RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie I
Heidelberg, 8.1.2013

EXERCISE SHEET 11

Jacobi Fields

To hand in by January 15, 14:00

Exercise 1. Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds. Let $M_1 \times M_2$ denote the Cartesian product of M_1 and M_2, with the standard structure of smooth manifold (as in Exercise sheet 2, §1). For $y \in M_2$, we will denote by $i_1^y : M_1 \to M_1 \times M_2$ the map $x \to (x, y)$. Similar definition for i_2^x, with $x \in M_1$.

(a) Prove that for $(x, y) \in M_1 \times M_2$, the tangent space $T_{(x,y)} M_1 \times M_2$ is the direct sum of the images of the differentials di_1^y in the point x and di_2^x in the point y. Conclude that there is a natural way to identify $T_{(x,y)} M_1 \times M_2$ with $T_x M_1 + T_y M_2$.

(b) For every vector $v \in T_{(x,y)} M_1 \times M_2$, we denote by v_1, v_2 the components in the subspaces $T_x M_1, T_y M_2$. Show that the formula $g(v, w) = g_1(v_1, w_1) + g_2(v_2, w_2)$ defines a Riemannian metric on $M_1 \times M_2$, called the product Riemannian metric.

(c) Find the first and second fundamental forms of the immersions i_1^y, i_2^x.

(d) Compute the Riemann tensor of g.

(e) Consider the case when $M_1 = M_2 = S_1^n$, with the standard metric induced by the immersion in $\mathbb{R}^{n+1}$. For $n \geq 1$, is the product Riemannian metric on $S_1^n \times S_1^n$ a metric with constant sectional curvature?

Exercise 2. Let (M, g) be an n-dimensional Riemannian manifold with non positive sectional curvature. Let J be a Jacobi field along a geodesic for (M, g).

(a) Show that $g(J, \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} J)$ is a non-negative function.

(b) Show that the second derivative of $g(J, J)$ is non-negative.

(c) Show that, if J is not identically zero, it has at most one zero.

Exercise 3. Let (S, g) be a 2-dimensional Riemannian manifold, and $\gamma : I \to S$ a geodesic parametrized by arc-length. Let Y be a vector field along γ, with $g(Y(t), \dot{\gamma}(t)) = 0$ and $g(Y(t), Y(t)) = 1$ for all $t \in I$.

(a) Show that Y is parallel along γ.

(b) Let $f : I \to \mathbb{R}$ a smooth function. Show that $f \cdot Y$ is a Jacobi field along γ if and only if f satisfies $K \cdot f + f'' = 0$, where K is the sectional curvature of (S, g).