RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 01.05.2013

Exercise sheet 2

Riemannian geometry

To hand in by Wednesday May 8, 13:00

Exercise 1. Prove that if a Riemannian manifold has constant sectional curvature, then it is locally symmetric.

Exercise 2. Let $(M, g), (N, h)$ be Riemannian manifolds of dimension bigger or equal to 3. A point $x \in M$ is an isotropic point if the sectional curvatures at the point x don't depend on the chosen plane. A diffeomorphism $f : M \to N$ is conformal if there exist a positive function $\phi : M \to \mathbb{R}_{>0}$ such that $f^*h = \phi g$. A diffeomorphism $f : M \to N$ preserves the sectional curvatures if for every point $x \in M$ and plane $P \subset T_xM$, the sectional curvature of $d_xf(P)$ is equal to the sectional curvature of P.

Prove that, if the non-isotropic points of M are dense, then every diffeomorphism $f : M \to N$ that preserves the sectional curvatures is conformal.

(Hint) This is mostly a matter of linear algebra, for every non-isotropic point x, you can study what happens in T_xM. You can find an orthonormal basis $e_1, \ldots, e_n$ such that for every triple of distinct indices i, j, k, the sectional curvatures $K(e_i, e_j), K(e_j, e_k), K(e_k, e_i)$ are all distinct. Denote $\bar{e_i} = d_xf(e_i)$, and $a_{ij} = h(\bar{e_i}, \bar{e_j})$. Use the symmetries of the Riemann tensor of (N, h) to prove that the $\bar{e_i}$ are pairwise orthogonal and with the same lenght.

Exercise 3. Given a vector space V, denote by $T^{p,q}(V)$ the tensor product of p copies of V and q copies of V^*. Recall that every linear isomorphism $A : V \to W$, induces a linear map $A^{p,q} : T^{p,q}(V) \to T^{p,q}(W)$, defined by

$$A^{p,q}(v_1 \otimes \cdots \otimes v_p \otimes \alpha_1 \otimes \cdots \otimes \alpha_q) = A(v_1) \otimes \cdots \otimes A(v_p) \otimes (A^t)^{-1}(\alpha_1) \otimes \cdots \otimes (A^t)^{-1}(\alpha_q)$$

In the same way, every diffeomorphism $\phi : M \to M$ induces a map $d\phi^{p,q} : \mathcal{T}^{p,q}(M) \to \mathcal{T}^{p,q}(M)$, where $d\phi : V(M) \to V(M)$ is the ordinary differential of ϕ.

Given a vector field X, let ϕ_t be the local flow generated by X. For every tensor field $K \in \mathcal{T}^{p,q}(M)$, we define the Lie derivative of K in the direction X by

$$L_X K = \lim_{t \to 0} \frac{1}{t} \left[K - d\phi_t^{p,q} K\right]$$

Prove that the Lie derivative

(a) Prove that the Lie derivative satisfies the following Leibniz rule:

$$L_X(K \otimes H) = L_X K \otimes H + K \otimes L_X H$$

(b) Prove that the Lie derivative commutes with contractions.

(c) What happens for $p = q = 0$ (when $\mathcal{T}^{p,q}(M) = \mathcal{F}(M)$)?

(d) What happens for $p = 1, q = 0$ (when $\mathcal{T}^{p,q}(M) = V(M)$)?