RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
Heidelberg, 08.05.2013

EXERCISE SHEET 4
Isometry groups
To hand in by Wednesday May 22, 13:00

Exercise 1. Let (M, g) be a Riemannian manifold and OM its frame bundle.

1. Give a definition of the smooth structure on OM. Use coordinate charts and proceed as in the definition of the tangent bundle or the tensor bundles. Verify that, for your definition, the natural projection map $\pi : OM \to M$ is smooth, and that for every smooth manifold N, a map $\psi : N \to OM$ is smooth if and only if for every vector $v \in \mathbb{R}^n$, the map $N \ni p \to \psi(q)(v) \in TM$ is smooth.

2. Consider the group $\mathrm{Isom}(M, g)$, with the compact open topology, fix a point $\phi \in O_pM$, and consider the map
$$\mathrm{Isom}(M, g) \ni f \to df_p \circ \phi \in OM$$
Prove that this map is a homeomorphism with the image. (Hint: the Ascoli-Arzelà theorem can be useful.)

Exercise 2.

1. Find an example of a Riemannian manifold such that the group of isometries is discrete.

2. Consider the flat torus $S^1 \times S^1$ with the product metric, and prove that its group of isometries has dimension 2. (Hint: consider the universal covering $\mathbb{R}^2 \to S^1 \times S^1$, you can use without proof the fact that every isometry of $\mathbb{R}^2$ with the Euclidean metric is an affine map.)

Exercise 3. Let G be a compact Lie group, endowed with the Riemannian metric induced by the Killing form. Consider the group $I = G \times G$, acting on G by $(g_1, g_2)g = g_1 g g_2$. Prove that all the elements of I act isometrically on G. Denote by H_e the subgroup of H that stabilizes the identity element of G. The embedding in OM gives a map $H_e \to O(T_eG) = O(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. In the case when $G = O(n)$ and $U(n)$, compute explicitely the representation $H_e \to O(\mathfrak{g})$. See exercise sheet 3, §1.

Exercise 4. Let G be a topological group, and H a closed subgroup of G. Denote by G/H the set of left cosets gH, for $g \in G$, and denote the projection by $\pi : G \ni g \to gH \in G/H$.

1. Prove that there exists an unique topology on G/H such that π is continuous and open.

2. Prove that G/H is a Hausdorff space.

3. For $g \in G$, denote by $\tau_g : G/H \to G/H$ the map defined by $\tau_g(hH) = ghH$. Prove that this makes G a transformation group of G/H.

4. Prove that G/H is a homogeneous space, i.e. for every $x, y \in G/H$ there exists an element $g \in G$ such that $\tau_g(x) = y$.