RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
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Exercise sheet 5

Compact subgroups

To hand in by Wednesday May 29, 13:00

Let G be a Lie group and K be a compact subgroup of G. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G, $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{z}$ be the center of $\mathfrak{g}$, i.e. $\mathfrak{z} = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}, [x, y] = 0\}$. Recall that every element $g \in G$ defines an automorphism of $\Psi_g : G \to G$ by $\Psi_g(h) = ghg^{-1}$. The diffarential of Ψ_g at the identity is a Lie-algebra automorphism of $\mathfrak{g}$, $d\Psi_g|_e : \mathfrak{g} \to \mathfrak{g}$. The map $\mathrm{Ad} : G \ni g \to d\Psi_g|_e \in GL(\mathfrak{g})$ is called the adjoint representation of G. Recall also that for $x, y \in \mathfrak{g}$, $\mathrm{ad}_x(y) = [x, y]$, and ad is a Lie-algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\mathfrak{gl}(\mathfrak{g})$ is the set of all linear endomorphisms of $\mathfrak{g}$. The map ad is called the adjoint representation of $\mathfrak{g}$. The kernel of ad is $\mathfrak{z}$. The Lie algebra $\mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of the group $GL(\mathfrak{g})$, and the subalgebra $\mathrm{ad}(\mathfrak{g})$ is the Lie algebra of the group $\mathrm{Ad}(G)$. Let B denote the Killing form of $\mathfrak{g}$. A Lie algebra is called semi-simple if its Killing form is non-degenerate. An ideal $\mathfrak{i}$ of $\mathfrak{g}$ is a sub-algebra such that for every $x \in \mathfrak{g}$ and every $y \in i$, $[x, y] \in \mathfrak{i}$.

Exercise 1.

- Prove that $\mathfrak{z}$ is an ideal.
- Prove that the orthogonal space to $\mathfrak{z}$ for the Killing form B, denoted by $\mathfrak{z}^\perp$, is an ideal.
- Prove that if $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ and B' is the Killing form of $\mathfrak{i}$, then B' is the restriction of B to $\mathfrak{i}$.
- Prove that, if $\mathfrak{g}$ is semi-simple, then $\mathfrak{z}$ is trivial.
- Prove that, if B is negative definite, then the group $\mathrm{Ad}(G)$ is compact.

Exercise 2.

- Prove that, if G is compact, there exists a positive definite quadratic form Q on $\mathfrak{g}$ that is invariant by the action of G through the adjoint representation of G.
- Prove that, if G is compact, there exists a basis of $\mathfrak{g}$ such that every element of $\mathrm{Ad}(G)$ is represented by an orthogonal matrix, and every element of $\mathrm{ad}(\mathfrak{g})$ is represented by a skew-symmeric matrix.
- Prove that, if G is compact, then B is negative semi-definite.
- Prove that, if G is compact and $\mathfrak{g}$ is semi-simple, then B is negative definite.
- Prove that, if G is compact, then $\mathfrak{g}$ is the direct sum $\mathfrak{z} \oplus \mathfrak{z}^\perp$, and the Killing form of $\mathfrak{z}^\perp$ is negative definite.
- Prove that, if $\mathfrak{k} \cap \mathfrak{z} = (0)$, then the Killing form of G is strictly negative definite on $\mathfrak{k}$.

Exercise 3. Assume that (G, K) is a Riemannian symmetric pair and that $\mathfrak{k} \cup \mathfrak{z} = (0)$. Prove that the involutive automorphism σ of G such that $(K_\sigma)_0 \subset K \subset K_\sigma$ is unique. (Hint: assume there are two different involutive automorphisms σ_1, σ_2. Consider the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_\mathfrak{i}$, where $\mathfrak{p}_\mathfrak{i}$ is the eigenspace for the eigenvalue -1 of σ_i. Prove that $\mathfrak{p}_\mathfrak{i}$ is orthogonal to $\mathfrak{k}$. For $X_1 \in \mathfrak{p}_1$, decompose it as $X_2 + T$, $X_2 \in \mathfrak{p}_2, T \in \mathfrak{k}$. This implies T is orthogonal to $\mathfrak{k}$. Use previous exercise to conclude.)