RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
Heidelberg, 30.10.2012

Exercise sheet 3
Vector fields
To hand in by November 6, 14:00

Let M be a manifold and $X, Y \in V(M)$ be vector fields.

Exercise 1. Recall that for every point $p \in M$, X_p is value of the field X at the point p, and for every function $\varphi \in \mathcal{F}(M)$, $X\varphi = X(\varphi)$ is the function in $\mathcal{F}(M)$ that to every point $p \in M$ associate the value $X_p(\varphi)$.

(a) For every point $p \in M$, consider the application $D_p : \mathcal{F}(M) \to \mathbb{R}$ that to every function $\varphi \in \mathcal{F}(M)$ associate the number $X_p(Y\varphi)$. Is the application D_p a tangent vector at the point p? Does the formula $p \to D_p$ define a vector field on M?

(b) Let (x, U) be a coordinate patch for M, and assume that in this coordinate patch, the vector fields X, Y can be expressed by $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i}$, with $\xi_i, \eta_i : U \to \mathbb{R}$ smooth functions. Prove that the vector field $Z = [X, Y]$ can be expressed by $Z = \sum_{i=1}^n \zeta_i \frac{\partial}{\partial x_i}$, where

$$\zeta_i = \sum_{j=1}^n \xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j}$$

Exercise 2. Let p be a point such that $X_p \neq 0$.

(a) Prove that there is a neighborhood V of p such that X never vanishes in V.

(b) Prove that there is a coordinate patch (x, U) around the point p such that $x(p) = 0$ and X_p is equal to the vector $\frac{\partial}{\partial x_1}$ in the point 0.

(c) Consider the subset $H = \{x \in \mathbb{R}^n \mid x_1 = 0\} = \{(0, x_2, \ldots, x_n)\}$. Prove that the inverse image $x^{-1}(H)$ is a sub-manifold of U.

(d) Recall that there exists an $\varepsilon > 0$ and a neighborhood U' of p contained in U such that the flow f^t of the field X is well defined in $(-\varepsilon, \varepsilon) \times U'$. Consider the application $\Phi : (-\varepsilon, \varepsilon) \times (x(U') \cap H) \to M$ defined by $\Phi(t, h) = f^t(x^{-1}(h))$. Prove that there exists an $\varepsilon' < \varepsilon$ and a neighborhood U'' of 0 in $(x(U') \cap H)$ such that Φ restricted to $(-\varepsilon', \varepsilon') \times U''$ is a diffeomorphism.

(e) Prove that there is a coordinate patch (x, U) around the point p such that X is equal to the vector field $\frac{\partial}{\partial x_1}$ in every point of U.

Exercise 3. Let f^t be the flow of X, g^t be the flow of Y, and $\mathcal{D}_X, \mathcal{D}_Y$ the domains of definition of the two flows. Assume that $\mathcal{L}_X(Y) = [X, Y] = 0$ in M.

(a) Prove that for all $(t, x) \in \mathcal{D}_X$, we have $df^t|_x(Y(x)) = Y(f^t(x))$.

(b) Let $c : (-\varepsilon, \varepsilon) \to M$ be an integral curve for Y such that $c(0) = x$. Prove that, if $(t, x) \in \mathcal{D}_X$, $f^t \circ c$ is an integral curve for Y such that $c(0) = f^t(x)$.

(c) Prove that for all $x \in M$, and for all $s, t \in \mathbb{R}$ small enough, $f^{-t} \circ g^{-s} \circ f^t \circ g^s(x) = x$. (Hint: follow the integral curves for Y, and move them with f^t, f^{-t}.)