Vorlesung Differentialgeometrie I
Heidelberg, 04.11.2012

Exercise sheet 8
Geodesics II
To hand in by December 11, 14:00

Exercise 1. Let (M, g) be a pseudo-Riemannian manifold with its Levi-Civita connection ∇.

(a) Given a function $\phi \in \mathcal{F}(M)$, the gradient of ϕ, denoted by grad ϕ, is defined as the vector field corresponding to the 1-form $d\phi$, or, in other words, as the vector field such that for every $X \in V(M)$, $g(X, \text{grad } \phi) = X(\phi)$. Prove that, in local coordinates, grad $\phi = \sum_i \eta^i \frac{\partial}{\partial x_i}$, where $\eta^j = \sum_i g^{ij} \frac{\partial \phi}{\partial x_i}$ and (g^{ij}) is the inverse of the matrix (g_{ij}).

(b) Given $X \in V(M)$, consider the application $(\nabla X)_p : T_pM \ni Y \to \nabla_Y X \in T_pM$, for every $p \in M$. The function $M \ni p \to \text{tr}(\nabla X)_p \in \mathbb{R}$ is called the divergence of X, denoted by div(X). Prove that, in local coordinates, if $X = \sum \xi^i \frac{\partial}{\partial x_i}$, then $\text{div}(X) = \sum_i \frac{\partial \xi^i}{\partial x_i} + \sum_{i,j} \xi^j \Gamma^i_{ij}$.

(c) Given a function $\phi \in \mathcal{F}(M)$, the Laplacian of ϕ, is defined by $\Delta\phi = \text{div}(\text{grad } \phi)$. Prove that, in local coordinates, $\Delta\phi = \sum_{i,j} g^{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{ij} \frac{\partial g^{ij}}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sum_{ijkl} g^{kl} g^{ij} \frac{\partial g_{ij}}{\partial x_l} \frac{\partial \phi}{\partial x_k}$

Exercise 2. Let $(M, g), (N, h)$ be two pseudo-Riemannian manifolds, and $f : M \to N$ be a local isometry. Prove that $\gamma : (-1, 1) \to M$ is a geodesic if and only if $f \circ \gamma : (-1, 1) \to N$ is a geodesic.

Exercise 3. Let M, N be smooth manifolds, and $f : N \to M$ be an immersion. Let g be a pseudo-Riemannian metric for M, with Levi-Civita connection ∇. Recall that if $X, Y \in V(N)$, $f_*(X) \in V_f(M)$, and $\nabla_Y(f_*(X)) \in V_f(M)$ is the covariant derivative of vector fields along f.

(a) For every $p \in N$, let π_p be the orthogonal projection $\pi_p : T_{f(p)}M \to df_p(T_pN)$. For $Z \in V_f(M)$, let $\pi(Z) \in V(N)$ be the vector field that associates to the point $p \in N$ the vector $df_p^{-1}(\pi_p(Z(p)))$. Verify that $\pi(Z)$ is a well defined vector field.

(b) Given vector fields $X, Y \in V(N)$, define $\nabla^f_Y(X) = \pi(\nabla_Y(f_*(X)))$. Prove that $\nabla^f_Y(X)$ is a connection on N.

(c) Let f^*g be the pull-back of the metric g by the map f: $f^*g(p)(v, w) = \langle df_p(v), df_p(w)\rangle$. Prove that $\nabla^f_Y(X)$ is the Levi-Civita connection of the metric f^*g.

Exercise 4.

(a) Let $a \in (0, \infty)$ be a parameter, and K_a be the cone $\{(x, y, z) \in \mathbb{R}^3 \mid z = a\sqrt{x^2 + y^2}, z \neq 0\}$ and m_a be the line $(0, t, at) \mid t > 0$. K_a is a submenifold, and it inherits a Riemannian metric g from the ambient space $\mathbb{R}^3$. Prove that $K_a \setminus m_a$ is isometric to a subset of the plane $\mathbb{R}^2$.

(b) Let γ be a curve on K_a with constant z coordinate and constant speed (i.e. $g(\dot\gamma(t), \dot\gamma(t))$ is constant.) Let U be a parallel vector field along γ. Recall that the angle between two vectors v, w is defined as $\frac{g(v,w)}{\sqrt{g(v,v)g(w,w)}}$. Prove that the angle between $U(t)$ and $\dot\gamma(t)$ changes with constant speed.

(c) Compute the total variation of the angle between $U(t)$ and $\dot\gamma(t)$, when $\gamma(t)$ makes a complete tour around the cone.

(d) Let S^2_1 be the sphere in $\mathbb{R}^3$. Let γ be a curve on S^2_1 with constant z coordinate and constant speed. Let U be a parallel vector field along γ. Compute the total variation of the angle between $U(t)$ and $\dot\gamma(t)$, when $\gamma(t)$ makes a complete tour around the sphere. (Hint: find a cone that is tangent to S^2_1 around γ, i.e. the cone and the sphere have the same tangent space at every point of γ. Use exercise 3 to prove that a vector field along γ is parallel for the cone if and only if it is parallel for the sphere.)