RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie I
Heidelberg, 18.10.2012

EXERCISE SHEET 1

Manifolds and differentiable structures

To hand in by October 23, 14:00 Uhr

Exercise 1. Which ones of the following topological subspaces of $\mathbb{R}^n$ are topological manifolds?

(a) $\{(x,y) \in \mathbb{R}^2 \mid x^3 + x^2 - y^2 = 0\}$.

(b) $\{(x,y) \in \mathbb{R}^2 \mid x^3 - y^2 = 0\}$.

(c) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0\}$.

(d) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 0 \text{ and } z \geq 0\}$.

(e) $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0 \text{ or } y = 0\}$.

(f) $\{(x,y,z) \in \mathbb{R}^3 \mid \max(|x|,|y|,|z|) = 1\}$.

Exercise 2.

(a) Find a C^∞-atlas, for the real line $\mathbb{R}$, that gives a different smooth structure than the usual atlas $\mathcal{A} = \{(\mathrm{Id}, \mathbb{R})\}$.

(b) Find a C^∞-atlas for the torus

$$\left\{(x,y,z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\right\}$$

Exercise 3.

(a) The sphere of radius r is the set $S^n_r = \left\{(x_0,\dots,x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n x_i^2 = r^2\right\}$. Denote $N = (1,0,\dots,0)$, $S = (-1,0,\dots,0)$. Consider the atlas $\{(p_N,\mathcal{U}_N),(p_S,\mathcal{U}_S)\}$, where $\mathcal{U}_N = S^n_r \setminus N$, $\mathcal{U}_S = S^n_r \setminus S$, the function $p_N : \mathcal{U}_N \to \mathbb{R}^n$ is defined by

$$p_N(x_0,\dots,x_n) = \frac{r}{r - x_0}(x_1,\dots,x_n),$$

and the function $p_S : \mathcal{U}_S \to \mathbb{R}^n$ is defined by

$$p_S(x_0,\dots,x_n) = \frac{r}{r + x_0}(x_1,\dots,x_n).$$

Prove that $\{(p_N,\mathcal{U}_N),(p_S,\mathcal{U}_S)\}$ is a C^∞-atlas for S^n_r.

(b) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Two points $x, y \in \mathbb{K}^{n+1} \setminus \{0\}$ are projectively equivalent ($x \sim y$) if they lie in the same 1-dimensional vector subspace. The projective space $\mathbb{KP}^n$ is the quotient space $\left(\mathbb{K}^{n+1} \setminus \{0\}\right) / \sim$, with the quotient topology. The image of the point $(x_0,\dots,x_n)$ will be denoted by $[x_0,\dots,x_n]$ (homogeneous coordinates). Consider the atlas $\{(b_i,\mathcal{U}_i) \mid i \in \{0,\dots,n\}\}$, where $\mathcal{U}_i = \{[x_0,\dots,x_n] \in \mathbb{KP}^n \mid x_i \neq 0\}$, and the function $b_i : \mathcal{U}_i \to \mathbb{K}^n$ is defined by

$$b_i([x_0,\dots,x_n]) = \frac{1}{x_i}(x_0,\dots,\widehat{x_i},\dots,x_n)$$

Prove that $\{(b_i,\mathcal{U}_i)\}$ is a C^∞-atlas for $\mathbb{KP}^n$.