Theorem. *The sets S and S^* are (left and right) Ore sets, i.e. the localisations Λ_S and Λ_{S^*} of Λ exist and the following holds:*

(i) *The category of all finitely generated S^*-torsion $\Lambda(\mathcal{G})$-modules coincides with $\mathfrak{M}_{\mathcal{H}}(\mathcal{G})$.*

(ii) *There is an long exact localisation sequence of K-groups*

$$\begin{tikzcd} & \Lambda(\mathcal{G})_{S^*} \arrow[d] & & \\ K_1(\Lambda(\mathcal{G})) \arrow[r] & K_1(\Lambda(\mathcal{G})_{S^*}) \arrow[r, "\partial"] & K_0(\mathfrak{M}_{\mathcal{H}}(\mathcal{G}) \arrow[r] & 0 \end{tikzcd}$$

and analogously for $\Lambda(\mathcal{G})_S$ and the category of finitely generated S-torsion modules.

(iii) *There is a canonical way of evaluation an element $f \in K_1(\Lambda(\mathcal{G})_{S^*})$ at any continuous representation $\rho : \mathcal{G} \to GL_n(\mathcal{O})$ with $n \geq 1$ and $\mathcal{O}$ the ring of integers of a finite extension (depending of ρ) of $\mathbb{Q}_p$:*

$$f(\rho) \in \mathbb{C}_p \cup \{\infty\},$$

i.e. f can be considered as a map on the set of such representations.

Conjecture 1 (Torsion-conjecture)**.** *The dual of the Selmer group is S^*-torsion:*

$$X(E/F_\infty) \in \mathfrak{M}_{\mathcal{H}}(\mathcal{G}).$$

$K(F_\infty)$	max. abelian ext. of $\mathbb{Q}$ inside F_∞ in which p does *not* ramify
$L = K(F_\infty)_{\mathfrak{P}}$	for some $\mathfrak{P}\vert p$, $[L : \mathbb{Q}_p] < \infty$!
$D = \mathcal{O}_L$	

Conjecture 2 (p-adic L-function)

There is a (unique) $\mathcal{L}_E \in K_1(\Lambda(G)_{S^*})$ such that

$$\mathcal{L}_E(\overset{\vee}{\varrho}) = \frac{L_R(E,\varrho,1)}{\Omega_+^{d^+(\varrho)}\Omega_-^{d^-(\varrho)}}\, e_p(\overset{\vee}{\varrho})\frac{P_p(\overset{\vee}{\varrho},u^{-1})}{P_p(\varrho,w^{-1})}u^{-\mathfrak{f}_p(\overset{\vee}{\varrho})}$$

for all Artin representations ϱ of G, where

$\Omega_\pm = \int\limits_{\gamma^\pm}\omega$, ω Neron differential

$R = \{q \text{ prime}, |j(E)|_q > 1\} \cup \{p\}$

$1 - a_pT + pT^2 = (1-uT)(1-wT)$, $u \in \mathbb{Z}_p^\times$

$p^{\mathfrak{f}_p(\varrho)}$ p-part of conductor of ϱ

$P_p(\varrho, T) = \det(1 - Frob_p^{-1}\, T | V_\varrho^{I_p})$ Euler-factor of ρ

$d^\pm(\varrho) = \dim_{\mathbb{C}} V_\varrho^\pm$

$e_p(\varrho)$ local ϵ-factor of ϱ at p

$(\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ are fixed$)$

Conjecture 2.

Conjecture 3 (Main Conjecture)**.** *The p-adic L-function $\mathcal{L}_E$ is a characteristic element of $X(E/F_\infty)$:*

$$\partial\mathcal{L}_E = [X(E/F_\infty)]_D.$$

Theorem (Manin-Višik, Katz, Yager, de Shalit, ...)

There is a unique $\mathcal{L}_{\bar{\psi}} := \mu \in D[\![G]\!]$ such that

$$\mathcal{L}_{\bar{\psi}}(\bar{\chi}) = \int_G \bar{\chi} d\mu = \frac{\Omega_p}{\Omega} e_{\mathfrak{p}}(\bar{\chi}) P_{\mathfrak{p}}(\bar{\chi}, u^{-1}) P_{\bar{\mathfrak{p}}}(\chi, u^{-1}) L(\bar{\psi}\chi, 1)$$

for all Artin-character χ of G, where

$\Omega \in \mathbb{C}^\times$	complex period s.t. $\Lambda_E = \Omega \mathcal{O}_K$
$\Omega_p \in D^\times$	p-adic period
$u = \psi(\bar{\mathfrak{p}}) = \bar{\pi}$	
$\mathfrak{p}^{\mathfrak{f}_{\mathfrak{p}}(\chi)}$	$\mathfrak{p}$-part of conductor of χ
$e_{\mathfrak{p}}(\chi)$	epsilon-factor of χ at $\mathfrak{p}$
$D = \widehat{\mathbb{Z}_p^{nr}}$	

Remark

(i) $\frac{\Omega_p}{\Omega}$ is independent of choices

(ii) We do not know whether $\frac{\mathcal{L}_{\bar{\psi}}(\bar{\chi})}{\Omega_p} \in K_{\mathfrak{p}}(\chi)$ for all χ.

Theorem (Bouganis, V.). *Assume Conjecture 1. Then*

$$\mathcal{L}_E \in K_1(\mathbb{Z}_p[[\mathcal{G}]]_S)$$

and Conjecture 2 and 3 hold.

Remark. Similar results hold for Grössencharacters ψ of type $(k, 0)$.

Lemma. *(Deligne's period conjecture, Blasius, ...)*

$$\mathcal{L}_E(\rho) \in \mathbb{Q}_p(\rho)$$

for all $\rho \in \mathrm{Irr}\mathcal{G}$.

$L/\mathbb{Q}_p$	finite, unramified
$\mathcal{O} := \mathcal{O}_L$	
L^{nr}	max. unramified ext. of L
$D := \widehat{\mathcal{O}_{L^{nr}}}$	

$\iota : K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_{S^*}) \to K_1(\Lambda_D(\mathcal{G})_{S^*})$

Theorem. *Assume that*

(i) $L(\rho)/L$ *is totally ramified (or trivial) for all* $\rho \in \mathrm{Irr}\mathcal{G}$,

(ii) $\mathcal{L} \in K_1(\Lambda_D(\mathcal{G})_{S^*})$ *is induced from an element in* $\Lambda_D(\mathcal{G}) \cap (\Lambda_D(\mathcal{G})_{S^*})^\times$

(iii) $\mathcal{L}(\rho) \in L(\rho)$ *for all* $\rho \in \mathrm{Irr}\mathcal{G}$,

(iv) *there is an* $F \in K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_{S^*})$ *such that*

$$\partial(\mathcal{L} \cdot \iota(F)^{-1}) = 0$$

(e.g. if $\mathcal{L}$ *is the characteristic element of the base change from a module in* $\mathfrak{M}_{\mathcal{O},\mathcal{H}}(\mathcal{G})$*).*

Then there exists $\mathcal{L}' \in K_1(\Lambda_{\mathcal{O}}(\mathcal{G})_{S^*})$ *with*

$$\mathcal{L}'(\rho) = \mathcal{L}(\rho) \text{ for all } \rho \in \mathrm{Irr}\mathcal{G}$$

and

$$\partial(\mathcal{L}') = \partial(F),$$

in particular

$$\iota\partial(\mathcal{L}') = \partial(\mathcal{L}).$$

The proof needs a generalisation of a result of M. Taylor:

Theorem. *(Izychev, Snaith, V.)*

(i) $\mathrm{Det}(K_1(\Lambda_D(\mathcal{G})))^{\mathrm{Frob}_p=1} = \mathrm{Det}(K_1(\Lambda_{\mathcal{O}}(\mathcal{G})))$,

(ii) $K_1(\Lambda_D(\mathcal{G}))^{\mathrm{Frob}_p=1} = K_1(\Lambda_{\mathcal{O}}(\mathcal{G}))$.

work in progress!