



EXERCISE SHEET 8

More examples*To hand in by Wednesday June 26, 13:00*

Exercise 1. Fubini-Study metric. Consider the vector space \mathbb{C}^{n+1} with the standard positive definite Hermitian form:

$$h(x, y) = \sum_{i=0}^n x_i \bar{y}_i$$

Denote by $\pi : \mathbb{C}^{n+1} \setminus 0 \rightarrow \mathbb{C}\mathbb{P}^n$ the usual projection into the projective space. Recall also that you can identify the tangent space at every point of $\mathbb{C}\mathbb{P}^n$ with the vector space \mathbb{C}^{n+1} .

1. Show that, $d\pi_v$ identifies the orthogonal space at v for the form h with the tangent space at $x = \pi(v)$.
2. Define a Riemannian metric on $\mathbb{C}\mathbb{P}^n$ using the real part of the restriction of h to these orthogonal subspaces. This metric is usually called the Fubini-Study metric.
3. Show that every element of $U(n+1)$ induces an isometry of this metric.
4. Show that the isotropy group at any point for the action of $U(n+1)$ is isomorphic to $U(1) \times U(n)$.
5. Show that, with this metric, the space $\mathbb{C}\mathbb{P}^n$ is a symmetric space of compact type.
6. Find the dual symmetric space.

Exercise 2. Real structures. Consider the complex vector space \mathbb{C}^n , with the standard positive definite Hermitian form. It can be seen as a real vector space \mathbb{R}^{2n} , with a linear map $J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ with $J^2 = -1$. The real part of h gives a scalar product \langle, \rangle on \mathbb{R}^{2n} . A totally real subspace is a real vector subspace V of dimension n such that V is orthogonal to $J(V)$ w.r.t. \langle, \rangle . A linear map A is complex antilinear if it anticommutes with J , i.e. if $A(J(v)) = -J(A(v))$.

1. Verify that, for the scalar product \langle, \rangle on \mathbb{R}^{2n} , the map J is orthogonal.
2. Prove that the map that associates to a complex anti-linear reflection its set of fixed points is a bijection between the set of complex anti-linear reflections and the set of totally real subspaces.
3. The set $TR(n)$ of totally real subspaces is a subset of the Grassmannian $P(n, 2n)$. Prove that it is a totally geodesic submanifold. (Hint: identify the Grassmannian with a subset of the set of reflections $R(2n)$, defined in exercise sheet 7, and show that $TR(n)$ is the set of fixed points of the isometry $X \rightarrow JXJ$).
4. Verify that the real span of an unitary basis is a totally real subspace, and that an orthogonal basis of a totally real subspace is an unitary basis.
5. Describe the set $TR(n)$ as $U(n)/O(n)$, and give it the structure of a symmetric space of compact type.
6. Find the dual symmetric space.