Let \( G \) be a Lie group and \( K \) be a compact subgroup of \( G \). Let \( g = T_e G \) be the Lie algebra of \( G \), \( t \) be the Lie algebra of \( K \) and \( \mathfrak{z} \) be the center of \( g \), i.e. \( \mathfrak{z} = \{ x \in g \mid \forall y \in g, [x, y] = 0 \} \). Recall that every element \( g \in G \) defines an automorphism of \( \Psi_g : G \to G \) by \( \Psi_g(h) = ghg^{-1} \). The differential of \( \Psi_g \) at the identity is a Lie-algebra automorphism of \( g \), \( d\Psi_g|_e : g \to g \). The map \( \text{Ad} : G \ni g \to d\Psi_g|_e \in \text{GL}(g) \) is called the adjoint representation of \( G \). Recall also that for \( x, y \in g \), \( \text{ad}_x(y) = [x, y] \), and \( \text{ad} \) is a Lie-algebra homomorphism \( \text{ad} : g \to \text{gl}(g) \), where \( \text{gl}(g) \) is the set of all linear endomorphisms of \( g \). The map \( \text{ad} \) is called the adjoint representation of \( g \). The kernel of \( \text{ad} \) is \( \mathfrak{z} \). The Lie algebra \( \text{gl}(g) \) is the Lie algebra of the group \( \text{GL}(g) \), and the subalgebra \( \text{ad}(g) \) is the Lie algebra of the group \( \text{Ad}(G) \). Let \( B \) denote the Killing form of \( g \). A Lie algebra is called semi-simple if its Killing form is non-degenerate. An ideal \( i \) of \( g \) is a sub-algebra such that for every \( x \in g \) and every \( y \in i \), \( [x, y] \in i \).

Exercise 1.

• Prove that \( \mathfrak{z} \) is an ideal.
• Prove that the orthogonal space to \( \mathfrak{z} \) for the Killing form \( B \), denoted by \( \mathfrak{z}^\perp \), is an ideal.
• Prove that if \( i \) is an ideal of \( g \) and \( B' \) is the Killing form of \( i \), then \( B' \) is the restriction of \( B \) to \( i \).
• Prove that, if \( g \) is semi-simple, then \( \mathfrak{z} \) is trivial.
• Prove that, if \( B \) is negative definite, then the group \( \text{Ad}(G) \) is compact.

Exercise 2.

• Prove that, if \( G \) is compact, there exists a positive definite quadratic form \( Q \) on \( g \) that is invariant by the action of \( G \) through the adjoint representation of \( G \).
• Prove that, if \( G \) is compact, there exists a basis of \( g \) such that every element of \( \text{Ad}(G) \) is represented by an orthogonal matrix, and every element of \( \text{ad}(g) \) is represented by a skew-symmetric matrix.
• Prove that, if \( G \) is compact, then \( B \) is negative semi-definite.
• Prove that, if \( G \) is compact and \( g \) is semi-simple, then \( B \) is negative definite.
• Prove that, if \( G \) is compact, then \( g \) is the direct sum \( \mathfrak{z} \oplus \mathfrak{z}^\perp \), and the Killing form of \( \mathfrak{z}^\perp \) is negative definite.
• Prove that, if \( t \cap \mathfrak{z} = (0) \), then the Killing form of \( G \) is strictly negative definite on \( t \).

Exercise 3. Assume that \((G, K)\) is a Riemannian symmetric pair and that \( t \cup \mathfrak{z} = (0) \). Prove that the involutive automorphism \( \sigma \) of \( G \) such that \((K_\sigma)_0 \subset K \subset K_\sigma \) is unique. (Hint: assume there are two different involutive automorphisms \( \sigma_1, \sigma_2 \). Consider the decomposition \( g = t + p_1 \), where \( p_1 \) is the eigenspace for the eigenvalue \(-1\) of \( \sigma_1 \). Prove that \( p_1 \) is orthogonal to \( t \). For \( X_1 \in p_1 \), decompose it as \( X_2 + T, X_2 \in p_2, T \in t \). This implies \( T \) is orthogonal to \( t \). Use previous exercise to conclude.)