Exercise 1. Prove that if a Riemannian manifold has constant sectional curvature, then it is locally symmetric.

Exercise 2. Let $(M,g), (N,h)$ be Riemannian manifolds of dimension bigger or equal to 3. A point $x \in M$ is an isotropic point if the sectional curvatures at the point $x$ don’t depend on the chosen plane. A diffeomorphism $f : M \to N$ is conformal if there exist a positive function $\phi : M \to \mathbb{R}_{>0}$ such that $f^* h = \phi g$. A diffeomorphism $f : M \to N$ preserves the sectional curvatures if for every point $x \in M$ and plane $P \subset T_x M$, the sectional curvature of $d_x f(P)$ is equal to the sectional curvature of $P$.

Prove that, if the non-isotropic points of $M$ are dense, then every diffeomorphism $f : M \to N$ that preserves the sectional curvatures is conformal.

(Hint) This is mostly a matter of linear algebra, for every non-isotropic point $x$, you can study what happens in $T_x M$. You can find an orthonormal basis $e_1, \ldots, e_n$ such that for every triple of distinct indices $i, j, k$, the sectional curvatures $K(e_i, e_j), K(e_j, e_k), K(e_k, e_i)$ are all distinct. Denote $\bar{e}_i = d_x f(e_i)$, and $a_{ij} = h(\bar{e}_i, \bar{e}_j)$. Use the symmetries of the Riemann tensor of $(N,h)$ to prove that the $\bar{e}_i$ are pairwise orthogonal and with the same length.

Exercise 3. Given a vector space $V$, denote by $T^{p,q}(V)$ the tensor product of $p$ copies of $V$ and $q$ copies of $V^*$. Recall that every linear isomorphism $A : V \to W$, induces a linear map $A^{p,q} : T^{p,q}(V) \to T^{p,q}(W)$, defined by

$$A^{p,q}(v_1 \otimes \cdots \otimes v_p \otimes \alpha_1 \otimes \cdots \otimes \alpha_q) = A(v_1) \otimes \cdots \otimes A(v_p) \otimes (A^t)^{-1}(\alpha_1) \otimes \cdots \otimes (A^t)^{-1}(\alpha_q)$$

In the same way, every diffeomorphism $\phi : M \to M$ induces a map $d\phi^{p,q} : T^{p,q}(M) \to T^{p,q}(M)$, where $d\phi : V(M) \to V(M)$ is the ordinary differential of $\phi$.

Given a vector field $X$, let $\phi_t$ be the local flow generated by $X$. For every tensor field $K \in T^{p,q}(M)$, we define the Lie derivative of $K$ in the direction $X$ by

$$L_X K = \lim_{t \to 0} \frac{1}{t} [K - d\phi_t^{p,q}K]$$

Prove that the Lie derivative

(a) Prove that the Lie derivative satisfies the following Leibniz rule:

$$L_X (K \otimes H) = L_X K \otimes H + K \otimes L_X H$$

(b) Prove that the Lie derivative commutes with contractions.

(c) What happens for $p = q = 0$ (when $T^{p,q}(M) = \mathcal{F}(M)$)?

(d) What happens for $p = 1, q = 0$ (when $T^{p,q}(M) = V(M)$)?