:

$$< x, y >= x_1 y_{n+k} + \cdots + x_k y_{n+1} + x_{k+1} y_{k+1} + \cdots + x_n y_n + x_{n+1} y_k + \cdots + x_{n+k} y_1$$

Let B be the matrix associated with this bilinear form of signature (k, n). Consider the indefinite orthogonal group

$$\mathfrak{o}(k, n) = \{A \in \mathfrak{gl}(n + k, \mathbb{R}) \mid BA + A^T B = 0\}$$

Consider the involution $\sigma(A) = -A^T$.

1. Verify that the signature of this bilinear form is actually (k, n)

2. Compute the decomposition $\mathfrak{o}(k, n) = \mathfrak{k} + \mathfrak{p}$.

3. Prove that the set

$$\left\{ \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_k & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & -\lambda_k & & \\ & & & & & & & \ddots & \\ & & & & & & & & -\lambda_1 \end{pmatrix} \mid \lambda_1, \dots, \lambda_k \in \mathbb{R} \right\}$$

is a maximal abelian subalgebra of $\mathfrak{p}$.

4. Find the roots and the root spaces.

5. Verify that $\mathfrak{g}_0$ (the root space associated with the root 0) strictly contains $\mathfrak{a}$.

6. Compute a Weil chamber.