Extensions of Profinite Duality Groups

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September 25, 2008

Let G be a profinite group and let p be a prime number. By $Mod_p(G)$ we denote the category of discrete p-primary G-modules. For $A \in Mod_p(G)$ and $i \geq 0$, let

$$D_i(G,A) = \varinjlim_{U} H^i(U,A)^*,$$

where * is $\mathrm{Hom}(-,\mathbb{Q}_p/\mathbb{Z}_p)$, the direct limit is taken over all open subgroups U of G and the transition maps are the duals of the corestriction maps. $D_i(G,A)$ is a discrete G-module in a natural way. Assume that $n = cd_p\, G$ is finite. Then the G-module

$$I(G) = \varinjlim_{\nu \in \mathbb{N}} D_n(G, \mathbb{Z}/p^\nu\mathbb{Z})$$

is called the **dualizing module** of G at p. Its importance lies in the functorial isomorphism

$$H^n(G,A)^* \cong \mathrm{Hom}_G(A, I(G))$$

for all $A \in Mod_p(G)$. This isomorphism is induced by the cup-products $(V \subseteq U)$

$$H^n(G,A)^* \times {}_{p^\nu}A^U \longrightarrow H^n(V,\mathbb{Z}/p^\nu\mathbb{Z})^*,\ \ (\phi,a) \longmapsto \big(\alpha \mapsto \phi(cor\,{}_G^V(\alpha \cup a))\big)$$

by passing to the limit over ν and V, and then over U. The identity-map of $I(G)$ gives rise to the homomorphism

$$tr : H^n(G, I(G)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p\,,$$

called the **trace map**.

The profinite group G is called a **duality group at p of dimension n** if for all $i \in \mathbb{Z}$ and all finite G-modules $A \in Mod_p(G)$, the cup-product and the trace map

$$H^i(G, \mathrm{Hom}(A, I(G)) \times H^{n-i}(G, A) \xrightarrow{\cup} H^n(G, I(G)) \xrightarrow{tr} \mathbb{Q}_p/\mathbb{Z}_p$$

yield an isomorphism

$$H^i(G, \mathrm{Hom}(A, I(G))) \cong H^{n-i}(G, A)^*.$$

Remark: In [Ve], J.-L. Verdier used the name **strict Cohen-Macaulay at p** for what we call a profinite duality group at p here. In [Pl], A. Pletch defined D_p^n-groups (and called them duality groups at p of dimension n). The D_p^n-groups of Pletch are exactly the duality groups at p (in our sense) which, in addition, satisfy the following finiteness condition:

$FC(G,p)$: *$H^i(G, A)$ is finite for all finite $A \in Mod_p(G)$ and for all $i \geq 0$.*

Since any finite, discrete G-module is trivialized by an open subgroup U of G, condition $FC(G,p)$ can also be rephrased in the form:

$FC(G,p)$: *$H^i(U, \mathbb{Z}/p\mathbb{Z})$ is finite for all open subgroups U of G and all $i \geq 0$.*

By a duality theorem due to J. Tate, see [Ta] Thm. 3 or [Ve] Prop. 4.3 or [NSW] (3.4.6), a profinite group G of cohomological p-dimension n is a duality group at p if and only if

$$D_i(G, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } 0 \leq i < n.$$

As a consequence we see that every open subgroup of a duality group at p is a duality group at p as well (of the same cohomological dimension), and if an open subgroup of G is a duality group at p and $cd_p\, G < \infty$, then G is a duality group at p of the same cohomological dimension (use [NSW] (3.3.5)(ii)). Furthermore, any profinite group of cohomological p-dimension 1 is a duality group at p.

We call a profinite group G **virtually a duality group at p of (virtual) dimension $vcd_p\, G = n$** if an open subgroup U of G is a duality group at p of dimension n.

The objective of this paper is to give a proof of Theorem 1 below, which states that the class of duality groups is closed under group extensions $1 \to H \to G \to G/H \to 1$ if the kernel satisfies $FC(H,p)$. Weaker forms of Theorem 1 were first proved by A. Pletch (for D_p^n-groups, see [Pl][1]) and by the second author (for Poincaré groups, see [Wi]).

[1] The proof given by Pletch in [Pl] is only correct for pro-p-groups as the author assumes that finitely generated projective modules over the complete group ring $\mathbb{Z}_p[\![G]\!]$ are free.

Theorem 1. *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

be an exact sequence of profinite groups such that condition $FC(H,p)$ is satisfied. Then the following assertions hold:

(i) *If G is a duality group at p, then H is a duality group at p and G/H is virtually a duality group at p.*

(ii) *If H and G/H are duality groups at p, then G is a duality group at p.*

Moreover, in both cases we have:

$$cd_p\, G = cd_p\, H + vcd_p\, G/H,$$

and there is a canonical G-isomorphism

$$I(G)^\vee \cong I(H)^\vee \,\hat{\otimes}_{\mathbb{Z}_p}\, I(G/H)^\vee,$$

where $^\vee$ is the Pontryagin dual and $\hat{\otimes}_{\mathbb{Z}_p}$ is the tensor-product in the category of compact $\mathbb{Z}_p$-modules.

Remark: The assumption $FC(H,p)$ is necessary, as the following examples show:

1. Let G be the free pro-p-group on two generators x, y and let $H \subset G$ be the normal subgroup generated by x. Then H is free of infinite rank, G/H is free of rank one and $1 \to H \to G \to G/H \to 1$ is an exact sequence in which all three groups are duality groups of dimension one.

2. Let D be a duality group at p of dimension 2, F a duality group at p of dimension 1 and $G = F * D$ their free product. The kernel of the projection $G \twoheadrightarrow D$ has cohomological p-dimension 1, hence is a duality group a p of dimension 1. The group G has cohomological p-dimension 2 but is is not a duality group at p.

In the proof of Theorem 1, we make use of the following

Proposition 2. *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

be an exact sequence of profinite groups. Assume that FC(H,p) holds. Then there is a spectral sequence of homological type

$$E^2_{ij} = D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \Longrightarrow D_{i+j}(G, \mathbb{Z}/p\mathbb{Z}).$$

Proof. Let g run through the open normal subgroups of G. Then $gH/H \cong g/g\cap H$ runs through the open normal subgroups of G/H. For a G-module $A \in Mod_p(G)$, we consider the Hochschild-Serre spectral sequence

$$E(g, g\cap H, A): \ E_2^{ij}(g, g\cap H, A) = H^i(g/g\cap H, H^j(g\cap H, A)) \Longrightarrow H^{i+j}(g, A).$$

If $g' \subseteq g$ is another open normal subgroup of G, then the corestriction yields a morphism

$$cor\ : E(g', g'\cap H, A) \longrightarrow E(g, g\cap H, A)$$

of spectral sequences. The map

$$E_2^{ij}(g', g'\cap H, A) \longrightarrow E_2^{ij}(g, g\cap H, A)$$

is the composite of the maps

$$H^i\big(g'/g'\cap H, H^j(g'\cap H, A)\big) \overset{cor\,{}^{g'\cap H}_{g\cap H}}{\longrightarrow} H^i\big(g'/g'\cap H, H^j(g\cap H, A)\big)$$

$$\overset{cor\,{}^{g'/g'\cap H}_{g/g\cap H}}{\longrightarrow} H^i\big(g/g\cap H, H^j(g\cap H, A)\big)$$

and the map between the limit terms is the corestriction

$$cor\,{}^{g'}_{g} : H^{i+j}(g', A) \longrightarrow H^{i+j}(g, A).$$

For $2 \le r \le \infty$ we set

$$E_{ij}^2 = D_{ij}^r(G, H, A) := \varinjlim_{g} E_r^{ij}(g, g\cap H, A)^*.$$

As taking duals and direct limits are exact operations, the terms $D_{ij}^r(G, H, A)$, $2 \le r \le \infty$, establish a homological spectral sequence which converges to $D_n(G, A)$. If h runs through the open subgroups of H which are normal in G, then the cohomology groups $H^j(h, A)$ are G-modules in a natural way. If g is open in G with $g\cap H \subseteq h$, then these groups are $g/g\cap H$-modules. We see that

$$D_{ij}^2(G, H, A) = \varinjlim_{\substack{h\subseteq H\\ h\trianglelefteq G}} \ \varinjlim_{\substack{g\subseteq G\\ g\cap H\subseteq h}} \ H^i(g/g\cap H, H^j(h, A))^*,$$

where for both limits the transition maps are (induced by) cor^*. In order to conclude the proof of the proposition, it remains to construct isomorphisms

$$D_{ij}^2(G, H, \mathbb{Z}/p\mathbb{Z}) \cong D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z})$$

for all i and j. To this end note that all occurring abelian groups are $\mathbb{F}_p$-vector spaces, so that * is $\mathrm{Hom}(-,\mathbb{F}_p)$. Further note that for vector spaces V, W over a field k the homomorphism

$$V^* \otimes W^* \longrightarrow (V \otimes W)^*, \ \phi \otimes \psi \longmapsto \big(v \otimes w \mapsto \phi(v)\psi(w)\big)$$

is an isomorphism provided that V or W is finite-dimensional. Let h be an open subgroup of H which is normal in G and let $g' \subseteq g$ be open subgroups of G such that g acts trivially on the finite group $H^j(h,\mathbb{Z}/p\mathbb{Z})$. Then, by [NSW] (1.5.3)(iv), the diagram

$$\begin{array}{ccc} H^i(g'/g'\cap H,\mathbb{Z}/p\mathbb{Z}) \otimes H^j(h,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow[\sim]{\cup} & H^i\big(g'/g'\cap H, H^j(h,\mathbb{Z}/p\mathbb{Z})\big) \\ \big\downarrow{\scriptstyle cor\otimes id} & & \big\downarrow{\scriptstyle cor} \\ H^i(g/g\cap H,\mathbb{Z}/p\mathbb{Z}) \otimes H^j(h,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow[\sim]{\cup} & H^i\big(g/g\cap H, H^j(h,\mathbb{Z}/p\mathbb{Z})\big) \end{array}$$

commutes. For fixed h, we therefore obtain isomorphisms

$$\begin{aligned} & D_i(G/H,\mathbb{Z}/p\mathbb{Z}) \otimes H^j(h,\mathbb{Z}/p\mathbb{Z})^* \\ & \qquad \cong \big(\varinjlim_g H^i(g/g\cap H,\mathbb{Z}/p\mathbb{Z})^*\big) \otimes H^j(h,\mathbb{Z}/p\mathbb{Z})^* \\ & \qquad \cong \varinjlim_g \ H^i(g/g\cap H,\mathbb{Z}/p\mathbb{Z})^* \otimes H^j(h,\mathbb{Z}/p\mathbb{Z})^* \\ & \qquad \cong \varinjlim_g \big(H^i(g/g\cap H,\mathbb{Z}/p\mathbb{Z}) \otimes H^j(h,\mathbb{Z}/p\mathbb{Z})\big)^* \\ & \qquad \cong \varinjlim_g H^i\big(g/g\cap H, H^j(h,\mathbb{Z}/p\mathbb{Z})\big)^*. \end{aligned}$$

Passing to the limit over h, we obtain the required isomorphism

$$D_i(G/H,\mathbb{Z}/p\mathbb{Z}) \otimes D_j(H,\mathbb{Z}/p\mathbb{Z}) \cong D^2_{ij}(G,H,\mathbb{Z}/p\mathbb{Z}).$$

□

Corollary 3. *Under the assumptions of Proposition* 2*, let* i_0 *and* j_0 *be the smallest integers such that* $D_{i_0}(G/H,\mathbb{Z}/p\mathbb{Z}) \neq 0$ *and* $D_{j_0}(H,\mathbb{Z}/p\mathbb{Z}) \neq 0$*, respectively. Then* $D_{i_0+j_0}(G,\mathbb{Z}/p\mathbb{Z}) \neq 0$.

Proof. The spectral sequence constructed in Proposition 2 induces an isomorphism

$$D_{i_0+j_0}(G,\mathbb{Z}/p\mathbb{Z}) \cong D_{i_0}(G/H,\mathbb{Z}/p\mathbb{Z}) \otimes D_{j_0}(H,\mathbb{Z}/p\mathbb{Z}) \neq 0.$$

□

Proof of Theorem 1. Assume that G is a duality group at p of dimension d. Let $cd_p\, H = m$ and $n = d - m$. Then there exists an open subgroup H_1 of H such that $H^m(H_1, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let G_1 be an open subgroup of G such that $H_1 = G_1 \cap H$. Then G_1 is a duality group at p of dimension d, $cd_p\, H_1 = m$ and G_1/H_1 is an open subgroup of G/H. We consider the exact sequence

$$1 \longrightarrow H_1 \longrightarrow G_1 \longrightarrow G_1/H_1 \longrightarrow 1.$$

As $H^m(H_1, \mathbb{Z}/p\mathbb{Z})$ is finite and nonzero, we have $vcd_p\, G_1/H_1 = n$, see [NSW] (3.3.9). Furthermore, $D_i(G_1, \mathbb{Z}/p\mathbb{Z}) = 0$, $i < n + m$. Using Corollary 3, we see that $D_i(G_1/H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i < n$ and $D_j(H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $j < m$. Thus G_1/H_1, hence G/H, is virtually a duality group at p of dimension n, and H_1, and so H, is a duality group at p of dimension m. This shows (i).

Assume now that H and G/H are duality groups at p of dimension m and n. Then, $cd_p G = n + m$ by [NSW] (3.3.8), and in the spectral sequence of Proposition 2 we have $E^2_{ij} = 0$ for $(i, j) \neq (n, m)$. Hence $D_r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \neq n + m$ showing that G is a duality group at p of dimension $n + m$.

In order to prove the assertion about the dualizing modules, let h run through all open subgroups of H which are normal in G and g runs through the open subgroups of G. Since $m = cd_p\, H$, the Hochschild-Serre spectral sequence induces isomorphisms

$$H^{m+n}(g, \mathbb{Z}/p^\nu\mathbb{Z}) \cong H^n(g/g \cap H, H^m(g \cap H, \mathbb{Z}/p^\nu\mathbb{Z})),$$

and we obtain

$$\begin{aligned}
I(G) &\cong \varinjlim_{\nu} \varinjlim_{g} H^{m+n}(g, \mathbb{Z}/p^\nu\mathbb{Z})^* \\
&\cong \varinjlim_{\nu} \varinjlim_{h} \varinjlim_{g} H^n\big(g/g \cap H, H^m(h, \mathbb{Z}/p^\nu\mathbb{Z})\big)^* \\
&\cong \varinjlim_{\nu} \varinjlim_{h} \varinjlim_{g,res} H^0\big(g/g \cap H, \mathrm{Hom}\,(H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H))\big) \\
&\cong \varinjlim_{\nu} \varinjlim_{h} \mathrm{Hom}(H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H)) \\
&\cong \mathrm{Hom}_{cts}(\varprojlim_{\nu} \varprojlim_{h} H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H)) \\
&\cong \mathrm{Hom}_{cts}\big((\varinjlim_{\nu} \varinjlim_{h} H^m(h, \mathbb{Z}/p^\nu\mathbb{Z})^*)^\vee, I(G/H)\big) \\
&\cong \mathrm{Hom}_{cts}\,(I(H)^\vee, I(G/H)) \cong \big(I(H)^\vee \hat{\otimes}_{\mathbb{Z}_p} I(G/H)^\vee\big)^\vee
\end{aligned}$$

(see [NSW] (5.2.9) for the last isomorphism). This completes the proof of the theorem. □

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