RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I

Heidelberg, 6.11.2012

Exercise sheet 4

Flows and semi-Riemannian metrics

To hand in by November 13, 14:00

Exercise 1. Let M be a manifold and $X, Y \in V(M)$ be vector fields. Fix a point $p \in M$, an open neighborhood $U \subset M$ and $\varepsilon > 0$ such that the flow f^t of the field X is defined on $(-\varepsilon, \varepsilon) \times U$. Recall that $\mathcal{L}_X(Y)_p$ is defined as $\frac{d}{dt}|_{t=0} \left(df^{-t}|_{f^t(p)} \left(Y_{f^t(p)} \right) \right) \in T_pM$. Recall also that for every $\phi \in \mathcal{F}(M)$, $[X, Y]_p(\phi)$ is defined as $X_p(Y\phi) - Y_p(X\phi)$.

(a) For every $\phi \in \mathcal{F}(M)$, consider the function $G : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}$ defined by $G_t(x) = \frac{1}{t}(\phi(f^{-t}(x)) - \phi(x))$, if $t \neq 0$, and by $G_0(x) = -X(\phi)$ when $t = 0$. Prove that G is smooth. (Hint: define $F(t, x) = \phi(f^{-t}(x)) - \phi(x)$, $F'(t, x) = \frac{\partial F}{\partial t}(t, x)$, and consider $\int_0^1 F'(ts, x)\, \mathrm{d}s$).

(b) Prove that $df^{-t}|_{f^t(p)} \left(Y_{f^t(p)} \right)(\phi) = Y_{f^t(p)}(\phi \circ f^{-t}) = Y_{f^t(p)}(\phi) + tY_{f(t)}(G_t)$.

(c) Prove that $\mathcal{L}_X(Y) = [X, Y]$. (Hint: Compute the derivative $\frac{d}{dt}|_{t=0}$ of the term in point (b). Remember that $c(t) = f^t(p)$ is an integral curve for X).

Exercise 2. Let M be a manifold, $V(M)$ be the vector space of all vector fields on M, and $[\cdot, \cdot] : V(M) \times V(M) \to V(M)$ be the Lie bracket.

(a) Prove that $(V(M), [\cdot, \cdot])$ is a Lie algebra. Namely, show that

  (i) $[\cdot, \cdot] : V(M) \times V(M) \to V(M)$ is $\mathbb{R}$-bilinear.

  (ii) For all $X \in V(M)$, $[X, X] = 0$.

  (iii) For all $X, Y \in V(M)$, $[X, Y] = -[Y, X]$.

  (iv) For all $X, Y, Z \in V(M)$, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (the Jacobi identity).

(b) Recall that if $f \in \mathcal{F}(M)$ and $X \in V(M)$, fX is the vector field that in the point $p \in M$ takes the value $f(p)X_p$, while $Xf = X(f)$ is the function that in the point $p \in M$ takes the value $X_p(f)$. Prove that for all $X, Y \in V(M)$ and for all $f, g \in \mathcal{F}(M)$,

$$[fX, gY] = fX(g)Y - gY(f)X + fg[X, Y].$$

Exercise 3. Let M be a manifold and $p \in M$. Let $c : (-1, 1) \to \mathbb{M}$ be a smooth curve such that $c(0) = p$ and $\dot{c}(0) = [c] = 0$. Choose a chart containing p, and define $\ddot{c}(0) \in T_pM$ in such a way that in this chart it corresponds to the usual second derivative of a curve in $\mathbb{R}^n$. Prove that your definition does not depend on the choosen chart.

Exercise 4. Consider the bilinear form on $\mathbb{R}^{n+1}$ defined by $\langle x, y \rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n$. Consider the subsets $H^n = \left\{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1, x_0 > 0 \right\}$ and $dS^n = \left\{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = 1 \right\}$. Prove that these subsets are submanifolds, and prove that for every point $x \in H^n$ (or $x \in dS^n$), the image of the tangent space T_xH^n (or T_xdS^n) by the differential of the identity map is the orthogonal vector space to x, i.e. $x^\perp = \{ v \in \mathbb{R}^{n+1} \mid \langle x, v \rangle = 0 \}$. Prove that the restriction of the bilinear form $\langle x, y \rangle$ to the tangent space at every point x gives a smooth Riemannian metric in the case of H^n, and a smooth semi-Riemannian metric of index 1 in the case of dS^n.