Exercise 1. Let \((M, g)\) be a connected, non-compact, complete, Riemannian manifold, and let \(p \in M\).

(a) Show that there exists a sequence \((p_i) \subset M\) such that \(d(p, p_i) \to \infty\) for \(i \to \infty\).

(b) Show that there is a complete geodesic ray \(\gamma : [0, \infty) \to M\) with \(\gamma(0) = p\) and with the property that for every \(s, t \in [0, \infty)\), \(d(\gamma(s), \gamma(t)) = L(\gamma|_{[s,t]})\). [Hint: for every \(i\), consider the shortest curve from \(p\) to \(p_i\)].

Exercise 2.

(a) Let \(S^n_1 \subset \mathbb{R}^{n+1}\) be the sphere of radius 1, and let \(d_{S^n_1}\) be the distance induced on the sphere by the standard Riemannian metric. Show that for every two points \(x, y \in S^n_1\), the distance \(d_{S^n_1}(x, y)\) is equal to the angle between the vectors \(x\) and \(y\).

(b) Let \(d_{\mathbb{R}^{n+1}}\) be the standard distance on \(\mathbb{R}^{n+1}\). Prove that for every two distinct points \(x, y \in S^n_1\), \(d_{\mathbb{R}^{n+1}}(x, y) < d_{S^n_1}(x, y)\).

(c) Consider the hyperbolic space \(H^n\), as in exercise sheet 4, §4. Prove that for every two points \(x, y \in H^n\), the distance \(d(x, y)\) is equal to \(\cosh^{-1}(\langle x, y \rangle)\), where \(\langle , \rangle\) is the bilinear form defined there.

Exercise 3. Let \((M, g)\) be a connected, complete, Riemannian manifold, and \(N \subset M\) a closed submanifold. Let \(m\) be a point of \(M\) that does not lie in \(N\).

(a) Show that there is a point \(p \in N\) such that \(d(m, p) = \inf_{x \in N} d(m, x)\).

(b) Show that there is a geodesics \(\gamma\) from \(m\) to \(p\) with length \(d(m, p)\).

(c) Show that \(\gamma\) is orthogonal to the submanifold \(N\) at the point \(p\).

Exercise 4. [Bonus exercise]\(^1\) Let \((M, g)\) be a semi-Riemannian manifold, and for every \(s \in [0, 1]\), let \(\gamma_s : \mathbb{R} \to M\) be a geodesics with the property that for every \(t \in \mathbb{R}\), \(\gamma_s(t+1) = \gamma_s(t)\). Assume that the function \([0, 1] \times \mathbb{R} \ni (s, t) \to \gamma_s(t) \in M\) is smooth. Show that the energy \(E[\gamma_s|_{[0,1]}]\) does not depend on \(s\). [Hint: consider the first variation of the energy functional].

\(^1\)This exercise is not officially part of the exercise sheet, and it doesn’t influence the count of points you need to be considered over 50%. People who solve exercises 1, 2, 3 correctly will already get the usual 60 points given by an exercise sheet. People who try to solve exercise 4, may get some extra points added to their total.