RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
Heidelberg, 11.12.2012

Exercise sheet 9

Curvature

To hand in by December 18, 14:00

Exercise 1. (First Bianchi Identity) Let M be a manifold with a torsion-free connection ∇.

(a) Define $F(X,Y,Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z]$. Prove that this is tensorial in Y and Z. (This is sometimes denoted by $(\mathcal{L}_X \nabla)_Y Z$, and called the Lie derivative of the connection.)

(b) Prove that $F(X,Y,Z)$ is symmetric in Y, Z. (Hint: Write $F(X,Y,Z) - F(X,Z,Y)$, regroup the six terms pairwise, and use the Jacobi identity from exercise sheet 4.)

(c) Prove that $F(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_{\nabla_Y Z} X - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X$.

(d) Prove the first Bianchi identity: $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X$. (Hint: don't forget that $R(X,Y)Z = -R(Y,X)Z$.)

Exercise 2. Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection ∇. Let $X, Y, Z, W \in V(M)$. Recall that $g(Z, W) \in \mathcal{F}(M)$.

(a) Compute $X(Y(g(Z,W))), Y(X(g(Z,W))$ and $[X,Y](g(Z,W))$ using the connection ∇.

(b) Compute the function $g(R(X,Y)Z, W) + g(R(X,Y)W, Z)$ using the connection ∇.

(c) Prove the identity $R(X,Y,Z,W) = -R(X,Y,W,Z)$.

Exercise 3. Let V be a real vector space of dimension m, and $\langle , \rangle$ be a non-degenerate symmetric bilinear form. Recall that T_pV is canonically identified with V, hence a vector field on V is just a smooth function $V \to V$. Let $k \neq 0$ be a constant. Consider the open set

$$V_k = \{ p \in V \mid k \langle p, p \rangle \neq -1 \}$$

On V_k, consider the following Riemannian metric: for $v, w \in T_pV = V$, set:

$$g_p(v, w) = \frac{4 \langle v, w \rangle}{(1 + k \langle p, p \rangle)^2}$$

i.e. g_p is the standard scalar product multiplied by the function $f_k(p) = \frac{4}{(1+k\langle p,p \rangle)^2}$.

(a) Compute the Levi-Civita connection of (V_k, g) using the Koszul formula (i.e. compute $2g_p(\nabla_X Y, Z)))$.

(b) Compute $(\nabla_X Y)(p) - (dX \cdot Y)(p)$ (where $dX \cdot Y$ is the standard connection on V, as in exercise sheet 5, §3).

(c) For every $p \in V_k$ and $u, v, w \in T_pV$, prove that $R(u,v)w = k(\langle v, w \rangle u - \langle u, w \rangle v)$. (Hint: choose vector fields X, Y, Z such that $X(p) = u, Y(p) = v, Z(p) = w$, and use the definition).

(d) For every $p \in V_k$ and $u, v \in T_pV$ independent vectors, compute the sectional curvature of the plane spanned by u, v.