RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie I
Heidelberg, 27.11.2012

EXERCISE SHEET 7

Geodesics

To hand in by December 4, 14:00

Exercise 1. Consider again the catenoid and helicoid of exercise sheet 5, §1. Recall that the metrics g_1 and g_2 are expressed by the matrices $\begin{pmatrix} \cosh^2 s & 0 \\ 0 & \cosh^2 s \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & s^2+1 \end{pmatrix}$.

(a) Compute the Christoffel symbols for the two metrics on $\mathbb{R}^2$, in the identity chart.

(b) Are the two curves $\mathbb{R} \to \mathbb{R}^2$ given by $s \to (s, 0)$ and $t \to (0, t)$ geodesics for the two metrics?

Exercise 2. Let Ω be an open subset of $\mathbb{R}^2$ with the restriction of the standard metric of $\mathbb{R}^2$.

(a) Find an explicit expression for the exponential map for the points of the domain $\mathcal{D} \subset T\Omega$ where it is well defined.

(b) When $\Omega = \mathbb{R}^2 \setminus \{0\}$, find explicitely the domain $\mathcal{D}$.

(c) When $\Omega = \mathbb{R}^2 \setminus \{0\}$, find two points that are not connected by a geodesic segment.

(d) Prove that $\mathcal{D} = T\Omega$ if and only if $\Omega = \mathbb{R}^2$.

Exercise 3. Let $S^2_1 \subset \mathbb{R}^3$ be the sphere of radius 1. Fix an angle $\theta \in (0, \pi)$, and consider the three curves $\alpha : [0, \pi/2] \ni t \to (-\sin t, 0, \cos t) \in S^2_1$, $\beta : [0, \theta] \ni t \to (-\cos t, -\sin t, 0) \in S^2_1$, $\gamma : [0, \pi/2] \ni t \to (-\cos\theta \sin t, -\sin\theta \sin t, \cos t) \in S^2_1$. The curve γ' that is the concatenation of α and β, and the curve γ are both piece-wise smooth curves from $x = (0, 0, 1)$ to $y = (-\cos\theta, -\sin\theta, 0)$. Show that the parallel transport linear isometries $P_\gamma, P_{\gamma'} : T_x S^2_1 \to T_y S^2_1$ differ by rotation by an angle θ. (Hint: you may use, without proving it, that the three curves α, β, γ are geodesic segments).

Exercise 4. Let $\pi : E \to M$ be a vector bundle over a manifold M, with fiber modeled over a vector space V (for the notation, see exercise sheet 5, §4). The symbol $\bigwedge^k(V^*)$ denotes as usual the space of alternating k-linear forms from $V \times \cdots \times V$ to $\mathbb{R}$. For $p \in M$, denote by E_p the inverse image $\pi^{-1}(p)$, with its structure of vector space isomorphic to V. Denote by $\bigwedge^k(E^*)$ the disjoint union $\bigsqcup_{p\in M} \bigwedge^k(E^*_p)$.

(a) Using the charts for E, construct a structure of vector bundle for $\bigwedge^k(E^*)$ with a projection $\pi^k : \bigwedge^k(E^*) \to M$. (Hint: given the open covering $\{U_\alpha, \alpha \in A\}$ used to define E, replace $U_\alpha \times V$ with $U_\alpha \times \bigwedge^k(V^*)$. You may use the fact that every linear automorphism of V induces in a natural way a linear automorphism of $\bigwedge^k(V^*)$).

(b) For the definition of a section, refer to exercise sheet 6, §4. Prove that a map $\omega : M \to \bigwedge^k(E^*)$ with $\pi^k(\omega(x)) = x$ is smooth (hence a section) if and only if for every sections $s_1, \ldots s_k \in \Gamma(E)$, the function $\omega(x)(s_1(x), \ldots, s_k(x)) : M \to \mathbb{R}$ is smooth.

(c) Let ∇ be a connection on E. We define the symbol:

$$(\nabla_X \omega)(s_1, \ldots, s_k) = X(\omega(s_1, \ldots, s_k)) - \omega(\nabla_X s_1, \ldots, s_k) - \cdots - \omega(s_1, \ldots, \nabla_X s_k)$$

for $X \in V(M)$, $s_1, \ldots, s_k \in \Gamma(E), \omega \in \Gamma(\bigwedge^k(E^*))$. Show that $\nabla_X \omega$ is tensorial in $s_1, \ldots, s_n$, hence it can be viewed as an element of $\Gamma(\bigwedge^k(E^*))$.

(d) Show that $\nabla_X \omega$ is a connection on $\bigwedge^k(E^*)$ (for the definition see exercise sheet 6, §4).