Review of perfectoid fields and almost mathematics

Matthias Wulkau, University of Muenster
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We follow closely [1, Chapter 2] and [2, Chapters 3,4].

Part I
Perfectoid fields

Let $p$ be a prime.

1 Introduction

Definition. Let $(K,|\cdot|)$ be a valued field, complete with respect to a non-archimedean absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. It is a perfectoid field if

- $|K^*| \subset \mathbb{R}_{\geq 0}$ is non-discrete,
- $\text{char}(K^*/m) = p$, where $K^*$ resp. $m$ are the valuation ring resp. the valuation ideal of $K$ with respect to $|\cdot|$,
- the Frobenius $\Phi : K^*/(p) \to K^*/(p)$, $x \mapsto x^p$ is surjective.

Remark. Examples include

- in characteristic 0 : the $p$-adic completions of $\mathbb{Q}_p^{\text{alg}}, \mathbb{Q}_p(p^{p^{-\infty}}) := \bigcup_{n \geq 1} \mathbb{Q}_p(p^{1/p^n})$ and $\mathbb{Q}_p(\mu_{p^{\infty}}) := \bigcup_{n \geq 1} \mathbb{Q}_p(\mu_{p^n})$;
- in characteristic $p$ : the $t$-adic completions of $\mathbb{F}_p((t))^{sp}$ and $\mathbb{F}_p((t))(t^{p^{-\infty}})$ for a variable $t$.

Non-examples are $p$-adic completions of algebraic unramified extensions of $\mathbb{Q}_p$ (the value group is discrete).

Observation. 1. If $K$ is perfectoid of characteristic 0 and $p \in m$, then it naturally contains $\mathbb{Q}_p$.

2. $K$ is perfectoid of characteristic $p$ if and only if $K$ is a perfect complete nonarchimedean field of characteristic $p$. 

1
2 The tilting construction for perfectoid fields

Definition. Let $A$ be a unital commutative ring, which is annihilated by $p$, $pA = 0$. Then $A \to A, a \mapsto a^p$ is a ring homomorphism. Set $R(A) := \varprojlim_{a \to a^p} A$, the perfection of $A$.

This yields a perfect ring (meaning that $x \to x^p$ is bijective) with $pR(A) = 0$. Applying this construction to the valuation ring of a perfectoid field, we obtain the following interesting result.

Proposition. Let $K$ be a perfectoid field and $\pi \in K^\circ \setminus \{0\}$ such that $|p| \leq |\pi| < 1$. For $y \in K^\circ/(\pi)$, denote by $\hat{y}$ a lift to $K^\circ$.

1. The maps
   
   \[ R(K^\circ/(\pi)) \cong \varprojlim_{x \to x^p} K^\circ \]
   
   \[ (x_n)_{n \geq 0} \mapsto (\lim_{m \to \infty} \hat{x}_{n+m}^m)_{n \geq 0} \]
   
   \[ (y^{(n)} + (\pi))_{n \geq 0} \mapsto (y^{(n)})_{n \geq 0} \]

   are mutually inverse and establish a well-defined and natural bijection between multiplicative monoids respecting the zero elements on both sides. In particular, the limits $\lim_{m \to \infty} \hat{x}_{n+m}$ exist and are independent of the choices of the lifts.

2. The ring $R(K^\circ/(\pi))$ is an integral domain, independent of the choice of $\pi$.

3. Denote by $\#^{-1}$ the map
   
   \[ K^\circ := Q(R(K^\circ/(\pi))) \xrightarrow{\eta} \varprojlim_{x \to x^p} K \xrightarrow{\theta_0} K \]

   where $\theta_0((y^{(n)})_n) := y^{(0)}$. Then
   
   \[ |x|_{K^\circ} := |x^p|_K \]

   defines a non-archimedean absolute value
   
   - which makes $K^\circ$ into a perfectoid field of characteristic $p$ and
   - with respect to which $K^\circ = R(K^\circ/(\pi))$.

4. If $\text{char} \ (K) = p$, then $K^\circ \simeq K$.

Definition. $K^\circ$ is called the tilt of $K$.

From the previous proposition, we see that, in particular, the tilt of a perfectoid field is again a perfectoid field (note that, for example, $Q^\circ_p = \mathbb{F}_p$). The properties of the tilt of a perfectoid field are summarized in the following theorem:
Theorem. Let $K$ be a perfectoid field with tilt $K^\flat$.

1. The map

\[
\text{set of equivalence classes of continuous absolute values on } K \rightarrow \text{set of equivalence classes of continuous absolute values on } K^\flat
\]

\[
\text{equivalence class of } |\cdot| \rightarrow \text{equivalence class of } |\cdot|^\flat = |\cdot|^\flat
\]

is bijective.

2. Any finite extension of $K$ is a perfectoid field.

3. The association $-^\flat$ induces an equivalence of categories

\[L \mapsto L^\flat\]

with $[L : K] = [L^\flat : K^\flat]$. In particular, $K$ is algebraically closed if $K^\flat$ is algebraically closed.
Part II
Almost mathematics

3 Almost category theory

Fix a perfectoid field \((K, | \cdot |, K^\circ, m)\), let \(\text{char}(K^\circ/m) = p\). Note that \(m\) is a flat \(K^\circ\)-module (since torsionfree modules over valuation rings are flat) and that \(m^2 = m \cong m \otimes_{K^\circ} m\).

**Definition.** Let \(M\) be a \(K^\circ\)-module.

- An element \(x \in M\) is almost zero if \(m x = 0\). \(M\) is almost zero if \(m M = 0\).
- A morphism \(f \in \text{Hom}_{K^\circ}(M, N)\) is an almost isomorphism if \(\ker(f)\) and \(\text{coker}(f)\) are almost zero.

**Example.** \(K^\circ/m\) is almost zero, whereas \(K^\circ/(p)\) is not.

Denote by \(\text{Ann}(m)\) the full subcategory of almost zero objects in \(K^\circ\)-mod.

**Lemma.** The category \(\text{Ann}(m)\) is thick (i.e. closed under subobjects, quotients and extensions).

**Proof.** Let \(M_1 \subseteq M\) and \(M/M_1\) be almost zero. Then \(m M \subseteq M_1\), hence \(0 = m^2 M = m M\).

Therefore we can form the quotient category \(K^\circ\text{-mod} / \text{Ann}(m) = K^{\text{ao}}\text{-mod}\). Denote the exact canonical functor \(K^\circ\text{-mod} \to K^{\text{ao}}\text{-mod}\) by \(M \mapsto M^a\). The latter object is \(M\), seen as an object of \(K^{\text{ao}}\text{-mod}\). We record the following facts about this abelian category:

- Let \(f : M \to N\) be a morphism in \(K^\circ\text{-mod}\). \(f^a\) is an isomorphism if and only if \(f\) is an almost isomorphism. Hence, \(M\) lies in \(\text{Ann}(m)\) \(\iff u : M \to 0\) is an almost isomorphism \(\iff u^a : M^a \to 0\) is an isomorphism.

- For any two \(K^\circ\)-modules \(M, N\), there is an equality of Hom-sets
  \[
  \text{Hom}_{K^\circ}(M^a, N^a) = \text{Hom}_{K^\circ}(m \otimes_{K^\circ} M, N)
  \]
  by which the left hand side obtains the natural structure of a \(K^\circ\)-module.

- There is no non-zero \(m\)-torsion in \(\text{Hom}_{K^{\text{ao}}}(X, Y)\) for any two objects \(X, Y\) in \(K^{\text{ao}}\text{-mod}\) (indeed, writing \(X = M^a, Y = N^a\), then for \(f \in \text{Hom}_{K^\circ}(m \otimes_{K^\circ} M, N)\) with \(mf = 0\), one has \(0 = mf(m \otimes_{K^\circ} M) = f(m^2 \otimes_{K^\circ} M) = f(m \otimes_{K^\circ} M)\)).
Set \( \text{alHom}_{K^a}(X,Y) := \text{Hom}_{K^a}(X,Y)^a \).

**Proposition.**

- The tensor product in \( K^a\text{-mod} \) induces a bifunctor \( -\otimes_{K^a} - \) on \( K^{a}\text{-mod} \), making it an abelian tensor category. There is a functorial isomorphism
  \[
  \text{Hom}_{K^a}(X \otimes_{K^a} Y, Z) \cong \text{Hom}_{K^a}(X, \text{alHom}_{K^a}(Y, Z)).
  \]

- The functor \( -^a \) has a right adjoint \( -_* \) and a left adjoint \( -_! \) with
  \[
  (X^a) \cong X \cong (X_!)^a
  \]
  and
  \[
  (M^a)_* \cong \text{Hom}_{K^a}(m, M) \quad \text{resp.} \quad (M^a)_! \cong m \otimes_{K^a} M
  \]
  for all objects \( X \) in \( K^{a}\text{-mod} \) and all objects \( M \) in \( K^a\text{-mod} \). Moreover, \( -_* \) is left exact and \( -_! \) is exact. Both are fully faithful.

In particular, the functors are defined as \( X_* := \text{Hom}_{K^a}(K^a, X) \) and \( X! := m \otimes_{K^a} X_* \). We have, for example, \( (K^a)_! \cong m \).

### 4 Almost commutative algebra

**Definition.** A \( K^{a}\text{-algebra} \) \( A \) is a commutative unitary monoid object in \( K^{a}\text{-mod} \) (i.e. there are morphisms \( \mu : A \otimes_{K^a} A \rightarrow A \) and \( \eta_A : K^a \rightarrow A \) satisfying certain associativity and commutativity constraints). Denote the category of \( K^{a}\text{-algebras} \) by \( K^{a}\text{-alg} \).

We collect several properties about almost algebras:

- \( -^a \) restricts to an essentially surjective functor \( K^a\text{-alg} \rightarrow K^{a}\text{-alg} \).

- Let \( A \) be an object of \( K^{a}\text{-alg} \). Then there is the notion of an \( A\)-module, the category of which is denoted by \( A\text{-mod} \). It is again an abelian tensor category.

- An \( A\)-algebra \( B \) is a \( K^{a}\)-algebra \( B \) together with a morphism of \( K^{a}\)-algebras \( A \rightarrow B \).

**Definition.** Let \( R^a = A \) be a \( K^{a}\text{-algebra} \) and \( X \) be an \( A\)-module. Then \( X \) is flat resp. almost projective if \( X \otimes_{A^a} - \) resp. \( \text{alHom}_A(X,-) \) is an exact functor. The module \( X = M^a \) is almost finitely presented if for all \( \epsilon \in m \), there exists a finitely presented \( R\)-module \( M_\epsilon \) and a homomorphism \( f_\epsilon : M_\epsilon \rightarrow M \), such that \( \epsilon \text{ ker}(f_\epsilon) = 0 = \epsilon \text{ coker}(f_\epsilon) \).

Let \( B \) be an \( A\)-algebra. The morphism \( A \rightarrow B \) is étale if the following two properties are satisfied:
• there exists $e \in (B \otimes_A B)_\ast$ such that $e$ is idempotent, $\mu \circ e = \eta_B$ and $\ker(\mu)_\ast \cdot e = 0$ (note: this is the definition of an unramified morphism);

• $B$ is flat as an $A$-module.

The morphism is finite étale if it is étale and $B$ is almost finitely presented as an $A$-module.

Denote by $A_{\text{f\acute{e}t}}$ the category of finite étale $A$-algebras.

An intermediate step towards proving the main theorem about tilting, namely the equivalence of the categories $K_{\text{f\acute{e}t}}$ and $K_{\text{f\acute{e}t}}^\flat$, is the following.

Theorem. Let $A$ be a $K^{\text{aa}}$-algebra, which is flat over $K^{\text{aa}}$ and which is isomorphic with $\lim \leftarrow A/\left(\pi \right)^n$. Then $B \mapsto B \otimes_A A/\left(\pi \right)$ induces an equivalence of categories $A_{\text{f\acute{e}t}} \simeq (A/\left(\pi \right))_{\text{f\acute{e}t}}$. 


References
