RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
Heidelberg, 13.11.2012

Exercise sheet 5

Riemannian metrics

To hand in by November 20, 14:00

Exercise 1.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be an immersion, and let g be the pull-back of the standard Euclidean metric of $\mathbb{R}^3$ by the map f: $g(p)(v, w) = \langle df_p(v), df_p(w)\rangle$. Compute the fundamental matrix g_{ij} in function of f.

(b) The catenoid is the image of the immersion $f_1 : \mathbb{R}^2 \ni (s, t) \to (\cosh s \cos t, \cosh s \sin t, s) \in \mathbb{R}^3$. The helicoid is the image of the immersion $f_2 : \mathbb{R}^2 \ni (s, t) \to (s \cos t, s \sin t, t) \in \mathbb{R}^3$. Let g_i be the pull-back of the standard Euclidean metric of $\mathbb{R}^3$ by the map f_i. Compute explicitely the metrics g_1 and g_2.

Exercise 2. Let g, h be Riemannian metrics on the manifolds M, N respectively. Let $f : M \to N$ be a local diffeomorphism. Prove that f is a local isometry if and only if for every chart (x, U) of M and (y, V) of N such that $f(U) \subset V$, the application $F : x(U) \to y(V)$ given by $F = y \circ f \circ x^{-1}$ satisfies:

$$g_{ij} = \sum_{k,l} h_{kl} \frac{\partial F^k}{\partial x_i} \frac{\partial F^l}{\partial x_j}$$

where g_{ij}, h_{kl} are the expressions for the metrics in the choosen charts.

Exercise 3. Recall that if X, Y are two vector fields in $\mathbb{R}^n$, we denote by $Z = dY \cdot X$ the directional derivative of Y in the direction X. In coordinates, if $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n), Z = (Z_1, \dots, Z_n)$ we have $Z_i = \sum_{j=1}^n \frac{\partial Y_i}{\partial x_j} X_j$.
Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bi-linear application. Consider the function $\nabla : V(\mathbb{R}^n) \times V(\mathbb{R}^n) \to V(\mathbb{R}^n)$ defined by

$$\nabla_X Y = dY \cdot X + B(X, Y).$$

(a) Prove that ∇ is a connection on $\mathbb{R}^n$.

(b) Compute the torsion $T(X, Y)$ and the curvature $R(X, Y)Z$ of ∇.

Exercise 4. Let M be a manifold, $\{U_\alpha | \alpha \in A\}$ an open covering of M and V a finite dimensional real vector space. For every pair $(\alpha, \beta) \in A \times A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, is given a smooth application $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$, with the following properties: for every $\alpha \in A$ and for every $p \in U_\alpha$, $g_{\alpha\alpha}(p) = \mathrm{Id}_V$, and that for all $\alpha, \beta, \gamma \in A$, $(g_{\alpha\beta} \cdot g_{\beta\gamma})|_{U_\alpha \cap U_\beta \cap U_\gamma} = g_{\alpha\gamma}|_{U_\alpha \cap U_\beta \cap U_\gamma}$.
For $\alpha \in A$, let $Y_\alpha = U_\alpha \times V$. On the disjoint union $\bigsqcup_{\alpha \in A} Y_\alpha$, we put the following equivalence relation $\sim$: $(p, v) \in Y_\alpha$ is equivalent to $(q, w) \in Y_\beta$ if and only if $p = q$ and $w = g_{\beta\alpha}(p) \cdot v$.

(a) Prove that the quotient space $E = \bigsqcup_{\alpha \in A} Y_\alpha$ has a structure of smooth manifold such that the application $\pi : E \to M$ defined by $\pi([(p, v)]) = p$ is a smooth map.

(b) For every $p \in M$ the inverse image $\pi^{-1}(p)$ can be identified with V using any open set U_α with $p \in U_\alpha$. Prove that the structure of vector space on $\pi^{-1}(p)$ does not depend on the choice of U_α.

(c) A space E that can be obtained with an open covering $\{U_\alpha | \alpha \in A\}$ and applications $g_{\alpha\beta}$ as above is called a vector bundle. For every manifold M, construct a structure of vector bundle on the tangent and on the cotangent bundles.