On the Embedding Problem with Non-abelian Kernel

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An embedding problem for a profinite group G is a diagram

$$\begin{tikzcd} & & & G \arrow[d, "\varphi"] & \\ 1 \arrow[r] & N \arrow[r] & E \arrow[r] & \bar{G} \arrow[r] & 1 \end{tikzcd}$$

with an exact sequence of profinite groups and a surjective homomorphism φ. A solution of this problem is a homomorphism $\psi\colon G \to E$ such that the diagram commutes. A solution is called proper if ψ is surjective. In the following we denote an embedding problem for G only by its exact line.

In this note we consider pro-p groups and problems with not necessarily abelian kernel N and prove a theorem which generalizes a result of Lur'e for the Galois group of the maximal p-extension of a $\mathfrak{p}$-adic field [1]. We will follow the proof given in [3].

Let G be a pro-p group of finite cohomological dimension $\mathrm{cd}_p\, G = n$. By $d(G)$ we denote the generator rank $\dim_{\mathbb{F}_p} H^1(G, \mathbb{Z}/p\mathbb{Z})$ of G. We recall the notion of the dualizing module of G, [2] (3.4.4):

$$D(G) = \varinjlim_{\nu \in \mathbb{N}} \varinjlim_{U} H^n(U, \mathbb{Z}/p^\nu\mathbb{Z})^\vee,$$

where $^\vee$ denotes the Pontryagin-dual, the second direct limit is taken over all open subgroups U of G and the transition maps are the duals of the corestriction maps. $D(G)$ is a discrete G-module in a natural way. We have a natural isomorphism

$$H^n(G, A)^\vee \xrightarrow{\sim} \mathrm{Hom}_G(A, D(G))$$

for every finite p-primary G-module A.

Theorem 1 *Let G be a pro-p group of cohomological dimension $\mathrm{cd}_p G = 2$ with dualizing module $D = D(G)$. Let H be a normal subgroup of G and let E be a pro-p group with $d(E) = d(G/H)$. Assume that one of the following assumptions is fulfilled:*

(i) *$D \cong \mathbb{Q}_p/\mathbb{Z}_p$ as abelian group.*

(ii) *$D = D^H$.*

Then the embedding problem

$$1 \longrightarrow N \longrightarrow E \longrightarrow G/H \longrightarrow 1 \tag{1}$$

has a proper solution if and only if the abelian problem

$$1 \longrightarrow N/N^{p^s}[N,N] \longrightarrow E/N^{p^s}[N,N] \longrightarrow G/H \longrightarrow 1 \tag{2}$$

has a proper solution, where p^s is the order of D^H, $s \leq \infty$.

Here we put $N^{p^\infty} = 1$. We will need the following

Lemma 2 *Let G be a pro-p group of cohomological dimension $\mathrm{cd}_p G = 2$ with dualizing module D. Then the embedding problem*

$$1 \longrightarrow A \longrightarrow E \longrightarrow G/H \longrightarrow 1$$

with finite p-primary abelian kernel A is solvable if and only if the problem

$$1 \longrightarrow A/B \longrightarrow E/B \longrightarrow G/H \longrightarrow 1$$

is solvable, where B is a normal subgroup of E, contained in A, such that

$$\mathrm{Hom}_G(A, D) = \mathrm{Hom}_G(A/B, D).$$

Proof: Let $\pi\colon A \twoheadrightarrow A/B$ be the natural surjection. Then by assumption

$$H^0(G, \mathrm{Hom}(A/B, D)) \xrightarrow[\sim]{\pi_*} H^0(G, \mathrm{Hom}(A, D)).$$

It follows that $\pi_*\colon H^2(G, A) \xrightarrow{\sim} H^2(G, A/B)$ is an isomorphism. Using [2](3.5.9), the commutative diagram

$$\begin{tikzcd} H^2(G,A) \arrow[r, "\pi_*", "\sim"'] & H^2(G,A/B) \\ H^2(G/H,A) \arrow[u, "\mathrm{inf}"] \arrow[r, "\pi_*"] & H^2(G/H,A/B) \arrow[u, "\mathrm{inf}"'] \end{tikzcd}$$

gives the result. $\square$

Proof of the theorem: Passing to the projective limit by using standard arguments, we may assume that N is a finite p-group.

If $D = D^H$, then $s = \infty$. Otherwise, using the lemma and since

$$\mathrm{Hom}_G(N^{ab}, D) = \mathrm{Hom}_G(N^{ab}, D^H) = \mathrm{Hom}_G(N/N^{p^s}[N,N], D),$$

the embedding problem (2) is solvable if and only if

$$1 \longrightarrow N^{ab} \longrightarrow E/[N,N] \longrightarrow G/H \longrightarrow 1 \tag{3}$$

is solvable.

In order to show that the solvability of (3) implies the solvability of (1), we use induction on the order of N. Since $d(E) = d(G/H)$, every solution will be proper.

If the dualizing module D is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ as abelian group, then we get for the kernel of the homomorphism

$$\varphi : G \longrightarrow \mathrm{Aut}(D) \cong \mathbb{Z}_p^\times,$$

which is given by the action of G on D, the inclusion $G/\ker\varphi \subseteq \mathbb{Z}_p^\times$. We may assume that $G/\ker\varphi \subseteq 1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$ (for $p \neq 2$ this is always the case) and that $s \geq 2$, if $p \neq 2$, and $s \geq 3$, if $p = 2$, i.e. $G/\ker\varphi \subseteq 1 + p^2\mathbb{Z}_p$. otherwise we consider the embedding problem

$$1 \longrightarrow N \longrightarrow E' \longrightarrow G/H' \longrightarrow 1,$$

where $E' = E \times \mathbb{Z}/p\mathbb{Z}$ and H' is defined by $D^{H'} = {}_{p^2}D$ if $p \neq 2$ (resp. $E' = E \times (\mathbb{Z}/2\mathbb{Z})^\varepsilon$, where $\varepsilon \leq 2$ such that $D^{H'} = {}_8D$); hence $G/H' = G/H \times \mathbb{Z}/p\mathbb{Z}$ (resp. $G/H' = G/H \times (\mathbb{Z}/2\mathbb{Z})^\varepsilon$); observe that $d(E') = d(G/H')$.

Now we assume that (3) is solvable. Let $\tilde{N} = N^p[N,E]$. If $N \supseteq N' \supseteq \tilde{N}$, then N' is normal in E and we get a commutative and exact diagram

$$\begin{tikzcd}
1 \arrow[r] & N^{ab} \arrow[r] & E/[N,N] \arrow[r] & G/H \arrow[r] & 1 \\
1 \arrow[r] & N'/[N,N] \arrow[r] \arrow[u, hook] & E/[N,N] \arrow[r] \arrow[u, equal] & E/N' \arrow[r] \arrow[u, two heads] & 1 \\
 & & & N/N'. \arrow[u, hook] &
\end{tikzcd}$$

Since the solution of (3) is proper, we get a normal subgroup H' of G contained in H such that $E/N' \cong G/H'$. We consider two cases:

1. Assume that there exists a subgroup $N \supseteq N' \supseteq \tilde{N}$, $(N : N') = p$ such that $D^H = D^{H'}$. Since N/N' is cyclic, we obtain

$$\begin{aligned}
\mathrm{Hom}_G((N')^{ab}, D) &= \mathrm{Hom}_G((N')^{ab}, D^{H'}) \\
&= \mathrm{Hom}_G((N')^{ab}, D^H) \\
&= \mathrm{Hom}_G(N'/[N',N], D^H) \;=\; \mathrm{Hom}_G(N'/[N,N], D).
\end{aligned}$$

Using the lemma, the solvability of (3) implies the solvability of

$$1 \longrightarrow (N')^{ab} \longrightarrow E/[N', N'] \longrightarrow G/H' \longrightarrow 1$$

By the induction hypothesis the result follows. This finishes already the proof of the theorem in the case that $D^H = D$.

2. The assumption of 1. is not fulfilled, i.e. $D^H \neq D^{H'}$ for all subgroups $N \supseteq N' \supseteq \tilde{N}$. Hence we assume that $D \cong \mathbb{Q}_p/Z_p$ and we have seen above that $\varphi(H) \cong \mathbb{Z}_p$, $\varphi : G \to \mathrm{Aut}(D)$. It follows that $(N : \tilde{N}) = p$. As above let $\tilde{H}$ be obtained via a solution of (3).

Let $N =< \beta, \tilde{N} >$, and $f \in \mathrm{Hom}_G(\tilde{N}^{ab}, D^{\tilde{H}})$. It follows that

$$\tilde{N} =< \beta^p, \tilde{N}^p, [N, E] > .$$

If $\bar{\alpha}$ denotes the class $\alpha[\tilde{N}, \tilde{N}]$, $\alpha \in \tilde{N}$, then $f(\bar{\beta}^p)^\beta = f(\bar{\beta}^p)$, hence $f(\bar{\beta}^p) \in (D^{\tilde{H}})^{<\beta>} = D^H$. Furthermore, $f(\tilde{N}^p) \in (D^{\tilde{H}})^p = D^H$. Since

$$[N, E]^p \subseteq [N^p, E][[N, E], N] \subseteq [N^p[N, E], E] = [\tilde{N}, E],$$

we obtain

$$f([N, E]/[\tilde{N}, \tilde{N}])^p \subseteq f(\tilde{N}^{ab})^{E-1} \subseteq (D^{\tilde{H}})^{G-1} \subseteq (D^{\tilde{H}})^{p^2} = (D^H)^p,$$

(since $s \geq 2$, if $p \neq 2$, and $s \geq 3$, if $p = 2$), hence $f([N, E]/[\tilde{N}, \tilde{N}]) \subseteq D^H$. It follows that

$$\mathrm{Hom}_G(\tilde{N}^{ab}, D) = \mathrm{Hom}_G(\tilde{N}/[\tilde{N}, N], D^H) = \mathrm{Hom}_G(\tilde{N}/[N, N], D).$$

As in case 1. the result follows. □

References

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[3] Wingberg, K. *Eine Bemerkung zum Einbettungsproblem mit nicht-abelschem Kern.* J. reine u. angew. Math. **331** (1982), 146-150

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