Exercise 1. Complex hyperbolic space. (This is another version of exercise sheet 3, §2, that was wrong). Consider the vector space $\mathbb{C}^{n+1}$ with the following Hermitian form of signature $(1, n)$:

$$h(x, y) = -x_0 \bar{y}_0 + \sum_{i=1}^{n} x_i \bar{y}_i$$

Denote by $U(1, n)$ the group of matrices preserving the form $h$. Denote by $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ the usual projection into the projective space. Recall also that you can identify the tangent space at every point of $\mathbb{C}^{n+1}$ with the vector space $\mathbb{C}^{n+1}$. The complex hyperbolic $n$-space $\mathbb{CH}^n$ is defined as:

$$\mathbb{CH}^n = \{ x \in \mathbb{CP}^n \mid x = \pi(v), \text{ with } h(v, v) < 0 \}$$

• Show that, if $h(v, v) < 0$, $d\pi_v$ identifies the orthogonal space at $v$ for the form $h$ with the tangent space at $x = \pi(v)$.

• Define a Riemannian metric on $\mathbb{CH}^n$ using the real part of the restriction of $h$ to these orthogonal subspaces.

• Show that every element of $U(1, n)$ induces an isometry of this metric.

• Find an isometry that does not come from the group $U(1, n)$. (Hint: use complex conjugation).

• Show that, with this metric, the space $\mathbb{CH}^n$ is a symmetric space.

Exercise 2. Grassmannians. Consider the vector space $\mathbb{R}^n$ with the standard scalar product. Let $S(n)$ denote the set of all symmetric $n \times n$ matrices. Define the subset

$$P(n) = \{ p \in S(n) \mid p^2 = p \}$$

• Prove that every matrix in $P(n)$ has integer trace between 0 and $n$. Denote by $P(k, n)$ the set of elements of $P(n)$ with trace $k$.

• Show that the map that associates to every element of $P(k, n)$ its image is a bijection with the set of all vector subspaces of dimension $k$.

• Prove that the group $O(n)$ acts on $P(k, n)$ by conjugation, show that the action is transitive and use this to define a structure of $C^\infty$ manifold on $P(k, n)$.

• Prove that the isotropy group of a point is isomorphic to $O(k) \times O(n-k)$.

• Use the description $P(k, n) = O(n)/O(k) \times O(n-k)$ to define a structure of symmetric space on the Grassmannian $P(k, n)$.
Exercise 3. Orthogonal complex structures. Consider $\mathbb{R}^{2n}$ with the standard scalar product. A complex structure on $\mathbb{R}^{2n}$ is a multiplication by $i$, i.e. a linear map $J$ such that $J^2 = -1$. A complex structure $J$ is orthogonal if $J$ is an orthogonal linear map.

- Prove that the set $S$ of orthogonal complex structures coincides with the set of all matrices that are orthogonal and antisymmetric.
- Prove that the group $O(2n)$ acts by conjugation on $S$, that the action is transitive and that the isotropy group is isomorphic to the unitary group $U(n)$.
- Put a structure of symmetric space on $S$. Proceed as in exercise §2.