RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie I
*Heidelberg, 23.10.2012*

# Exercise sheet 2

## Smooth maps and submanifolds

*To hand in by October 30, 14:00 Uhr*

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**Exercise 1.**

(a) Consider the sphere $S_r^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = r^2\right\}$, and the real projective space $\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim$, with $C^\infty$ structures defined by the atlases in Exercise sheet 1, §3. Prove that the map $\phi : S_r^n \to \mathbb{RP}^n$ defined by $\phi(x_0, \ldots, x_n) = [x_0, \ldots, x_n]$ is a smooth map. Is it locally invertible? Is it a diffeomorphism?

(b) Let $M, N$ be diffrentiable manifolds with atlases $\{(\phi_i, U_i) \mid i \in I\}, \{(\psi_j, V_j) \mid j \in J\}$ respectively. Describe an atlas for the product $M \times N$ such that the two projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are smooth maps.

(c) Prove that the map $\phi : \mathbb{R}^2 \to (\mathbb{R}_{>0})^2$ defined by $\phi(x, y) = (e^x, e^y)$ is a diffeomorphism.

**Exercise 2.**

(a) Prove that the identity map $i : S_r^n \to \mathbb{R}^{n+1}$ is an embedding of $S_r^n$ into a submanifold of $\mathbb{R}^{n+1}$.

(b) Recall that for every point $p \in \mathbb{R}^{n+1}$, the tangent space $T_p\mathbb{R}^{n+1}$ can be identified with $\mathbb{R}^{n+1}$ by the identity chart. For every point $q \in S_r^n$, show that the image of the differential of $i$ at the point $q$ (i.e. the subspace $di_q(T_q S_r^n)$) is equal to the orthogonal vector subspace to $i(q)$, with reference to the standard scalar product, $i(q)^\perp = \{v \in \mathbb{R}^{n+1} \mid \langle i(q), v\rangle = 0\}$.

**Exercise 3.**

(a) Let $N$ be a differentiable manifold and $M \subset N$ a submanifold. Show that $f : M \to \mathbb{R}$ is a smooth function if and only if the following hold: for every point $p \in M$ there is an open neighborhood $U$ of $p$ in $N$ and a smooth function $\phi : U \to \mathbb{R}$ such that $\phi|_{U \cap M} = f|_{U \cap M}$.

(b) Let $M$ and $N$ be smooth manifolds and $h : M \to N$ be a submersion. Show that a function $f : N \to \mathbb{R}$ is smooth if and only if $f \circ h$ is smooth. Show also that for every point $p \in M$, we have $\mathrm{rank}_{h(p)} f = \mathrm{rank}_p(f \circ h)$.

**Exercise 4.**

(a) Prove that a connected topological manifold is path-connected. (Hint: Fix a point $p$ and consider the set of all points that can be joined to $p$ with a continuous curve. Prove that this set is open and closed.)

(b) Let $M$ be a differentiable manifold. If $[a, b] \subset \mathbb{R}$ is a closed interval, a map $\gamma : [a, b] \to M$ is called smooth if there exists $\varepsilon > 0$ and a smooth map $\gamma' : (a - \varepsilon, b + \varepsilon) \to M$ such that the restriction of $\gamma'$ to $[a, b]$ coincides with $\gamma$. A map $\gamma : [a, b] \to M$ is called piece-wise smooth if there exists a finite number of points $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$ such that the restriction of $\gamma$ to every interval $[x_i, x_{i+1}]$ is smooth. Prove that in a connected differentiable manifold, for every two points there is a piece-wise smooth curve joining them.