Exercise 1. Let \((M, g)\) be a symmetric space, either of compact type or of non-compact type. We use the usual notation, \(G = \text{Isom}(M, g)\), \(K < G\) the isotropy subgroup of a point, \(\mathfrak{g}\) the Lie algebra of \(G\), \(\mathfrak{t} \subset \mathfrak{g}\) the Lie algebra of \(K\), \(\mathfrak{p} \subset \mathfrak{g}\) the complementary subspace. Prove that \(\mathfrak{p}\) generates \(\mathfrak{g}\) as a Lie algebra (i.e. the smallest Lie subalgebra containing \(\mathfrak{p}\) is \(\mathfrak{g}\)).

Exercise 2. The open Grassmannian. Consider the vector space \(\mathbb{R}^n\), with a non-degenerate symmetric bilinear form \(b\) of signature \((k, n-k)\). A vector subspace of \(\mathbb{R}^n\) is positive definite if the restriction of \(b\) to the subspace is a positive definite form. Consider the set \(X\) of all positive definite vector subspaces of \(\mathbb{R}^n\) of dimension \(k\).

1. Prove that \(X\), considered as a subset of the Grassmannian \(P(k, n)\) defined in exercise sheet 6, is open.

2. Define a transitive action of the group \(O(k, n-k)\) on \(X\), with isotropy subgroup isomorphic to \(O(k) \times O(n-k)\).

3. Define a structure of symmetric space on \(X\), as in exercise sheet 6.

4. Compute the OILA and the decomposition \(\mathfrak{g} = \mathfrak{t} + \mathfrak{p}\) for the open Grassmannian \(X\) and for the standard Grassmannian \(P(k, n)\). Find the dual symmetric spaces, i.e. the symmetric spaces associated with the dual OILA.

Exercise 3. Other constructions of the Grassmannian.

1. Consider the vector space \(\mathbb{R}^n\) with the standard scalar product. Let \(S(n)\) be the set of all symmetric matrices, and \(O(n)\) be the set of all orthogonal matrices. Prove that the intersection \(R(n) = S(n) \cap O(n)\) is the set of all reflections, i.e. the orthogonal involutions. Construct a natural bijection between \(R(n)\) and \(P(n)\) (the space of projections defined in exercise sheet 6), and describe Grassmannians as subsets of \(R(n)\), in terms of traces.

2. Consider the action of \(GL(n, \mathbb{R})\) on the set of all \(k\)-dimensional vector subspaces of \(\mathbb{R}^n\). Verify that the action is transitive and that the stabilizer \(H\) of a point is non-compact. This identifies the Grassmannian with the homogeneous space \(GL(n, \mathbb{R})/H\).

3. Let \(Q\) be a vector subspace of \(\mathbb{R}^n\) of dimension \(n-k\). Consider the set \(U_Q\) of all vector subspaces of \(\mathbb{R}^n\) of dimension \(k\) that are transversal to \(Q\). The set \(U_Q\) can be parametrized by \(\mathbb{R}^{k(n-k)}\) in the following way: choose a base point \(P \in U_Q\) and consider the set \(\text{Hom}(P, Q)\) of all linear maps from \(P\) to \(Q\). Prove that the map that associates to every \(A \in \text{Hom}(P, Q)\) its graph \(L_A = \{x + Ax \mid x \in P\}\) is a bijection between \(\text{Hom}(P, Q)\) and \(U_Q\). Use this to prove that the Grassmannian is a topological manifold of dimension \(k(n-k)\).