RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 12.06.2013

Exercise sheet 7
More on Grassmannians
To hand in by Wednesday June 19, 13:00

Exercise 1. Let (M, g) be a symmetric space, either of compact type or of non-compact type. We use the usual notation, $G = \mathrm{Isom}(M, g)$, $K < G$ the isotropy subgroup of a point, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{k} \subset \mathfrak{g}$ the Lie algebra of K, $\mathfrak{p} \subset \mathfrak{g}$ the complementary subspace.

Prove that $\mathfrak{p}$ generates $\mathfrak{g}$ as a Lie algebra (i.e. the smallest Lie subalgebra containing $\mathfrak{p}$ is $\mathfrak{g}$).

Exercise 2. The open Grassmannian. Consider the vector space $\mathbb{R}^n$, with a non-degenerate symmetric bilinear form b of signature $(k, n-k)$. A vector subspace of $\mathbb{R}^n$ is positive definite if the restriction of b to the subspace is a positive definite form. Consider the set X of all positive definite vector subspaces of $\mathbb{R}^n$ of dimension k.

1. Prove that X, considered as a subset of the Grassmannian $P(k, n)$ defined in exercise sheet 6, is open.

2. Define a transitive action of the group $O(k, n-k)$ on X, with isotropy subgroup isomorphic to $O(k) \times O(n-k)$.

3. Define a structure of symmetric space on X, as in exercise sheet 6.

4. Compute the OILA and the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the open Grassmannian X and for the standard Grassmannian $P(k, n)$. Find the dual symmetric spaces, i.e. the symmetric spaces associated with the dual OILA.

Exercise 3. Other constructions of the Grassmannian.

1. Consider the vector space $\mathbb{R}^n$ with the standard scalar product. Let $S(n)$ be the set of all symmetric matrices, and $O(n)$ be the set of all orthogonal matrices. Prove that the intersection $R(n) = S(n) \cap O(n)$ is the set of all reflections, i.e. the orthogonal involutions. Construct a natural bijection between $R(n)$ and $P(n)$ (the space of projections defined in exercise sheet 6), and describe Grassmannians as subsets of $R(n)$, in terms of traces.

2. Consider the action of $GL(n, \mathbb{R})$ on the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. Verify that the action is transitive and that the stabilizer H of a point is non-compact. This identifies the Grassmannian with the homogeneous space $GL(n, \mathbb{R})/H$.

3. Let Q be a vector subspace of $\mathbb{R}^n$ of dimension $n-k$. Consider the set U_Q of all vector subspaces of $\mathbb{R}^n$ of dimension k that are transversal to Q. The set U_Q can be parametrized by $\mathbb{R}^{k(n-k)}$ in the following way: choose a base point $P \in U_Q$ and consider the set $\mathrm{Hom}(P, Q)$ of all linear maps from P to Q. Prove that the map that associates to every $A \in \mathrm{Hom}(P, Q)$ its graph $L_A = \{x + Ax \mid x \in P\}$ is a bijection between $\mathrm{Hom}(P, Q)$ and U_Q. Use this to prove that the Grassmannian is a topological manifold of dimension $k(n-k)$.