Exercise 1. Consider again the catenoid and helicoid of exercise sheet 5, §1. Recall that the metrics \( g_1 \) and \( g_2 \) are expressed by the matrices \( \begin{pmatrix} \cosh^2 s & 0 \\ 0 & \cosh^2 s \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & s^2 + 1 \end{pmatrix} \).

(a) Compute the Christoffel symbols for the two metrics on \( \mathbb{R}^2 \), in the identity chart.

(b) Are the two curves \( \mathbb{R} \to \mathbb{R}^2 \) given by \( s \to (s, 0) \) and \( t \to (0, t) \) geodesics for the two metrics?

Exercise 2. Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) with the restriction of the standard metric of \( \mathbb{R}^2 \).

(a) Find an explicit expression for the exponential map for the points of the domain \( \mathcal{D} \subset T\Omega \) where it is well defined.

(b) When \( \Omega = \mathbb{R}^2 \setminus \{0\} \), find explicitly the domain \( \mathcal{D} \).

(c) When \( \Omega = \mathbb{R}^2 \setminus \{0\} \), find two points that are not connected by a geodesic segment.

(d) Prove that \( \mathcal{D} = T\Omega \) if and only if \( \Omega = \mathbb{R}^2 \).

Exercise 3. Let \( S_1^2 \subset \mathbb{R}^3 \) be the sphere of radius 1. Fix an angle \( \theta \in (0, \pi) \), and consider the three curves \( \alpha : [0, \pi/2] \ni t \to (-\sin t, 0, \cos t) \in S_1^2 \), \( \beta : [0, \theta] \ni t \to (-\cos t, -\sin t, 0) \in S_1^2 \), \( \gamma : [0, \pi/2] \ni t \to (-\cos \theta \sin t, -\sin \theta \sin t, \cos t) \in S_1^2 \). The curve \( \gamma' \) that is the concatenation of \( \alpha \) and \( \beta \), and the curve \( \gamma \) are both piece-wise smooth curves from \( x = (0, 0, 1) \) to \( y = (-\cos \theta, -\sin \theta, 0) \). Show that the parallel transport linear isometries \( P_{\gamma}, P_{\gamma'} : T_xS_1^2 \to T_yS_1^2 \) differ by rotation by an angle \( \theta \). (Hint: you may use, without proving it, that the three curves \( \alpha, \beta, \gamma \) are geodesic segments).

Exercise 4. Let \( \pi : E \to M \) be a vector bundle over a manifold \( M \), with fiber modeled over a vector space \( V \) (for the notation, see exercise sheet 5, §4). The symbol \( \bigwedge^k(V^*) \) denotes as usual the space of alternating \( k \)-linear forms from \( V \times \cdots \times V \) to \( \mathbb{R} \). For \( p \in M \), denote by \( E_p \) the inverse image \( \pi^{-1}(p) \), with its structure of vector space isomorphic to \( V \). Denote by \( \bigwedge^k(E^*) \) the disjoint union \( \bigsqcup_{p \in M} \bigwedge^k(E_p^*) \).

(a) Using the charts for \( E \), construct a structure of vector bundle for \( \bigwedge^k(E^*) \) with a projection \( \pi^k : \bigwedge^k(E^*) \to M \). (Hint: given the open covering \( \{U_\alpha, \alpha \in A \} \) used to define \( E \), replace \( U_\alpha \times V \) with \( U_\alpha \times \bigwedge^k(V^*) \). You may use the fact that every linear automorphism of \( V \) induces in a natural way a linear automorphism of \( \bigwedge^k(V^*) \).

(b) For the definition of a section, refer to exercise sheet 6, §2.4. Prove that a map \( \omega : M \to \bigwedge^k(E^*) \) with \( \pi^k(\omega(x)) = x \) is smooth (hence a section) if and only if for every sections \( s_1, \ldots, s_k \in \Gamma(E) \), the function \( \omega(x)(s_1(x), \ldots, s_k(x)) : M \to \mathbb{R} \) is smooth.

(c) Let \( \nabla \) be a connection on \( E \). We define the symbol:

\[
(\nabla_X \omega)(s_1, \ldots, s_k) = X(\omega(s_1, \ldots, s_k)) - \omega(\nabla_X s_1, \ldots, s_k) - \cdots - \omega(s_1, \ldots, \nabla_X s_k)
\]

for \( X \in \mathfrak{X}(M) \), \( s_1, \ldots, s_k \in \Gamma(E), \omega \in \Gamma(\bigwedge^k(E^*)) \). Show that \( \nabla_X \omega \) is tensorial in \( s_1, \ldots, s_n \), hence it can be viewed as an element of \( \Gamma(\bigwedge^k(E^*)) \).

(d) Show that \( \nabla_X \omega \) is a connection on \( \bigwedge^k(E^*) \) (for the definition see exercise sheet 6, §2.4).