RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
Heidelberg, 24.04.2013

EXERCISE SHEET 1
Riemannian geometry
To hand in by Tuesday April 30, 14:00

Let (M, g) be a Riemannian manifold.

Exercise 1. A smooth map $f : M \to M$ is an affine transformation if for every geodesics $c : (a, b) \to M$, $f \circ c$ is a geodesics.

Assume that M is connected. Let f be an affine transformation with the additional property that there exists a point $x \in M$ such that $d_x f : T_x M \to T_{f(x)} M$ is a linear isometry. Prove that f is an isometry.

Exercise 2. Denote by $d : M \times M \to \mathbb{R}$ the distance induced by g. A distance-preserving map is a map $f : M \to M$ such that $d(f(x), f(y)) = d(x, y)$.

(a) Prove that a smooth distance preserving map is an isometry.

(b) (Bonus) The hypothesis that the map is smooth in the previous point is not necessary. As a bonus exercise, you can try to prove that a distance-preserving map is an isometry.

Exercise 3. Given a vector space V, denote by $T^{p,q}(V)$ the tensor product of p copies of V and q copies of V^*. Recall that, for $l \leq p$ and $m \leq q$, an (l, m)-contraction is a linear map $T^{p,q}(V) \to T^{p-1,q-1}(V)$ that associates to the tensor $v_1 \otimes \cdots \otimes v_p \otimes \alpha_1 \otimes \cdots \otimes \alpha_q$ the tensor $\alpha_m(v_l) v_1 \otimes \ldots v_{l-1} \otimes v_{l+1} \cdots \otimes v_p \otimes \alpha_1 \otimes \ldots \alpha_{m-1} \otimes \alpha_{m+1} \cdots \otimes \alpha_q$.

(a) Write a definition of the tensor bundle $T^{p,q}$, where the underlying set is the disjoint union of all the sets $T^{p,q}(T_x M)$, for $x \in M$. Use local charts and the vectors $\frac{\partial}{\partial x_i}$ and dx_i, as in the definition of the tangent bundle.

(b) Prove that for every connection ∇ on M, there exists a unique way to extend it to tensor fields with operators

$$\nabla : V(M) \times \mathcal{T}^{p,q}(M) \to \mathcal{T}^{p,q}(M)$$

such that they commute with contractions and satisfy the following Leibniz rule:

$$\nabla_X (K \otimes L) = \nabla_X K \otimes L + K \otimes \nabla_X L$$

(Here the symbol $\mathcal{T}^{p,q}(M)$ denotes the space of smooth sections of the bundle $T^{p,q}$.)

Exercise 4. Let ∇ be the Levi-Civita connection on (M, g), and R its curvature tensor. Prove that $\nabla R = 0$ if and only if for every smooth curve $c : I \to M$ and for every three parallel vector fields X, Y, Z along c, the vector field $R(X, Y)Z$ is a parallel vector field along c.