RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 05.06.2013

Exercise sheet 6

Examples of symmetric spaces

To hand in by Wednesday June 12, 13:00

Exercise 1. Complex hyperbolic space. (This is another version of exercise sheet 3, §2, that was wrong). Consider the vector space $\mathbb{C}^{n+1}$ with the following Hermitian form of signature $(1, n)$:

$$h(x, y) = -x_0\bar{y}_0 + \sum_{i=1}^{n} x_i\bar{y}_i$$

Denote by $U(1, n)$ the group of matrices preserving the form h. Denote by $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ the usual projection into the projective space. Recall also that you can identify the tangent space at every point of $\mathbb{C}^{n+1}$ with the vector space $\mathbb{C}^{n+1}$. The complex hyperbolic n-space $\mathbb{CH}^n$ is defined as:

$$\mathbb{CH}^n = \{x \in \mathbb{CP}^n \mid x = \pi(v), \text{ with } h(v, v) < 0\}$$

- Show that, if $h(v, v) < 0$, $d\pi_v$ identifies the orthogonal space at v for the form h with the tangent space at $x = \pi(v)$.
- Define a Riemannian metric on $\mathbb{CH}^n$ using the real part of the restriction of h to these orthogonal subspaces.
- Show that every element of $U(1, n)$ induces an isometry of this metric.
- Find an isometry that does not come from the group $U(1, n)$. (Hint: use complex conjugation).
- Show that, with this metric, the space $\mathbb{CH}^n$ is a symmetric space.

Exercise 2. Grassmannians. Consider the vector space $\mathbb{R}^n$ with the standard scalar product. Let $S(n)$ denote the set of all symmetric $n \times n$ matrices. Define the subset

$$P(n) = \{p \in S(n) \mid p^2 = p\}$$

- Prove that every matrix in $P(n)$ has integer trace between 0 and n. Denote by $P(k, n)$ the set of elements of $P(n)$ with trace k.
- Show that the map that associates to every element of $P(k, n)$ its image is a bijection with the set of all vector subspaces of dimension k.
- Prove that the group $O(n)$ acts on $P(k, n)$ by conjugation, show that the action is transitive and use this to define a structure of C^∞ manifold on $P(k, n)$.
- Prove that the isotropy group of a point is isomorphic to $O(k) \times O(n - k)$.
- Use the description $P(k, n) = O(n)/O(k) \times O(n - k)$ to define a structure of symmetric space on the Grassmannian $P(k, n)$.

Exercise 3. Orthogonal complex structures. Consider $\mathbb{R}^{2n}$ with the standard scalar product. A complex structure on $\mathbb{R}^{2n}$ is a multiplication by i, i.e. a linear map J such that $J^2 = -1$. A complex structure J is orthogonal if J is an orthogonal linear map.

- Prove that the set S of orthogonal complex structures coincides with the set of all matrices that are orthogonal and antisymmetric.
- Prove that the group $O(2n)$ acts by conjugation on S, that the action is transitive and that the isotropy group is isomorphic to the unitary group $U(n)$.
- Put a structure of symmetric space on S. Proceed as in exercise §2.