Exercise 1. Let \((M_1, g_1), (M_2, g_2)\) be Riemannian manifolds. Let \(M_1 \times M_2\) denote the Cartesian product of \(M_1\) and \(M_2\), with the standard structure of smooth manifold (as in Exercise sheet 2, §1). For \(y \in M_2\), we will denote by \(i^y_1 : M_1 \to M_1 \times M_2\) the map \(x \mapsto (x, y)\). Similar definition for \(i^x_2\), with \(x \in M_1\).

(a) Prove that for \((x, y) \in M_1 \times M_2\), the tangent space \(T((x, y)) M_1 \times M_2\) is the direct sum of the images of the differentials \(di^y_1\) in the point \(x\) and \(di^x_2\) in the point \(y\). Conclude that there is a natural way to identify \(T((x, y)) M_1 \times M_2\) with \(T_x M_1 + T_y M_2\).

(b) For every vector \(v \in T((x, y)) M_1 \times M_2\), we denote by \(v_1, v_2\) the components in the subspaces \(T_x M_1, T_y M_2\). Show that the formula \(g(v, w) = g_1(v_1, w_1) + g_2(v_2, w_2)\) defines a Riemannian metric on \(M_1 \times M_2\), called the product Riemannian metric.

(c) Find the first and second fundamental forms of the immersions \(i^y_1, i^x_2\).

(d) Compute the Riemann tensor of \(g\).

(e) Consider the case when \(M_1 = M_2 = S^n_1\), with the standard metric induced by the immersion in \(\mathbb{R}^{n+1}\). For \(n \geq 1\), is the product Riemannian metric on \(S^n_1 \times S^n_1\) a metric with constant sectional curvature?

Exercise 2. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with non positive sectional curvature. Let \(J\) be a Jacobi field along a geodesic for \((M, g)\).

(a) Show that \(g(J, \nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} J)\) is a non-negative function.

(b) Show that the second derivative of \(g(J, J)\) is non-negative.

(c) Show that, if \(J\) is not identically zero, it has at most one zero.

Exercise 3. Let \((S, g)\) be a 2-dimensional Riemannian manifold, and \(\gamma : I \to S\) a geodesic parametrized by arc-length. Let \(Y\) be a vector field along \(\gamma\), with \(g(Y(t), \dot{\gamma}(t)) = 0\) and \(g(Y(t), Y(t)) = 1\) for all \(t \in I\).

(a) Show that \(Y\) is parallel along \(\gamma\).

(b) Let \(f : I \to \mathbb{R}\) a smooth function. Show that \(f \cdot Y\) is a Jacobi field along \(\gamma\) if and only if \(f\) satisfies \(K \cdot f + f'' = 0\), where \(K\) is the sectional curvature of \((S, g)\).