Let $M$ be a manifold and $X,Y \in V(M)$ be vector fields.

**Exercise 1.** Recall that for every point $p \in M$, $X_p$ is value of the field $X$ at the point $p$, and for every function $\varphi \in C^\infty(M)$, $X \varphi = X(\varphi)$ is the function in $C^\infty(M)$ that to every point $p \in M$ associate the value $X_p(\varphi)$.

(a) For every point $p \in M$, consider the application $D_p : C^\infty(M) \to \mathbb{R}$ that to every function $\varphi \in C^\infty(M)$ associate the number $X_p(\varphi)$. Is the application $D_p$ a tangent vector at the point $p$? Does the formula $p \to D_p$ define a vector field on $M$?

(b) Let $(x,U)$ be a coordinate patch for $M$, and assume that in this coordinate patch, the vector fields $X,Y$ can be expressed by $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i}$, with $\xi_i, \eta_i : U \to \mathbb{R}$ smooth functions. Prove that the vector field $Z = [X,Y]$ can be expressed by $Z = \sum_{i=1}^n \zeta_i \frac{\partial}{\partial x_i}$, where

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\zeta_i = \sum_{j=1}^n \xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j}.
$$

**Exercise 2.** Let $p$ be a point such that $X_p \neq 0$.

(a) Prove that there is a neighborhood $V$ of $p$ such that $X$ never vanishes in $V$.

(b) Prove that there is a coordinate patch $(x,U)$ around the point $p$ such that $x(p) = 0$ and $X_p$ is equal to the vector $\frac{\partial}{\partial x_1}$ in the point 0.

(c) Consider the subset $H = \{x \in \mathbb{R}^n \mid x_1 = 0\} = \{(0,x_2,\ldots,x_n)\}$. Prove that the inverse image $x^{-1}(H)$ is a sub-manifold of $U$.

(d) Recall that there exists an $\varepsilon > 0$ and a neighborhood $U'$ of $p$ contained in $U$ such that the flow $f^t$ of the field $X$ is well defined in $(-\varepsilon,\varepsilon) \times U'$. Consider the application $\Phi : (-\varepsilon,\varepsilon) \times (x(U') \cap H) \to M$ defined by $\Phi(t,h) = f^t(x^{-1}(h))$. Prove that there exists an $\varepsilon' \leq \varepsilon$ and a neighborhood $U''$ of 0 in $(x(U') \cap H)$ such that $\Phi$ restricted to $(-\varepsilon',\varepsilon') \times U''$ is a diffeomorphism.

(e) Prove that there is a coordinate patch $(x,U)$ around the point $p$ such that $X$ is equal to the vector field $\frac{\partial}{\partial x_1}$ in every point of $U$.

**Exercise 3.** Let $f^t$ be the flow of $X$, $g^t$ be the flow of $Y$, and $\mathcal{D}_X, \mathcal{D}_Y$ the domains of definition of the two flows. Assume that $\mathcal{L}_X(Y) = [X,Y] = 0$ in $M$.

(a) Prove that for all $(t,x) \in \mathcal{D}_X$, we have $df^t|_x(Y(x)) = Y(f^t(x))$.

(b) Let $c : (-\varepsilon,\varepsilon) \to M$ be an integral curve for $Y$ such that $c(0) = x$. Prove that, if $(t,x) \in \mathcal{D}_X$, $f^t \circ c$ is an integral curve for $Y$ such that $c(0) = f^t(x)$.

(c) Prove that for all $x \in M$, and for all $s,t \in \mathbb{R}$ small enough, $f^{-t} \circ g^{-s} \circ f^t \circ g^s(x) = x$.

(Hint: follow the integral curves for $Y$, and move them with $f^t,f^{-t}$.)