RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
*Heidelberg, 15.01.2013*

Exercise sheet 12

# Distance function

*To hand in by January 22, 14:00*

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**Exercise 1.** Let $(M, g)$ be a connected, non-compact, complete, Riemannian manifold, and let $p \in M$.

(a) Show that there exists a sequence $(p_i) \subset M$ such that $d(p, p_i) \to \infty$ for $i \to \infty$.

(b) Show that there is a complete geodesic ray $\gamma : [0, \infty) \to M$ with $\gamma(0) = p$ and with the property that for every $s, t \in [0, \infty), d(\gamma(s), \gamma(t)) = L(\gamma|_{[s,t]})$. [Hint: for every $i$, consider the shortest curve from $p$ to $p_i$].

**Exercise 2.**

(a) Let $S_1^n \subset \mathbb{R}^{n+1}$ be the sphere of radius 1, and let $d_{S_1^n}$ be the distance induced on the sphere by the standard Riemannian metric. Show that for every two points $x, y \in S_1^n$, the distance $d_{S_1^n}(x, y)$ is equal to the angle between the vectors $x$ and $y$.

(b) Let $d_{\mathbb{R}^{n+1}}$ be the standard distance on $\mathbb{R}^{n+1}$. Prove that for every two distinct points $x, y \in S_1^n$, $d_{\mathbb{R}^{n+1}}(x, y) < d_{S_1^n}(x, y)$.

(c) Consider the hyperbolic space $H^n$, as in exercise sheet 4, §4. Prove that for every two points $x, y \in H^n$, the distance $d(x, y)$ is equal to $\cosh^{-1}(\langle x, y\rangle)$, where $\langle , \rangle$ is the bilinear form defined there.

**Exercise 3.** Let $(M, g)$ be a connected, complete, Riemannian manifold, and $N \subset M$ a closed submanifold. Let $m$ be a point of $M$ that does not lie in $N$.

(a) Show that there is a point $p \in N$ such that $d(m, p) = \inf_{x \in N} d(m, x)$.

(b) Show that there is a geodesics $\gamma$ from $m$ to $p$ with length $d(m, p)$.

(c) Show that $\gamma$ is orthogonal to the submanifold $N$ at the point $p$.

**Exercise 4. [Bonus exercise[1]]** Let $(M, g)$ be a semi-Riemannian manifold, and for every $s \in [0, 1]$, let $\gamma_s : \mathbb{R} \to M$ be a geodesics with the property that for every $t \in \mathbb{R}$, $\gamma_s(t+1) = \gamma_s(t)$. Assume that the function $[0, 1] \times \mathbb{R} \ni (s, t) \to \gamma_s(t) \in M$ is smooth. Show that the energy $E\left[\gamma_s|_{[0,1]}\right]$ does not depend on $s$. [Hint: consider the first variation of the energy functional].

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[1] This exercise is not officially part of the exercise sheet, and it doen't influence the count of points you need to be considered over 50%. People who solve exercises 1,2,3 correclty will already get the usual 60 points given by an exercise sheet. People who try to solve exercise 4, may get some extra points added to their total