RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
Heidelberg, 20.11.2012

Exercise sheet 6
Connections
To hand in by November 27, 14:00

Exercise 1. Let M be a manifold, and ∇, ∇' be connections on M.

(a) Prove that the difference $\nabla - \nabla'$, defined by

$$V(M) \times V(M) \ni (X, Y) \to \nabla_X(Y) - \nabla'_X(Y) \in V(M)$$

is a $(2, 1)$ tensor.

(b) Conversely, given a $(2, 1)$ tensor S, prove that $\nabla + S$ is a connection.

(c) A $(2, 1)$ tensor S is symmetric if $S(X, Y) = S(Y, X)$. Prove that if S is symmetric, ∇ and $\nabla + S$ have the same torsion.

Exercise 2. On $\mathbb{R}^2$, with cartesian coordinates (x_1, x_2), consider the two vector fields

$$V_1(x_1, x_2) = (\cos x_1)\frac{\partial}{\partial x_1} + (\sin x_1)\frac{\partial}{\partial x_2}$$
$$V_2(x_1, x_2) = (-\sin x_1)\frac{\partial}{\partial x_1} + (\cos x_1)\frac{\partial}{\partial x_2}$$

(a) Prove that V_1, V_2 form a parallelization of $\mathbb{R}^2$, and compute $[V_1, V_2]$.

(b) Let ∇ be the connection associated to the parallelization given by V_1 and V_2. Compute the Christoffel symbols of ∇ with reference to the identity chart.

(c) Is ∇ torsionfree?

Exercise 3. Let M, N be manifolds, $f : N \to M$ a smooth map, and ∇ a connection on M. Show that for $X, Y \in V(N)$:

$$\nabla_X(f_*Y) - \nabla_Y(f_*X) - f_*[X, Y] = T(f_*X, f_*Y)$$

where $\nabla_X(f_*Y)$ is the covariant derivative of vector fields along f.

Exercise 4. Let M be a manifold and E a vector bundle (as in Exercise sheet 5, §4), with projection $\pi : E \to M$. A section of E is a smooth map $s : M \to E$ such that for every $x \in M$, $\pi(s(x)) = x$. We denote the space of sections of E by $\Gamma(E)$. $\Gamma(E)$ is an $\mathbb{R}$-vector space and an $\mathcal{F}(M)$-module. For example vector fields are sections of the tangent bundle: $V(M) = \Gamma(TM)$. A connection on E is an $\mathbb{R}$-bilinear map $\nabla : V(M) \times \Gamma(E) \to \Gamma(E)$ satisfying the following:

$$\forall f \in \mathcal{F}(M), \nabla_{fX}(s) = f\nabla_X(s) \text{ and } \nabla_X(fs) = X(f)s + f\nabla_X(s)$$

(a) Given $p \in M$ and a neighborhood U of p, prove that $\forall X, Y \in V(M)$ and $\forall s, t \in \Gamma(E)$, if $X(p) = Y(p)$ and $s|_U = t|_U$, then $\nabla_X(s)(p) = \nabla_Y(t)(p)$.

(b) Prove that for every point $p \in M$ there exists a neighborhood U of p such that $\pi^{-1}(U) \to U$ is a vector bundle, and there exist sections $s_1, \dots s_n \in \Gamma(\pi^{-1}(U))$ such that for every point $q \in U$, $s_1(q), \dots s_n(q)$ are a basis of $\pi^{-1}(q)$.

(c) For every point p, use a chart around p and sections as above to define the analog of Christoffel symbols, and find a local expression for the connection.