Workshop "$G_{\mathbb{Q}_p}$ as a geom. fundamental group"

SC 8.5.2015

$X_{F,E}$ and $X^{ad}_{F,E}$ are "geometrically simply connected"

NOTATION. E a finite extension of $\mathbb{Q}_p$ (or $E = \mathbb{F}_q((t))$, $\pi = t$)
with uniformizer π,
residue class field $\mathbb{F}_q$

$F/\mathbb{F}_q$ complete algebraically closed

$\rightsquigarrow$ $X = X_{F,E}$, $X^{ad} = X^{ad}_{F,E}$ Fargues-Fontaine curve

VB_X vector bundles, simple objects: $\mathcal{O}(\lambda)$, $\lambda \in \mathbb{Q}$.

THEOREM. We have an equiv. of categories

$$(\mathrm{Spa}\, E)_{f\acute{e}t} = \begin{pmatrix}\text{finite étale}\\ E\text{-algebras}\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}\text{finite étale}\\ \mathcal{O}_X\text{-algebras}\end{pmatrix} = (X_{F,E})_{f\acute{e}t}$$

$$A \longmapsto \mathcal{O}_X \otimes_E A,$$

and analogously for $X^{ad}_{F,E}$: $(\mathrm{Spa}\, E)_{f\acute{e}t} \xrightarrow{\sim} (X^{ad}_{F,E})_{\acute{e}t}$.

<u>Proof</u>. Let $Y \xrightarrow{f} X$ be a finite étale covering, let $\mathcal{A} := f_* \mathcal{O}_Y$, a loc. free $\mathcal{O}_X$-module with an $\mathcal{O}_X$-algebra structure. Assume: Y connected (if not, consider the conn. components individually)

Classification of vector bundles on X $\rightsquigarrow$

$$\mathcal{A} \cong \bigoplus_{i=1}^{n} \mathcal{O}_X(d_i), \qquad d_1 \geq \cdots \geq d_n \in \mathbb{Q}$$

<u>Claim</u>. $d_i = 0 \ \forall i$.

Proof of claim. Let $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denote the multiplication. It is given by homomorphisms

$$m_{i,j,k} : \mathcal{O}(d_i) \otimes \mathcal{O}(d_j) \longrightarrow \mathcal{O}(d_k),$$
$$1 \leq i, j, k \leq n$$

Assume $d_1 > 0$. Then $m_{1,1,k} : \mathcal{O}(d_1)^{\otimes 2} \to \mathcal{O}(d_k)$ can be viewed as an element of

$$\cancel{H^0(X, \underbrace{\mathcal{O}(d_k - 2d_1)}_{<0}) = 0}. \quad H^0(X, \underbrace{\mathcal{O}(d_k) \otimes \mathcal{O}(-d_1)^{\otimes 2}}_{\text{has slopes } <0}) = 0$$

so we find $m|_{\mathcal{O}(d_i)\otimes\mathcal{O}(d_i)} = 0$.

This is a contradiction: Since Y is an integral scheme, the product of two non-zero sections is non-zero.

We have shown that $0 \geq d_1 \geq \cdots \geq d_n$.

Prop. Let $f: Y \to X$ be a finite étale morph. of schemes. Then there is a canonical isom.

$$\det(f_*\mathcal{O}_Y)^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$$

of invertible $\mathcal{O}_X$-modules

Proof. If A is a ring and $A \to B$ is a finite étale A-algebra, then B is a locally free A-module and the trace map $tr_{B/A}: B \to A$ induces a perfect pairing $B \otimes_A B \to A$,

$$b \otimes b' \mapsto tr_{B/A}(bb'),$$

hence an isomorphism $B \xrightarrow{\sim} B^* := \mathrm{Hom}_A(B, A)$ (*)

We can glue the local trace maps, pairings and isomorphisms (*) to obtain an isomorphism

$$f_* \mathcal{O}_Y \xrightarrow{\sim} (f_* \mathcal{O}_Y)^* := \mathcal{H}om_{\mathcal{O}_X}(f_* \mathcal{O}_Y, \mathcal{O}_X)$$

of locally free $\mathcal{O}_X$-modules, for any $f: X \to Y$ finite étale.

Taking the top exterior power, i.e. passing to the determinant of these locally free $\mathcal{O}_X$-modules, we get

$$\det f_* \mathcal{O}_Y \simeq \det((f_* \mathcal{O}_Y)^*) = \det(f_* \mathcal{O}_Y)^{-1},$$

so $\det(f_* \mathcal{O}_Y)^{\otimes 2} \simeq \mathcal{O}_X$.

<u>Remarks</u>. (i) The same proof shows that the same statement holds for finite étale morphisms $f: Y \to X$ of adic spaces.

(ii) In general, $\det(f_* \mathcal{O}_Y)$ is not isomorphic to $\mathcal{O}_X$.

(iii) For an alternative proof of the proposition, see [FF, Prop. 4.7].

We obtain that

$$\mathcal{O}_X = \det(\mathcal{A})^{\otimes 2} = \bigotimes_i \det \mathcal{O}(d_i)^{\otimes 2} = \mathcal{O}\left(2\sum a_i\right)$$

where $d_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathbb{Z}$, $b_i > 0$, $\gcd(a_i, b_i) = 1$.

and hence $d_i = 0 \ \forall i$. The claim is proved.

So $\mathcal{A} \cong \mathcal{O}_X \otimes A$ for some E-vector space A, as $\mathcal{O}_X$-modules.

Since $H^0(X, \mathcal{O}_X) = E$, the algebra structure on $\mathcal{A}$ comes from an algebra structure on A. Taking global sections of the above isomorphism, we see that

$A \cong H^0(X, \mathcal{A}) = H^0(Y, \mathcal{O}_Y)$.

This shows that the functors

$\operatorname{Spec} A \mapsto X \otimes_E A$

and $H^0(Y, \mathcal{O}_Y) \longleftarrow\!\!\!\mid \ Y$ are quasi-inverses of each other.

Remark. In the same way one obtains a proof that $\mathbb{P}^1$ over an alg. closed field is simply connected w.r.t. the étale topology which avoids using the Riemann-Hurwitz formula.

In terms of the étale fundamental gp: $\pi_1(X_{F,E}) = \mathrm{Gal}(\bar{E}/E)$

$\bar{E}$ sep. cl. of E

Using the short exact sequence

$$1 \to \pi_1(X \otimes_E \bar{E}) \to \pi_1(X) \to \mathrm{Gal}(\bar{E}/E) \to 1,$$

we see that the theorem is equivalent to

$$\pi_1(X \otimes_E \bar{E}) = \{1\}.$$

Similarly, for X^{ad} we have a short exact sequence

$$1 \to \pi_1(X^{ad} \hat{\otimes}_E \hat{\bar{E}}) \xrightarrow{\alpha} \pi_1(X^{ad}) \to \mathrm{Gal}(\bar{E}/E) \to 1.$$

To prove this, use that X^{ad} is quasi-compact together with an approximation argument as in talk TE, see [Scholze, Perfectoid Spaces, Prop. 7.4 / Gabber-Ramero Prop. 5.4.53] to show that every finite étale cover of $X^{ad} \hat{\otimes} \hat{\bar{E}}$ comes from a finite étale cover of some $X^{ad} \hat{\otimes}_E E'$, E'/E finite.

(This is needed to prove the injectivity of α.)

Cf. also [de Jong, Etale fundamental groups ..., Compositio Math. 97 (1995), 89-118] Prop 2.13.
and [Scholze lectures at MSRI, Prop. 17.3.13.