LOCALISATIONS AND COMPLETIONS OF SKEW POWER SERIES RINGS

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Abstract. This paper is a natural continuation of the study of skew power series rings $A = R[[t;\sigma,\delta]]$ initiated in [11]. We construct skew Laurent series rings B and show the existence of some canonical Ore sets S for the skew power series rings A such that a certain completion of the localisation A_S is isomorphic to B. This is applied to certain Iwasawa algebras. Finally we introduce subrings of overconvergent skew Laurent series rings.

Introduction

In [11] we introduced the general notion of a skew power series ring $A = R[[t;\sigma,\delta]]$ over a pseudocompact ring R (see section 1 for details) and we studied basic properties of it. This was motivated by and applied to the study of Iwasawa algebras of certain p-adic Lie groups. It is a very important step in non-commutative Iwasawa theory to pass from the Iwasawa algebra to its localisation in a specific rather big Ore subset (cf. [4]). The starting point of the present paper is the discovery that a natural completion of this localisation can be viewed as a skew Laurent series ring. This means in particular that this completed localisation can be constructed by inverting a single element t with subsequent completion.

In fact we will develop our results in the context of a general skew power series ring $A = R[[t;\sigma,\delta]]$. In section 1 we want to formally invert the element t, i.e. construct a skew Laurent series ring

$$B = R((t;\sigma,\delta)).$$

It turns out that for a (left of right) Artinian ring R the ring B exists in form of the localisation A_T of A with respect to $T = \{1, t, t^2, \ldots\}$, see subsection 1.1. But for general R and non-trivial δ one sees easily that a ring extension of A in which t is invertible in general contains series $\sum_{i\in\mathbb{Z}} r_i t^i$ with *infinite* negative part. Due to this observation we define in subsection 1.2 a, in both directions, infinite skew Laurent series ring

$$B = R\ll t;\sigma,\delta]]$$

consisting of formal infinite sums $\sum_{i\in\mathbb{Z}} r_i t^i$ such that the negative part satisfies some growth condition; in fact the commutative version, i.e. for trivial σ and δ, with for example $R = \mathbb{Z}_p$ is a well-known ring in p-adic analysis. The multiplication law is given by explicit formulae (1.9). In order to see that they actually define a (topological) ring structure we identify B

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with the projective limit of skew Laurent series rings with Artinian coefficients

$$B \cong \varprojlim_k (R/I_k)((t;\overline{\sigma},\overline{\delta}))$$

for a certain filtration $I_\bullet$ of R which is needed to grant the existence of B. In Proposition 1.16 we prove a criterion under which the ring B is Noetherian and flat as a left and right A-module.

Then we generalise and axiomatise the localisation technique from [4] so that it applies in our context. Assuming that $\delta(R)$ is contained in the Jacobson radical $\mathrm{Jac}(R)$ of R we consider the canonical projection

$$\pi : A = R[[t;\sigma,\delta]] \twoheadrightarrow (R/\mathrm{Jac}(R))[[t;\overline{\sigma}]].$$

Theorem 2.25, the main result of section 2, states that $S = \mathcal{C}_A(\ker(\pi))$ is an Ore set of A consisting of regular elements. In particular, the localisation A_S of A with respect to S exists. The proof is again reduced to the Artinian case which is dealt with in subsection 2.2 after discussing in subsection 2.1 a general method how to attach Ore sets to a ring homomorphism $R \to A$ of arbitrary rings R and A, which might be of its own interest.

As alluded to above the remarkable fact is that the two ring extensions B and A_S of A are in fact closely related: The filtration $I_\bullet$ induces a filtration on A_S such that the completion of A_S with respect to it is isomorphic to B, see Proposition 2.27, i.e. after completion it is in some sense sufficient to just invert $t \in S$ instead of the much bigger set S.

In section 3 we investigate the G- and K-theory of the localisation A_S in degrees 0 and 1 assuming that A is Noetherian, that $\delta(R)$ is contained in $\mathrm{Jac}(R)$, and that σ is induced by inner conjugation with a unit in A. In particular we show - following an idea of David Burns in the case of Iwasawa algebras - that the connecting homomorphism from degree 1 to 0 of the long exact localisation sequence splits, for details see Proposition 3.1 and Corollary 3.2.

In section 4 we apply our results to Iwasawa algebras. In this context the existence of the localisation A_S, of course, is known from [4], see also [2]. The main result is the existence of the skew Laurent series ring B in this setting and the fact that it is a pseudocompact Noetherian ring, flat over A and A_S, see Theorem 4.7.

Finally in section 5 we discuss different convergence conditions for our skew Laurent series which leads to the definition of *overconvergent* skew Laurent series rings generalising again a concept from p-adic Hodge theory as studied by Cherbonnier and Colmez. We expect that our constructions and results will have important applications in non-commutative Iwasawa theory, but also in the theory of p-adic representations (see the forthcoming work of the first author with M-.F. Vigneras).

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1. Infinite skew Laurent series rings

Let R be a Noetherian pseudocompact ring together with the following data:

(i) a topological ring automorphism σ of R,
(ii) a continuous left σ-derivation $\delta : R \to R$ which is σ-nilpotent in the sense of [11, §1].

In particular there exists the pseudocompact skew power series ring

$$A := R[[t;\sigma,\delta]],$$

see [11, §1], where the ring structure is given by the following relation

$$tr = \sigma(r)t + \delta(r), \quad (\text{ or by } rt = t\sigma'(r) + \delta'(r)) \tag{1.1}$$

and continuity. Here we set $\sigma' := \sigma^{-1}$ and $\delta' := -\delta \circ \sigma^{-1}$, i.e. δ' is a right σ'-derivation.

The aim of this section is to introduce also a skew Laurent series ring version which contains a formal inverse t^{-1} of the element t. For non-trivial δ this involves in general series $\sum_{i\in\mathbb{Z}} r_i t^i$ with infinitely many nonzero coefficients r_i for negative i, as can be seen easily from the formulae (1.6) below.

But if δ and δ' are nilpotent those formulae are actually finite and we will show in the following subsection that then the skew Laurent series ring with *finite negative part* exists. Afterwards we shall construct for rather general δ a skew Laurent series ring with *infinite negative part* but which satisfies a certain convergence condition.

1.1. **Skew Laurent series rings in the nilpotent case.** In this subsection we assume in addition that both δ and δ' are nilpotent, say of degree M, i.e.

$$\delta^n \equiv 0, \quad \delta'^n \equiv 0 \text{ for all } n \geq M.$$

Note that for a (left or right) Artinian ring R the nilpotence of δ and δ' follows already from σ-nilpotence of δ.

For any integers $k, l \geq 0$, $M_{k,l}(Y,Z)$ denotes the sum of all noncommutative monomials in two variables Y, Z with k factors Y and l factors Z. Consider the multiplicatively closed subset

$$T := \{1, t, t^2, t^3, \cdots\}$$

of A consisting of all powers of t.

Lemma 1.1. *Suppose that $\delta^M = 0$ for some $M \geq 0$. Then:*

(i) *For all $r \in R$, we have*

$$t^M \cdot r = \left(\sum_{k=0}^{M-1} t^{M-1-k}\sigma\delta^k(r)\right)\cdot t.$$

(ii) *For all $a \in A$ there exists $b \in A$ such that $t^M a = bt$.*

(iii) *T is a left Ore set.*

Proof. (i) This is proved by a simple telescope sum argument using the second relation in (1.1).

(ii) Writing $a \in A$ in the form $a = \sum_{i=0}^{\infty} t^i r_i$ we have $t^M a = bt$ by (i) where

$$b = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{M-1} \sigma\delta^k(r_{n+k+1-M}),$$

provided we interpret $r_l = 0$ for negative indices l; observe that since the multiplication in the pseudocompact ring A is continuous the multiplication from the left by t^M commutes with the summation in a.

(iii) Assume inductively that for all $a \in A$ and all integers $n \geq 1$ there exists $b \in A$ such that

$$t^{Mn} a = bt^n,$$

the case $n = 1$ being part (ii). By part (ii) again, $t^M b = ct$ for some $c \in A$, so

$$t^{M(n+1)}a = t^M bt^m = ct^{n+1}.$$

$\square$

Proposition 1.2. *The set T satisfies the (right and left) Ore condition, i.e. the localisation A_T of A at T exists. Moreover, for all $r \in R$ and all $j \geq 1$ we have*

$$t^j rt^{-j} = \sum_{k=0}^{\infty} t^{-k} \sigma M_{k,j-1}(\delta, \sigma)(r) \tag{1.2}$$

inside A_T.

Proof. By the previous lemma T satisfies the left Ore condition. The right Ore condition follows from an analogous argument, and thus the localisation exists by [9, thm. 2.1.12] and the fact that all elements in T are regular. For the asserted identity we first note that, for $k \gg 0$, any monomial containing k δ's and $j-1$ σ's must contain a string of at least M δ's, and is therefore zero. In fact the sum on the right hand side stops at $k = j(M-1)$ and therefore makes sense inside A_T. Now the case $j = 1$ follows immediately from part (i) of the above lemma, and the general case follows by induction as follows:

$$\begin{aligned} t^{j+1}rt^{-j-1} &= t\Big(\sum_{l=0}^{\infty} t^{-l}\sigma M_{l,j-1}(\delta,\sigma)(r)\Big)t^{-1} \\ &= \sum_{l=0}^{\infty} t^{-l} \sum_{k=0}^{\infty} t^{-k}\sigma\delta^k\sigma M_{l,j-1}(\delta,\sigma)(r) \\ &= \sum_{n=0}^{\infty} t^{-n}\sigma \sum_{k+l=n} \delta^k \sigma M_{l,j-1}(\delta,\sigma)(r) \\ &= \sum_{n=0}^{\infty} t^{-n}\sigma M_{n,j}(\delta,\sigma)(r) \end{aligned}$$

where we used the case $j = 1$ for the second equation and the relation

$$\sum_{k=0}^{n} Y^k Z M_{n-k,j-1}(Y,Z) = M_{n,j}(Y,Z)$$

for the last equation. $\square$

Clearly all elements in the *skew Laurent series ring*

$$B := R((t;\sigma,\delta)) := A_T$$

can be written as series $\sum_{i \gg -\infty} r_i t^i$ and $\sum_{i \gg -\infty} t^i r_i$ with *finite* negative part $\sum_{-\infty < i < 0} r_i t^i$ and $\sum_{-\infty < i < 0} t^i r_i$, respectively.

Remark 1.3. The same argument shows that T is also an Ore set in the skew polynomial ring $R[t;\sigma,\delta]$ and thus also the *skew Laurent polynomial* ring

$$R(t;\sigma,\delta) := R[t;\sigma,\delta]_T$$

exists under the hypothesis of this subsection.

It follows from the identity (1.2) that in the ring $R((t;\sigma,\delta))$ the relations

$$rt^j = \sum_{m\le j} t^m \sigma M_{j-m,-1-j}(\delta,\sigma)(r) \tag{1.3}$$

and

$$t^j r = \sum_{m\le j} \sigma' M_{j-m,-1-j}(\delta',\sigma')(r) t^m \tag{1.4}$$

hold for $j<0$. In particular, we have

$$rt^{-1} = \sum_{i\le -1} t^i \sigma \delta^{-i-1}(r), \tag{1.5}$$

and

$$t^{-1} r = \sum_{i\le -1} \sigma' \delta'^{-i-1}(r) t^i. \tag{1.6}$$

For $j\ge 0$ we recall the formulae

$$t^j r = \sum_{0\le m\le j} M_{j-m,m}(\delta,\sigma)(r) t^m \tag{1.7}$$

and

$$rt^j = \sum_{0\le m\le j} t^m M_{j-m,m}(\delta',\sigma')(r). \tag{1.8}$$

Finally, the multiplication in $R((t;\sigma,\delta))$ and in $R(t;\sigma,\delta)$ is explicitly given by the following formula

$$(\sum_{j\in\mathbb{Z}} a_j t^j)(\sum_{l\in\mathbb{Z}} b_l t^l) = \sum_{m\in\mathbb{Z}} c_m t^m \quad \text{with} \tag{1.9}$$

$$\begin{aligned}
c_m &:= c_m^+ + c_m^-, && (1.10)\\
c_m^+ &:= \sum_{j\ge n\ge 0} a_j M_{j-n,n}(\delta,\sigma)(b_{m-n}) \quad \text{and} && (1.11)\\
c_m^- &:= \sum_{n\le j<0} a_j \sigma' M_{j-n,-1-j}(\delta',\sigma')(b_{m-n}). && (1.12)
\end{aligned}$$

An analogous formula holds for *right* Laurent series:

$$(\sum_{j\in\mathbb{Z}} t^j a_j)(\sum_{l\in\mathbb{Z}} t^l b_l) = \sum_{m\in\mathbb{Z}} t^m d_m \tag{1.13}$$

with

$$\begin{aligned}
d_m &:= d_m^+ + d_m^-, && (1.14)\\
d_m^+ &:= \sum_{j\ge n\ge 0} M_{j-n,n}(\delta',\sigma')(a_{m-n}) b_j \quad \text{and} && (1.15)\\
d_m^- &:= \sum_{n\le j<0} \sigma M_{j-n,-1-j}(\delta,\sigma)(a_{m-n}) b_j. && (1.16)
\end{aligned}$$

For the notion of the Krull-dimension κ we refer the reader to [9, chap. 6]. If we consider a ring S as left or right module over itself we write ${}_SS$ or S_S, respectively. For a left (right) module M we denote by $\mathcal{L}(M)$ the lattice of all left (right) submodules of M.

Lemma 1.4. *Suppose that $\delta = 0$. We then have $\kappa(B_B) \leq \kappa(R_R)$ and $\kappa({}_BB) \leq \kappa({}_RR)$. In particular, if R is a (left or right) Artinian ring, so is B.*

Proof. The 'left'-version following by symmetry we only consider the 'right'-version. For the purposes of this proof we call $i(b) := \min\{i : a_i \neq 0\}$ the order of any element $b = \sum_{i \gg -\infty} a_i t^i$ in B and $l(b) := a_{i(b)}$ its leading coefficient (with $l(0) := 0$). If J is a right ideal of B we write $l(J)$ for the set of leading coefficients of elements in J. Using that $\delta = 0$ it is easy to check that $l(J)$ is a right ideal of R, thus we obtain a map of partially ordered sets $l : \mathcal{L}(B_B) \to \mathcal{L}(R_R)$. We just have to show that this map l preserves 'proper containment'. Thus let $J_1 \subsetneqq J_2$ be right ideals of B. Then there exists an element $b \in J_2 \cap A$ which is not contained in J_1. We shall derive a contradiction assuming that $l(J_1) = l(J_2)$: then we find $j_1 \in J_1$ with $l(j_1) = l(b)$ and, after possibly multiplying from the right by a suitable power of t, with $i(j_1) = i(b)$; in particular j_1 belongs to $J_1 \cap A$. Similarly we find $j_2 \in J_1 \cap A$ such that $l(j_2) = l(b - j_1)$ and $i(j_2) = i(b - j_1)$ and inductively a sequence of elements $j_n \in J_1 \cap A$ with strictly increasing order and such that the series $\sum_{n\geq 1} j_n$ converges to b, which thus is an element of the closed ideal $J_1 \cap A$ (A is pseudocompact and Noetherian), a contradiction. □

Let I be a two-sided ideal of R which is σ-, σ'- and δ-stable. We define the left R-submodule I_B of B to consist of all $b = \sum b_i t^i \in B$ with $b_i \in I$ for all $i \in \mathbb{Z}$.

Lemma 1.5. *Let I, J be σ-, σ'- and δ-stable two-sided ideals of R. Then*

(i) *$I_B = IB = BI$ is a two-sided ideal of B.*
(ii) *$J_B \cdot I_B = (JI)_B$.*

Proof. (i) Since I is finitely generated as a right ideal in R we have $I_B = IB$. The formula (1.9) implies that $BI \subseteq I_B = IB$. By symmetry we must even have $BI = IB$. (ii) follows immediately from (i). □

In [11, §1] we constructed a descending exhaustive ring filtration I_k, $k \geq 0$, by two-sided ideals in R which are σ- and σ'-stable and satisfy $\delta(I_k) \subseteq I_{k+1}$, which we refer to as the *standard* filtration. By Lemma 1.5 the filtration $I_\bullet$ of R induces a ring filtration $J_\bullet$ of B by setting $J_k := (I_k)_B$ for all $k \geq 0$.

Suppose that R is (left or right) Artinian. Then the filtration $I_\bullet$ and thus also $J_\bullet$ stabilises. Furthermore, as R is Noetherian, the J_k are finitely generated (left and right) B-modules by Lemma 1.5. Hence the subquotients J_k/J_{k+1} are finitely generated modules over $B/J_1 = R/I_1 \otimes_R B \cong (R/I_1)((t;\bar{\sigma}))$ and thus have finite length by Lemma 1.4. We have shown the following

Proposition 1.6. *Suppose that the standard filtration is separated, i.e. $I_k = 0$ for $k \gg 0$. Then, if R is (left or right) Artinian, so is B.*

Remark 1.7. Let R be (left or right) Artinian. Note that the standard filtration is separated if and only if there exist *any* separated descending ring filtration I_k, $k \geq 0$, by two-sided ideals in R which are σ- and σ'-stable and satisfy $\delta(R) \subseteq I_1$.

Proof. Assume that such a filtration $I_\bullet$ is given and let k be any natural number. For $l \geq 1$ the lth ideal of the standard filtration is generated by all subgroups

$$M_{m_1}(R) \cdot \ldots \cdot M_{m_r}(R) \tag{1.17}$$

where $m_1 + \ldots + m_r = l$ is any partition of l with $m_j > 0$, $1 \leq j \leq r$, and M_{m_j} is a non-commutative monomial in δ, σ and σ' with at least m_j factors δ. Since R is Artinian and due to the σ-nilpotence of δ we find a positive number m such that $M(R) = 0$ for all such monomials M with at least m factors δ. Choosing $l > km$ we see that the lth step of the standard filtration is contained in I_k because the only non-zero contributions of the form (1.17) have at least k factors, which all belong to I_1 by assumption. $\square$

1.2. **Skew Laurent series rings with infinite negative part.** The following definition leads to a reasonable, in both directions infinite, skew Laurent series ring: Let

$$B := R\ll t; \sigma, \delta]]$$

consist of all formal infinite sums $\sum_{i\in\mathbb{Z}} r_i t^i$ such that r_j tends to zero in the pseudocompact topology of R for $j \leq 0$ running to $-\infty$.

B is naturally endowed with the exhaustive and separated descending filtration $(F^kB)_{k\geq 0}$ of left R-modules defined as

$$F^kB := \big(\prod_{i\in\mathbb{Z}} \mathrm{Jac}(R)^k t^i\big) \cap B,$$

where $\mathrm{Jac}(R)$ denotes the Jacobson radical of R. The topology induced by this filtration will be called the *strong* topology: There is another interesting topology on B but which will not be used in this paper. This *weak* topology is given by the system of open zero neighbourhoods $\{F^kB + At^m\}_{k,m\geq 0}$.

Remark 1.8. B is a complete R-module with respect to the strong topology.

Note that $\sigma(\mathrm{Jac}(R)) = \mathrm{Jac}(R)$, in particular the $\mathrm{Jac}(R)$-adic filtration is σ- and σ'-stable. In general we are not assuming that $\mathrm{Jac}(R)$ is also stable under δ, but the continuity of δ implies that there is a natural number $s \geq 1$ such that

$$\delta(\mathrm{Jac}(R)^s),\ \ \delta'(\mathrm{Jac}(R)^s) \subseteq \mathrm{Jac}(R). \tag{1.18}$$

By induction one shows immediately that

$$\delta(\mathrm{Jac}(R)^{sk}),\ \ \delta'(\mathrm{Jac}(R)^{sk}) \subseteq \mathrm{Jac}(R)^k \tag{1.19}$$

for all $k \geq 0$. If $M(Y,Z)$ denotes a noncommutative monomial with m factors Y and arbitrarily, but finitely many factors Z, we obtain

$$M(\delta,\sigma)(\mathrm{Jac}(R)^{s^m k}),\ \ M(\delta',\sigma')(\mathrm{Jac}(R)^{s^m k}) \subseteq \mathrm{Jac}(R)^k \tag{1.20}$$

for all $k \geq 0$.

Let I_n, $n \geq 0$, be a σ-, σ'- and δ-stable separated exhaustive descending filtration of R, in particular satisfying $I_k \cdot I_l \subseteq I_{k+l}$, consisting of (closed two-sided) ideals. We define the exhaustive and separated filtration

$$J_k := \big(\prod_{i\in\mathbb{Z}} I_k t^i\big) \cap B$$

of B consisting of strongly closed left R-submodules.

Assumption (I): There exists a filtration $(I_k)_k$ as above where the ideals I_k are all *open* in R.

A slightly stronger version is the following

Assumption ($\mathbf{SI_0}$): There exists a filtration $(I_k)_k$ as above where the ideals I_k are all *open* in R and such that $\delta(R) \subseteq I_1$.

Later we shall also consider the strong version in which the analogous condition holds in all degrees.

Assumption (SI): There exists a filtration $(I_k)_k$ as above where the ideals I_k are all *open* in R and such that $\delta(I_k) \subseteq I_{k+1}$ for all $k \geq 0$.

Remark 1.9. If $\delta(\mathrm{Jac}(R)) \subseteq \mathrm{Jac}(R)$ holds, then (I) is satisfied (with $I_\bullet$ the $\mathrm{Jac}(R)$-adic filtration). If, in addition, $\delta(R) \subseteq \mathrm{Jac}(R)$ or even both $\delta(R) \subseteq \mathrm{Jac}(R)$ and $\delta(\mathrm{Jac}(R)) \subseteq \mathrm{Jac}(R)^2$ hold, then ($\mathrm{SI_0}$) and (SI) are satisfied, respectively.

If we assume (I), then the filtrations J_k and F^kB are compatible by [5, IV.3, prop. 11], in particular also the filtration $(J_k)_k$ induces the strong topology on B and we obtain an isomorphism

$$B \cong \varprojlim_k B/J_k \tag{1.21}$$

of topological R-modules. On the other hand we then have an isomorphism of left R-modules

$$B/J_k \cong (R/I_k)((t;\overline{\sigma},\overline{\delta})), \tag{1.22}$$

where $\overline{\sigma}$, $\overline{\sigma'}$, $\overline{\delta}$ and $\overline{\delta'}$ denote the induced maps on R/I_k. Note that since I_k is open in R the σ-nilpotence of δ implies that both $\overline{\delta}$ and $\overline{\delta'}$ are nilpotent and thus the latter ring exists by proposition 1.2.

Below we shall show that the formula (1.9) defines an obviously distributive multiplication law on B, which, for every k, induces by construction the ring structure of $\big(R/I_k\big)((t;\overline{\sigma},\overline{\delta}))$ and thus coincides with the ring multiplication of the projective limit ring. In particular, the multiplication law on B is also associative.

Proposition 1.10. *If* (I) *is satisfied, then the formula* (1.9) *defines a topological ring structure on B with respect to the strong topology. Moreover, if* ($\mathrm{SI_0}$) *holds, then B is a pseudo-compact ring.*

Proof. We will first show that (1.9) (actually without even assuming (I)) gives a well-defined map $B \times B \to B$. To this end we check for the positive and negative parts c_m^+ and c_m^- separately, that the defining sums in (1.11) and (1.12) converge, independently of the order of summation, and that for m tending to $-\infty$ the $c_m^\pm$ converge to zero: Let k be any given positive number and m be any fixed integer. Then by the σ-nilpotence of δ there is a constant $N_1 \gg 0$ such that $M_{j-n,n}(\delta,\sigma)(R)$ (and for later purposes $M_{j-n,-1-j}(\delta',\sigma')(R)$) and thus also $a_j M_{j-n,n}(\delta,\sigma)(b_{m-n})$ lies in $\mathrm{Jac}(R)^k$ for all $j-n \geq N_1$. On the other hand, by the definition of B, there exists a constant $N_2 \gg 0$ such that $b_{m-n} \in \mathrm{Jac}(R)^{s^{N_1}k}$ for all $m-n \leq -N_2$. Thus it follows from (1.20) that the summand $a_j M_{j-n,n}(\delta,\sigma)(b_{m-n}) \in \mathrm{Jac}(R)^k$ whenever $n \geq N_2+m$ or $j-n \geq N_1$. Hence all but possibly the finitely many summands of (1.11) for $n \leq j < N_1+N_2+m$ and $0 \leq n < N_2+m$ lie in $\mathrm{Jac}(R)^k$. This implies the convergence of the positive part. Now we will show that c_m^+ belongs to $\mathrm{Jac}(R)^k$ if m is small enough: We have already seen that all summands outside this finite set of exceptions lie in $\mathrm{Jac}(R)^k$. Now we assume $m \leq -N_2$. Then $n \geq N_2+m$ for any $n \geq 0$ and so the exceptional set is empty.

For the negative part we assume m again to be fixed but arbitrary. By the definition of B there exists a constant $N_0 \gg 0$ such that a_j and thus the summand $a_j\sigma' M_{j-n,-1-j}(\delta',\sigma')(b_{m-n})$ lies in $\mathrm{Jac}(R)^k$ for all $j \leq -N_0$. On the other hand we have $M_{j-n,-1-j}(\delta',\sigma')(b_{m-n}) \in \mathrm{Jac}(R)^k$ for all $j-n \geq N_1$, see above. Thus apart from possibly the indices $-N_0-N_1 < n < 0$ all summands belong to $\mathrm{Jac}(R)^k$ which implies convergence of the negative part. Moreover, assuming $m \leq -N_0-N_1-N_2$ we have - as for the positive part - $b_{m-n} \in \mathrm{Jac}(R)^{s^{N_1}k}$ whence $M_{j-n,-1-j}(\delta',\sigma')(b_{m-n}) \in \mathrm{Jac}(R)^k$ for all $n > -N_0-N_1$. It follows that $c_m^- \in \mathrm{Jac}(R)^k$.

Finally, we show that the multiplication is continuous with respect to the strong topology. As the addition is continuous and the multiplication is distributive it suffices to check this in a neighbourhood of 0. But from the formula (1.9) one sees that $F^kB \cdot B \subseteq F^kB$ for any given $k \geq 0$. By (1.22), (1.21), Remark (1.7) and Proposition 1.6 the ring B is pseudocompact provided (SI_0) holds. $\square$

Remark 1.11. Assume (I) (respectively (SI_0)). Then, a posteriori the isomorphism

$$B \cong \varprojlim_k \big(R/I_k\big)((t;\overline{\sigma},\overline{\delta})) \tag{1.23}$$

induced by (1.21) and (1.22) is an isomorphism of *topological (pseudocompact) rings* if the rings $\big(R/I_k\big)((t;\overline{\sigma},\overline{\delta}))$ are endowed with the discrete topology. Moreover, the J_k are two-sided ideals of B because they are kernels of the natural ring homomorphisms $B \to \big(R/I_k\big)((t;\overline{\sigma},\overline{\delta}))$.

Henceforth we assume that (I) holds. Let I be a two-sided ideal of R which is σ-, σ'- and δ-stable. For $C = A$ or B we define as before the left R-submodule I_C of C to consist of all $c = \sum c_i t^i \in C$ with $c_i \in I$ for all $i \geq 0$, resp. $i \in \mathbb{Z}$.

Lemma 1.12. *Let I, J be σ-, σ'- and δ-stable two-sided ideals of R and let $C = A$ or B. Then*

(i) *$I_C = IC = CI$ is a two-sided ideal of C.*
(ii) *$J_C \cdot I_C = (JI)_C$.*

In particular, B is a filtered ring with respect to $J_\bullet$.

Proof. For $C = A$ the proof is identical with the one of Lemma 1.5. In case $C = B$ we have to modify the former argument. Fix generators $u_1,\ldots,u_m$ of I as a right ideal in R. All homomorphisms of pseudocompact R-modules are strict. Hence the quotient topology from $R^m \twoheadrightarrow I$ coincides with the subspace topology from $I \subseteq R$. We therefore find a strictly increasing sequence of natural numbers $j(1) < j(2) < \ldots$ such that

$$\sum_{i=1}^m u_i \mathrm{Jac}(R)^k \supseteq I \cap \mathrm{Jac}(R)^{j(k)} \qquad \text{for any } k \geq 1.$$

This implies that $I_B = IB$: for $b = \sum b_i t^i \in I_B$ we can write $b_i = \sum_{j=1}^m u_j c_i^{(j)}$, $i \in \mathbb{Z}$, such that $c^{(j)} := \sum_{i\in\mathbb{Z}} c_i^{(j)} t^i$ belongs to B, whence $b = \sum_{j=1}^m u_j c^{(j)} \in IB$; the other inclusion is obvious. The rest of the proof is exactly the same as the one of Lemma 1.5. $\square$

Remark 1.13. Using the construction in [8, chap. IV§1] it is not difficult to show that the filtered ring $(B, J_\bullet)$ is the algebraic microlocalisation of the filtered ring $(A, J_\bullet \cap A)$ in the multiplicative subset $\{1, t, t^2, \ldots\}$.

Lemma 1.14. *We have*

$$\mathrm{gr}_{J_\bullet} B \cong \mathrm{gr}_{I_\bullet} R \otimes_{R/I_1} (R/I_1)((t;\overline{\sigma},\overline{\delta})) \cong (R/I_1)((t;\overline{\sigma},\overline{\delta})) \otimes_{R/I_1} \mathrm{gr}_{I_\bullet} R$$

where the ring multiplication on the right hand side is given by formulae (1.9) *and* (1.13), *respectively, if we view the elements in the right side as Laurent series in the variable t over the ring* $\mathrm{gr}_{I_\bullet} R$.

Proof. Under the isomorphism (1.22) the ideal J_{k-1}/J_k of B/J_k corresponds to the ideal $(I_{k-1}/I_k)_{(R/I_k)((t;\overline{\sigma},\overline{\delta}))}$ which is isomorphic to $(I_{k-1}/I_k) \otimes_{R/I_1} (R/I_1)((t;\overline{\sigma},\overline{\delta}))$ as I_{k-1}/I_k is finitely generated over R/I_1. The result follows. □

Lemma 1.15. *B is faithfully flat as a left or right R-module.*

Proof. By symmetry it suffices to consider the left module case. We have to show that, for any proper right ideal $Q \subset R$, the natural map $Q \otimes_R B \longrightarrow B$ is injective but not surjective. The non-surjectivity is clear. To establish the injectivity we fix generators $u_1, \ldots, u_m$ of the right ideal Q. Then any element $x \in Q \otimes_R B$ can be written as

$$x = \sum_{j=1}^{m} u_j \otimes b^{(j)} \qquad \text{with } \ b^{(j)} = \sum_{i\in\mathbb{Z}} b_i^{(j)} t^i \in B.$$

We suppose now that the image $\sum_j u_j b^{(j)}$ of x under the above map is zero. Then the tuple $(b_i^{(1)}, \ldots, b_i^{(m)})$, for any $i \in \mathbb{Z}$, lies in the right submodule $N := \{(a^{(1)}, \ldots, a^{(m)}) \in R^m : \sum_{j=1}^m u_j a^{(j)} = 0\}$ of R^m. Since R is Noetherian N has finitely many generators $\alpha_1 = (a_1^{(1)}, \ldots, a_1^{(m)}), \ldots, \alpha_s = (a_s^{(1)}, \ldots, a_s^{(m)})$. Write

$$(b_i^{(1)}, \ldots, b_i^{(m)}) = \alpha_1 c_i^{(1)} + \ldots + \alpha_s c_i^{(s)} \qquad \text{for any } \ i \in \mathbb{Z}$$

with $c_i^{(1)}, \ldots, c_i^{(s)} \in R$. In fact, since by the pseudocompactness of R (cf. the comment in the proof of Lemma 1.12) we have a strictly increasing sequence $0 < j(1) < j(2) < \ldots$ such that

$$\sum_{k=1}^{s} \alpha_k \mathrm{Jac}(R)^\ell \supseteq N \cap (\mathrm{Jac}(R)^{j(\ell)} R^m) \qquad \text{for any } \ \ell \geq 1$$

(cf. [15, cor. 3.8]) we may choose the $c_i^{(k)}$ in such a way that $c^{(k)} := \sum_{i\in\mathbb{Z}} c_i^{(k)} t^i$ lies in B for each $1 \leq k \leq s$. Then $b^{(j)} = \sum_{k=1}^s a_k^{(j)} c^{(k)}$ in B and hence

$$x = \sum_{j=1}^{m} u_j \otimes b^{(j)} = \sum_{j=1}^{m} u_j \otimes \sum_{k=1}^{s} a_k^{(j)} c^{(k)} = \sum_{k=1}^{s} (\sum_{j=1}^{m} u_j a_k^{(j)}) \otimes c^{(k)} = 0.$$

□

Proposition 1.16. *Assume that* (SI) *holds, that σ induces the identity map on $\mathrm{gr}_{I_\bullet} R$, and that $\mathrm{gr}_{I_\bullet} R$ is an almost normalising extension of its subring R/I_1 in the sense of* [9, 1.6.10]. *Then $B = R\ll t; \sigma, \delta]]$ is Noetherian and is flat as a left and a right A-module.*

Proof. By (SI) we have $\overline{\delta} = 0$. Using Lemma 1.14 we see that $\mathrm{gr}_{J_\bullet} B \cong \mathrm{gr}_{I_\bullet} R \otimes_{R/I_1} (R/I_1)((t))$ is a subring of the Laurent series ring $(\mathrm{gr}_{I_\bullet} R)((t))$ and is an almost normalising extension of $(R/I_1)((t))$. The latter is well-known to be Noetherian. Hence the former is Noetherian as well by [9, thm. 1.6.14]. The first assertion now follows from [8, prop. II.1.2.3]. Similarly we have $\mathrm{gr}_{J_\bullet \cap A} A \cong \mathrm{gr}_{I_\bullet} R \otimes_{R/I_1} (R/I_1)[[t]]$ which is a Noetherian ring as well. Moreover it follows that $\mathrm{gr}_{J_\bullet} B$ is the localisation of $\mathrm{gr}_{J_\bullet \cap A} A$ in t which therefore is a flat ring extension. The flatness of B over A now follows from [8, prop. II.1.2.1 and prop. II.1.2.3]. □

2. Another canonical Ore set

Keeping the notations and assumptions of the previous section we assume in addition throughout this section that

$$\delta(R) \subseteq \mathrm{Jac}(R).$$

Then, by Remark 1.9, the assumption (SI_0) is satisfied for the $\mathrm{Jac}(R)$-adic filtration.

Note that $\mathrm{Jac}(R)_A$ is the kernel of the canonical projection

$$A = R[[t; \sigma, \delta]] \twoheadrightarrow (R/\mathrm{Jac}(R))[[t; \overline{\sigma}]].$$

The aim of this section is to show that

$$\mathcal{C}_A(\mathrm{Jac}(R)_A) \text{ is an Ore set of } A \text{ consisting of regular elements,} \tag{2.1}$$

i.e. that the localisation $A_{\mathcal{C}_A(\mathrm{Jac}(R)_A)}$ of A with respect to $\mathcal{C}_A(\mathrm{Jac}(R)_A)$ exists. In the first subsection we shall establish a general method how to attach to a ring homomorphism $R \to A$ of arbitrary rings R and A a left (respectively right) Ore set S_l (S_r). In the second subsection we return to our above setting under the additional hypothesis that R is Artinian (and later semisimple and even simple): we show statement (2.1) and that $\mathcal{C}_A(\mathrm{Jac}(R)_A)$ equals S_l and S_r. In the third and last subsection we use the results of the previous ones to lift (2.1) to the general case.

2.1. **Ore sets attached to homomorphisms of rings.** Let $\alpha : R \to A$ be a homomorphism of (unital) rings. A left or right ideal I of A is called *R-cofinite* if A/I is a Noetherian R-module (via α.) We define the set S_l of left R-cofinite elements of A as

$$S_l := S_l(\alpha) := \{s \in A |\ As \text{ is } R\text{-cofinite}\}.$$

The set $S_r := S_r(\alpha)$ of right R-cofinite elements of A is defined analogously.

Lemma 2.1. (i) *S_l and S_r are multiplicatively closed.*

(ii) *If $as \in S_l$ (resp. $sa \in S_r$) then $s \in S_l$ (resp. $s \in S_r$).*

Proof. We only discuss the "left" versions. The statement (i) follows immediately from the following exact sequence

$$A/As \xrightarrow{\cdot t} A/Ast \xrightarrow{\mathrm{pr}} A/At \longrightarrow 0\ .$$

The statement (ii) is a consequence of the surjection $A/Aas \twoheadrightarrow A/As$. □

Consider the following property

$\mathbf{PC_l}(\alpha)$**:** Any R-cofinite left ideal I of A is *principally cofinite*, i.e. contains a principal left R-cofinite ideal or, in other words, an element in S_l.

The property $\mathbf{PC_r}(\alpha)$ is defined similarly. In the following we only give "left" versions of assertions and proofs. But it is understood that in each case the corresponding "right" version holds as well.

Lemma 2.2. *Assume $PC_l(\alpha)$. Then, for any s in S_l the left A-module A/As is S_l-torsion.*

Proof. Let $c \in A$ be any element and consider the left ideal $L := \{b \in A : bc \in As\}$ in A. Since $A/L \xrightarrow{\cdot c} A/As$ is injective the left ideal L is R-cofinite. Our assumption $PC_l(\alpha)$ therefore ensures the existence of an element $t \in S_l \cap L$. Then $t(c + As) = 0$. $\square$

Proposition 2.3. *Assume $PC_l(\alpha)$ and let M be a Noetherian left A-module; then M is Noetherian as an R-module if and only if it is S_l-torsion.*

Proof. Since S_l is multiplicative an extension of two S_l-torsion A-modules is S_l-torsion as well. Secondly, since M is Noetherian over A there is a finite exhaustive and separated filtration on M whose subquotients are cyclic (and Noetherian) A-modules. These observations reduce us to the case where M is cyclic over A, i.e. of the form $M \cong A/L$ for some left ideal L of A. If M is S_l-torsion we find an $s \in S_l$ such that $s = s \cdot 1 \in L$. Then M is a quotient of the Noetherian R-module A/As and therefore is Noetherian over R. Suppose, vice versa, that M is Noetherian over R which means that L is R-cofinite. By our assumption $PC_l(\alpha)$ we have $As \subseteq L$ for some $s \in S_l$. According to the above lemma A/As and a fortiori $A/L \cong M$ are S_l-torsion. $\square$

Proposition 2.4. *If $\mathrm{PC}_l(\alpha)$ is satisfied, then S_l is a left Ore set of A.*

Proof. Let $s \in S$ and $b \in A$. By the lemma the A-module A/As is S-torsion. Hence $tb \in As$ for some $t \in S$. We therefore find a $b' \in A$ such that $tb = b's$. $\square$

Lemma 2.5. *Let $\alpha_i : R_i \to A_i$, for $1 \leq i \leq m$, be homomorphisms of arbitrary (unital) rings and let $\alpha : R := R_1 \times \ldots \times R_m \to A := A_1 \times \ldots \times A_m$ be their product. Then $PC_l(\alpha)$ holds if and only if $PC_l(\alpha_i)$ holds for all $1 \leq i \leq m$.*

Proof. Without loss of generality we assume $m = 2$. Then every left ideal L of A is of the form $L_1 \times L_2$ with left ideals $L_i = e_i L$ of A_i, where e_i denotes the central idempotent corresponding to A_i. Thus $A/L = A_1/L_1 \times A_2/L_2$ is Noetherian over R if and only if each $A_i/L_i = e_i(A/L)$ is Noetherian over $R_i = e_i R$. Now assume that $PC_l(\alpha_i)$ holds for $i = 1, 2$ and let L be an R-cofinite left ideal of A. Then L_i contains an element f_i such that $A_i f_i$ is R_i-cofinite for $i = 1, 2$. Putting $f := (f_1, f_2) \in L_1 \times L_2 = L$ we obtain an R-cofinite ideal $Af \subseteq L$, whence $PC_l(\alpha)$ follows. For the converse let L_1 be an R_1-cofinite left ideal of A_1. Then $L := L_1 \times A_2$ is an R-cofinite left ideal of A which thus by assumption contains an R-cofinite element f. Then $f_1 = e_1 f \in e_1 L = L_1$ is R_1-cofinite. Thus $PC_l(\alpha_1)$, and similarly $PC_l(\alpha_2)$, follows. $\square$

Lemma 2.6. *Let $R \xrightarrow{\beta} A_0 \xrightarrow{\gamma} A_1$ be ring homomorphisms and put $\alpha := \gamma \circ \beta$.*

(i) *Suppose that we have an surjection $A_0^m \twoheadrightarrow A_1$ of A_0-bimodules and $PC_l(\beta)$ holds; then $PC_l(\alpha)$ holds and*

$$S_l(\alpha) = \{s \in A_1 \mid as \in \gamma(S_l(\beta)) \text{ for some } a \in A_1\}. \tag{2.2}$$

In particular, if $A_0 \subseteq A_1$ then $S_l(\beta) \subseteq S_l(\alpha)$.

(ii) *If A_0 is Noetherian as a left R-module, then $PC_l(\alpha)$ holds if and only if $PC_l(\gamma)$ holds.*

Proof. Let L be an R-cofinite left ideal of A_1. Then $A_0/\gamma^{-1}(L) \subseteq A_1/L$ are Noetherian R-modules. Thus property $PC_l(\beta)$ grants the existence of an $f \in \gamma^{-1}(L)$ such that $A_0/A_0 f$ is Noetherian over R. Hence $A_1/A_1\gamma(f) = A_1 \otimes_{A_0} A_0/A_0 f$ is the image of $(A_0/A_0 f)^m$ for some m and thus is Noetherian over R. This proves the first part of (i). For (2.2) we first assume that $\gamma(f) = as$ belongs to $\gamma(S_l(\beta))$. Then by the same argument as above $A_1/A_1 as = A_1 \otimes_{A_0} A_0/A_0 f$ is Noetherian over R. Hence $as \in S_l(\alpha)$ and therefore $s \in S_l(\alpha)$ by Lemma 2.1(ii). For the converse suppose that $s \in S_l(\alpha)$. Since $A_0/\gamma^{-1}(A_1 s)$ is an R-submodule of the

Noetherian left R-module A_1/A_1s, the left ideal $\gamma^{-1}(A_1s)$ of B is R-cofinite. Thus property $PC_l(\beta)$ implies that $\gamma^{-1}(A_1s) \cap S_l(\beta) \neq \emptyset$.
Under the hypothesis of (ii) any finitely generated (left) A_0-module M is Noetherian over R, hence the statement is clear. □

2.2. **The Artinian case.** In this subsection we assume that, in addition to our standard hypothesis, R is (left and right) Artinian. Then $B = R((t;\sigma,\delta))$ is Artinian, too, by Proposition 1.6 and Remark 1.7 applied to the $\mathrm{Jac}(R)$-adic filtration of R. Hence, by [9, prop. 3.1.1] B is a (left and right) quotient ring, i.e. every regular element of B is a unit. Since t is regular in A, we have a canonical inclusion $A \subseteq A_T = B$. The regular elements of A are also regular in A_T, they thus are all units in B. It easily follows that $\mathcal{C}_A(0)$ is an Ore set and that

$$B = A_{\mathcal{C}_A(0)}.$$

Lemma 2.7. *A is Noetherian.*

Proof. Since R is Artinian, $\mathrm{Jac}(R)$ is nilpotent whence the standard filtration I_k is separated and thus stabilises at 0. It follows that $\mathrm{gr}_{I_\bullet} R$ is finitely generated over the Noetherian ring R/I_1 and thus is Noetherian itself. The claim follows from [11, lem. 1.5]. □

Now Small's theorem ([9, cor. 4.1.4]) combined with Lemma 2.7 tells us that

$$\mathcal{C}_A(0) = \mathcal{C}_A(\mathcal{N}(A)), \tag{2.3}$$

where, for a ring C, we write $\mathcal{N}(C)$ for its prime radical, i.e. the intersection of all prime ideals of C. In particular, it contains all nilpotent ideals of C. Since $\mathrm{Jac}(R)$ is nilpotent as R is Artinian, $\mathrm{Jac}(R)_A$ and $\mathrm{Jac}(R)_B$ are nilpotent by Lemma 1.12 and thus we obtain the inclusions

$$\mathrm{Jac}(R)_A \subseteq \mathcal{N}(A) \qquad \text{and} \qquad \mathrm{Jac}(R)_B \subseteq \mathcal{N}(B). \tag{2.4}$$

Next we need (ii) of the following Proposition; we are grateful to the referee for pointing out to us and providing a proof for the fact that even the stronger result (i) holds, thereby simplifying our earlier proof of (ii).

Proposition 2.8. (i) *Let $\tilde{R}$ be a (not necessarily Artinian) semiprime Noetherian ring with a ring automorphism $\tilde{\sigma}$. Then $\tilde{A} = \tilde{R}[[t,\tilde{\sigma}]]$ is also semiprime.*
(ii) *If R is an (Artinian) semisimple ring, then B is so, too.*

Proof. (i) First of all, $\tilde{A}$ is Noetherian by [9, thm. 1.4.5(iv)]. By [9, thm. 2.3.7], $\mathcal{N}(\tilde{A})^k = 0$ for some k. The automorphism $\tilde{\sigma}$ of $\tilde{R}$ extends to an automorphism $\tilde{\sigma}$ of $\tilde{A}$, and this is just conjugation by t. In particular, $\tilde{\sigma}(\mathcal{N}(\tilde{A})) = \mathcal{N}(\tilde{A})$, whence $t^i\mathcal{N}(\tilde{A}) = \mathcal{N}(\tilde{A})t^i$ for all $i \geq 0$. Since $\tilde{A}/\tilde{A}t \cong \tilde{R}$ is semiprime, we see that $\mathcal{N}(\tilde{A}) \subseteq \tilde{A}t$. Now

$$t^k \cdot (\mathcal{N}(\tilde{A})t^{-1})^k = \mathcal{N}(\tilde{A})^k = 0$$

so $(\mathcal{N}(\tilde{A})t^{-1})^k = 0$ as t is a regular element. Hence $\mathcal{N}(\tilde{A})t^{-1} \subseteq \mathcal{N}(\tilde{A})$, and thus $\mathcal{N}(\tilde{A}) = \mathcal{N}(\tilde{A})t$. But $t \in \mathrm{Jac}(\tilde{A})$ and $\tilde{A}$ is Noetherian, so $\mathcal{N}(\tilde{A}) = 0$ by Nakayama's Lemma.
(ii) We have already seen above that B is Artinian. By assumption $\mathrm{Jac}(R) = 0$ and hence $\delta = 0$ by our general hypothesis. So (i) applies and gives that A is semiprime. Then Goldie's theorem implies that $B = A_{\mathcal{C}_A(0)}$ is semisimple. □

The proposition implies that for R Artinian (but not necessarily semisimple), $B/\mathrm{Jac}(R)_B \cong (R/\mathrm{Jac}(R))((t,\overline{\sigma}))$ is semisimple, whence $\mathcal{N}(B/\mathrm{Jac}(R)_B) = \mathrm{Jac}(B/\mathrm{Jac}(R)_B) = 0$ and thus $\mathcal{N}(B) \subseteq \mathrm{Jac}(R)_B$. An argument in the proof of [9, thm. 4.1.4] shows that $\mathcal{N}(B) \cap A = \mathcal{N}(A)$. Thus we obtain

$$\mathcal{N}(A) = \mathcal{N}(B) \cap A \subseteq \mathrm{Jac}(R)_B \cap A = \mathrm{Jac}(R)_A,$$

which combined with (2.4) gives

$$\mathcal{N}(A) = \mathrm{Jac}(R)_A \qquad \text{and} \qquad \mathcal{N}(B) = \mathrm{Jac}(B) = \mathrm{Jac}(R)_B. \tag{2.5}$$

Taking (2.3) into account we have proven the following

Proposition 2.9. *$\mathcal{C}_A(\mathrm{Jac}(R)_A) = \mathcal{C}_A(0)$ is an Ore set of A (consisting of regular elements).*

For the rest of this subsection we assume that R is also semisimple, and so in particular $\delta = 0$ due to our general hypothesis in the present Section 2. Then, by Wedderburn theory, R decomposes into a product

$$R = R_1 \times \cdots \times R_m$$

of full matrix rings

$$R_i \cong M_{n_i}(D_i)$$

over skew fields D_i. The R_i are precisely the minimal ideals of R. Thus, any ring automorphism σ of R maps R_i again onto some $R_{\sigma(i)}$ where we write, by abuse of notation, σ also for the permutation on the set $\{1, \ldots, m\}$ induced in this way by the automorphism σ of R. By taking the products of those R_i which belong to the same σ-orbit, it follows that the pair (R, σ) decomposes into a product of pairs $(C_j, \sigma_j), 1 \leq j \leq \ell$, each consisting of a ring C_j with a ring automorphism σ_j of C_j, i.e.

$$R = C_1 \times \ldots \times C_\ell,$$

and

$$\sigma = \sigma_1 \times \ldots \times \sigma_\ell,$$

where σ_j denotes the restriction of σ to C_j.

The proof of the following result is obvious.

Lemma 2.10. *Let (R, σ) be the product of pairs $(C_j, \sigma_j), 1 \leq j \leq \ell$. Then there is a canonical isomorphism of rings*

$$R[[t;\sigma]] \cong C_1[[t, \sigma_1]] \times \ldots C_\ell[[t;\sigma_\ell]].$$

Because of Lemma 2.5 the crucial case to consider therefore is the *cyclic case,* i.e. the situation where R equals some C_j as above, i.e. we assume that $R = R_1 \times \ldots \times R_n$ (with R_i simple) and that σ is given by $\tau_i : R_i \to R_{i+1}$ for $i = 1, \ldots, n-1$ and $\tau_n : R_n \to R_1$ in the following way:

$$\sigma(r_1, \ldots, r_n) = (\tau_n(r_n), \tau_1(r_1), \ldots, \tau_{n-1}(r_{n-1})).$$

We now define isomorphisms of rings

$$\begin{aligned} \psi : R = R_1 \times \ldots \times R_n &\to R_1 \times \ldots \times R_1 \\ (r_1, \ldots, r_n) &\mapsto (r_1, \tau_n \circ \ldots \circ \tau_2(r_2), \ldots, \tau_n(r_n)) \end{aligned}$$

and

$$\phi = \phi_0 \times \mathrm{id}_{R_1} \times \ldots \times \mathrm{id}_{R_1} : R_1 \times \ldots \times R_1 \to R_1 \times \ldots \times R_1$$

where $\phi_0 := \tau_n \circ \tau_{n-1} \circ \ldots \circ \tau_1$ as well as the cyclic permutation

$$\begin{aligned} \pi : R_1 \times \ldots \times R_1 &\to R_1 \times \ldots \times R_1 \\ (r_1, \ldots, r_n) &\mapsto (r_n, r_1, \ldots, r_{n-1}). \end{aligned}$$

Then the following diagram of isomorphisms of rings is clearly commutative:

$$\begin{tikzcd} R = R_1 \times \ldots \times R_n \arrow[r, "\psi"] \arrow[dd, "\sigma"'] & R_1 \times \ldots \times R_1 \arrow[d, "\phi"] \\ & R_1 \times \ldots \times R_1 \arrow[d, "\pi"] \\ R = R_1 \times \ldots \times R_n \arrow[r, "\psi"] & R_1 \times \ldots \times R_1 \end{tikzcd}$$

Note that $\pi \circ \phi \neq \phi \circ \pi$ if $n \geq 2$, but that

$$(\pi \circ \phi)^n = \phi_0 \times \ldots \times \phi_0. \tag{2.6}$$

The isomorphism ψ induces therefore a natural identification of rings

$$R[[t; \sigma]] \cong (R_1 \times \ldots \times R_1)[[t; \pi \circ \phi]].$$

Furthermore, we obtain the injective homomorphism of rings

$$\begin{aligned} (R_1 \times \ldots \times R_1)[[x; (\pi \circ \phi)^n]] &\hookrightarrow (R_1 \times \ldots \times R_1)[[t; \pi \circ \phi]] \\ x &\mapsto t^n, \end{aligned}$$

and the latter ring is free of rank n over the former (from the left as well as right). By Lemma 2.10 and (2.6) the ring $(R_1 \times \ldots \times R_1)[[x; (\pi \circ \phi)^n]]$ can be identified with the product $R_1[[x; \phi_0]] \times \ldots \times R_1[[x; \phi_0]]$, and we have shown the following

Proposition 2.11. *In the cyclic case $R[[t, \sigma]]$ is a free (left or right) module of finite rank n over a subring isomorphic to $R_1[[x; \phi_0]] \times \ldots \times R_1[[x; \phi_0]]$ where x corresponds to t^n.*

By Proposition 2.11 the case where R is a simple Artinian ring, i.e. $R \cong M_n(D)$ for some skew field D, merits special attention. If γ denotes an automorphism of D we write $M_n(\gamma)$ for the induced automorphism of $M_n(D)$ given by applying γ to each matrix entry. For our purposes the following observation will be crucial.

Proposition 2.12. *Every automorphism σ of the ring $M_n(D)$ decomposes into the composite*

$$\sigma = Int(C) \circ M_n(\gamma)$$

for some automorphism γ of D and some inner automorphism $Int(C)$ corresponding to an invertible matrix $C \in M_n(D)$.

Proof. This is an easy consequence of the Isomorphism Theorem in [6, III. §5] which we shall explain briefly for the convenience of the reader: Let s be any ring automorphism of $\mathrm{End}_D(V)$ for some finite dimensional D-vector space V. Then there exists an automorphism γ of D and a γ-linear bijective map $S : V \to V$ such that $s(f) = SfS^{-1}$. We apply this to the standard D-vector space $V = D^n$ and to the automorphism s which corresponds to σ under the identification of $\mathrm{End}_D(D^n)$ with $M_n(D)$ by using the standard basis $\{e_1, \ldots, e_n\}$ of D^n. Consider the γ-linear bijective map $\Gamma : D^n \to D^n, (d_1, \ldots, d_n) \mapsto (\gamma d_1, \ldots, \gamma d_n)$. Clearly,

$S\circ\Gamma^{-1}$ is a D-linear map of V, say F, given with respect to the standard basis by an invertible matrix, say C. Thus we obtain that

$$s(f) = F\circ\Gamma\circ f\circ\Gamma^{-1}\circ F^{-1} \tag{2.7}$$

for all $f\in \mathrm{End}_D(V)$. Now it is easy to check that under the identification $\mathrm{End}_D(D^n) = M_n(D)$ the automorphism $\mathrm{End}_D(V)\to\mathrm{End}_D(V)$, $f\mapsto\Gamma\circ f\circ\Gamma^{-1}$, corresponds to $M_n(\gamma)$ and thus (2.7) becomes

$$\sigma(A) = CM_n(\gamma)(A)C^{-1} = Int(C)\circ M_n(\gamma)(A)$$

for all $A\in M_n(D)$. □

Before we can draw our promised conclusion from this proposition we have to prove the following

Lemma 2.13. *Let C be an arbitrary ring with ring automorphism σ and let u be a unit in C. If $Int(u)$ denotes the inner automorphism $c\mapsto ucu^{-1}$ of C then there are canonical isomorphisms of rings*

$$C[[t; Int(u)\circ\sigma]] \xrightarrow{\cong} C[[t;\sigma]]$$
$$\sum a_n t^n \mapsto \sum a_n(ut)^n$$

and

$$C[[t;\sigma\circ Int(u)]] \xrightarrow{\cong} C[[t;\sigma]]$$
$$\sum a_n t^n \mapsto \sum a_n(\sigma(u)t)^n = \sum a_n(tu)^n.$$

Proof. Both maps are obviously bijective and additive. The multiplicativity follows by a straightforward computation based on the multiplication formula (4) in [11]. E.g., under the first map the relation $ta = u\sigma(a)u^{-1}t$ corresponds to $uta = u\sigma(a)u^{-1}ut = u\sigma(a)t$, which is equivalent to the law $ta=\sigma(a)t$ that holds in $C[[t;\sigma]]$. □

Corollary 2.14. *Let σ be any automorphism of $M_n(D)$. Then there are isomorphisms of rings*

$$M_n(D)[[t;\sigma]] \cong M_n(D)[[t; M_n(\gamma)]] \cong M_n(D[[t;\gamma]])$$

where γ is as in Proposition 2.12.

Proof. The first isomorphism follows from the Proposition 2.12 combined with Lemma 2.13. The second one is easily checked. □

The following result is certainly well-known.

Proposition 2.15. *For a skew field D the power series $D[[t;\sigma]]$ form a principal left (and principal right) ideal domain. All its left (right) ideals are two-sided and they are precisely the ideals of the form $D[[t;\sigma]]t^n$ with $n\geq 0$.*

Proof. The first statement follows from [9, prop. 1.4.5]. For lack of a reference known to us for the second statement we give a complete proof using the Weierstrass preparation theorem [16, cor. 3.2], which says that any element in $D[[t;\sigma]]$ can be written as a unit times a distinguished polynomial (or vice versa) in t. But note that in this situation a distinguished polynomial is just a power t^n of t with $n\geq 0$. Now, if n is the minimal such exponent among all elements of a given left (or right) ideal I, then t^n certainly generates I. Since t is regular, it follows that D has no zero divisor, i.e. is a domain. □

Corollary 2.16. *Let σ be any automorphism of $M_n(D)$. Then $M_n(D)[[t;\sigma]]$ is a principal left ideal and principal right ideal ring.*

Proof. Combine Corollary 2.14 with [9, prop. 3.4.10]. $\square$

Remark 2.17. In the cyclic case (of order n) we have shown that $R[[t;\sigma]]$ is free of finite rank as a (left or right) $D[[x;\gamma]]$-module. Here $x = t^n$ and γ corresponds via Proposition 2.12 to ϕ_0 which in turn corresponds to σ.

Proposition 2.18. *Assuming that R is semisimple we have:*

(i) $\mathcal{C}_A(0) = S_l(R \subseteq A) = S_r(R \subseteq A) =: S$.
(ii) *The set S is an (left and right) Ore set of A consisting of regular elements; in particular the localisation A_S of A with respect to S exists.*

Proof. The first claim will follow from Lemma 2.19 below. The second claim is a consequence either of (i) and Proposition 2.9 or of Proposition 2.4 and Lemma 2.21 below. $\square$

Lemma 2.19. *If R is semisimple then, for any $f \in A$, the following implications hold:*

(i) *If f is left (right) regular, then Af (fA) is R-cofinite.*
(ii) *If Af (fA) is R-cofinite, then f is right (left) regular.*

In particular, the following statements are all equivalent:

(a) *f is regular,*
(b) *f is left regular,*
(c) *f is right regular,*
(d) *Af is R-cofinite,*
(e) *fA is R-cofinite.*

Proof. First note that all properties involved in this lemma behave well under finite products $(R,\sigma) = (R_1,\sigma_1) \times \ldots \times (R_m,\sigma_m)$, e.g. $f \in A$ being right regular is equivalent to $f_i = e_i f$ being right regular in $A_i = e_i A$ for all $i = 1,\ldots,m$. Indeed, if one f_{i_0} were a right zero divisor, say $f_{i_0} g_{i_0} = 0$ for some $0 \neq g_{i_0} \in A_i$, then f were a right zero divisor, too, as $f \cdot (0,\ldots,0,g_{i_0},0,\ldots,0) = 0$; the other direction is trivial. The compatibility of cofiniteness with products was shown during the proof of Lemma 2.5.
Thus, in order to prove (i) we may and do assume that we are in the cyclic case. Let $f \in A$ be left regular, i.e. the left A-module map $A \to A, a \mapsto af$, is injective. Thus its cokernel A/Af is a finitely generated (left) $D[[x;\gamma]]$-torsion module; here we use again Remark 2.17. By Lemma 2.20 below A/Af is finitely generated over D. To prove (ii) assume that Af is R-cofinite and that $g \in A$ satisfies $fg = 0$. Multiplication by g on the right induces a surjective map $A/Af \twoheadrightarrow Ag$ of A-modules, which shows that Ag is finitely generated over R and hence over D. Again by Lemma 2.20 below it follows that Ag is a $D[[x;\gamma]]$-torsion module. On the other hand Ag is a submodule of the free (Remark 2.17) and hence torsionfree $D[[x;\gamma]]$-module A, because $D[[x;\gamma]]$ is a domain by Proposition 2.15. Thus $g = 0$ which shows that f is right regular. The rest of the lemma follows by symmetry. $\square$

Lemma 2.20. *Let D be a skew field, γ any automorphism of D and N a finitely generated $D[[x;\gamma]]$-module. Then the following statements are equivalent:*

(i) *N is $D[[x;\gamma]]$-torsion,*
(ii) *N is annihilated by a power of x,*
(iii) *N is finite dimensional over D when considered as D-module via the natural inclusion $D \hookrightarrow D[[x;\gamma]]$.*

Proof. It follows immediately from Proposition 2.15 that (i) and (ii) are equivalent. Also the implication (ii)⇒(iii) is clear because $D[[x;\gamma]]/(x^n)$ is finite dimensional over D for any n. For the converse, since the unique simple $D[[x;\gamma]]$-module $D = D[[x;\gamma]]/(x)$ has annihilator ideal (x), it suffices to show that N has finite length, say n, as $D[[x;\gamma]]$-module, because then N is annihilated by (x^n). Assume the contrary. Then the Noetherian $D[[x;\gamma]]$-module N is not Artinian, i.e. it exists a descending sequence of $D[[x;\gamma]]$-submodules $N = N_0 \supsetneqq N_1 \supsetneqq \ldots \supsetneqq N_i \supsetneqq \ldots$ with $\dim_D(N_i/N_{i+1}) > 0$ for all $i \geq 0$. This implies that $\dim_D(N)$ is infinite, a contradiction. □

Lemma 2.21. *For semisimple R the properties $PC_l(R \subseteq A)$ and $PC_r(R \subseteq A)$ both hold.*

Proof. By Lemma 2.5 we may and do assume that we are in the cyclic case. Then A is a free $D[[x,\gamma]]$-module of finite rank by Remark 2.17. Now let L be an R-cofinite left ideal of A, i.e. $N := A/L$ is finitely generated over D. As seen in the proof of Lemma 2.20 the module N is annihilated by a power of x and thus by some power of t. In particular, $t^n \cdot 1 \in L$ for some $n \geq 0$. It follows that L contains the R-cofinite principal ideal At^n. The right version is shown similarly. □

2.3. **The general case.** We put $\bar{R} := R/\mathrm{Jac}(R)$, $\bar{A} := A/\mathrm{Jac}(R)_A$, $\bar{M} := M/\mathrm{Jac}(R)M$ for a (left) R-module M and we recall the topological Nakayama lemma from [14, lem. 4.9]:

Lemma 2.22. *Let M be a pseudocompact R-module. Then M is finitely generated over R if and only if $\bar{M}$ is finitely generated over $\bar{R}$.*

Lemma 2.23. *The property $PC_l(R \subseteq A)$ (respectively $PC_r(R \subseteq A)$) holds if and only if $PC_l(\bar{R} \subseteq \bar{A})$ ($PC_r(\bar{R} \subseteq \bar{A})$) holds.*

Proof. Note that for $f \in A$ the R-module Af is pseudocompact being the epimorphic image of the pseudocompact R-module A under the continuous R-linear map $A \to A, a \mapsto af$, thus also the quotient A/Af is a natural pseudocompact R-module. Hence it follows from the topological Nakayama lemma 2.22 that

$$A/Af \text{ is finitely generated over } R \Leftrightarrow \bar{A}/\bar{A}\bar{f} \text{ is finitely generated over } \bar{R}. \tag{2.8}$$

Here $\bar{f}$ denotes the image of f in $\bar{A}$. Now assume that $PC_l(\bar{R} \subseteq \bar{A})$ holds and let L be a R-cofinite left ideal of A. Then $A/\mathrm{Jac}(R)+L$ is finitely generated over $\bar{R}$ and thus there exists an element $f \in L$ such that $\bar{A}/\bar{A}\bar{f}$ is finitely generated over $\bar{R}$ whence Af is R-cofinite by the above equivalence. The opposite implication is trivial and the right version follows as usual by symmetry. □

Proposition 2.24. *$\mathcal{C}_A(\mathrm{Jac}(R)_A)$ consists of regular elements.*

Proof. Let a be in $\mathcal{C}_A(\mathrm{Jac}(R)_A)$ and assume that $ab = 0$ for some $b \in A$. Then we have also $\bar{a}\bar{b} = 0$ for the images $\bar{a}$ and $\bar{b}$ of a and b in $A_n := A/\mathrm{Jac}(R)^n_A \supseteq R_n := R/\mathrm{Jac}(R)^n$. Since by Proposition 2.9 $\bar{a} \in \mathcal{C}_{A_n}(\mathrm{Jac}(R_n)_{A_n}) = \mathcal{C}_{A_n}(0)$ is regular, $\bar{b}$ must belong to $\mathrm{Jac}(R)^n_A$ for all $n \geq 0$. But it is easily seen from the definition of $(-)_A$ and the fact that the Noetherian pseudocompact ring R is Hausdorff with respect to the $\mathrm{Jac}(R)$-adic topology (cf. [11, rem. 0.1 i]) that

$$\bigcap \mathrm{Jac}(R)^n_A \subseteq \Big(\bigcap \mathrm{Jac}(R)^n\Big)_A = 0.$$

Thus $b = 0$ and a is right regular. By symmetry we obtain also left regularity. □

Theorem 2.25. (i) $\mathcal{C}_A(\mathrm{Jac}(R)_A) = S_l(R \subseteq A) = S_r(R \subseteq A) =: S$.

(ii) *The set S is an (left and right) Ore set of A consisting of regular elements; in particular the localisation A_S of A with respect to S exists.*

Proof. We set as before $\bar{R} = R/\mathrm{Jac}(R)$ and $\bar{A} = A/\mathrm{Jac}(R)_A \cong \bar{R}[[t;\bar{\sigma}]]$ with $\bar{\sigma}$ the automorphism of $\bar{R}$ induced by σ. By Proposition 2.18 we know that $\mathcal{C}_{\bar{A}}(0) = S_l(\bar{R} \subseteq \bar{A}) = S_r(\bar{R} \subseteq \bar{A}) =: \bar{S}$. Since $\mathcal{C}_A(\mathrm{Jac}(R)_A)$ is the full preimage of $\mathcal{C}_{\bar{A}}(0)$ and since, by (2.8), $S_l(R \subseteq A)$ and $S_r(R \subseteq A)$ are the full preimages of $S_l(\bar{R} \subseteq \bar{A})$ and $S_r(\bar{R} \subseteq \bar{A})$, respectively, under the canonical projection $A \twoheadrightarrow \bar{A}$ the first claim follows and S is the full preimage of $\bar{S}$. Now S is an Ore set since the conditions $PC_l(R \subseteq A)$ and $PC_r(R \subseteq A)$ are satisfied by Lemmata 2.23 and 2.21. $\square$

For the sake of completeness we explicitly state the following

Proposition 2.26. (i) $\mathrm{Jac}(B) = \mathrm{Jac}(R)_B$.

(ii) $B/\mathrm{Jac}(B)$ *is the quotient ring of* $A/\mathrm{Jac}(R)_A$.

(iii) $\mathrm{Jac}(A_S) = \mathrm{Jac}(R)A_S$.

Proof. As a consequence of (2.5) the ideal $\mathrm{Jac}(R)_B$ is equal to the intersection of all open maximal left ideals in the pseudocompact ring B. But according to [5, IV.4 prop. 13.b] this intersection in fact coincides with the ordinary Jacobson radical $\mathrm{Jac}(B)$. This establishes the first claim. In view of (i) the second claim was discussed already at the beginning of the previous subsection. For the third claim we first note that since $S = \mathcal{C}_A(\mathrm{Jac}(R)_A)$ consists of regular elements we have $1 + \mathrm{Jac}(R)_A A_S \subseteq A_S^\times$ and hence that

$$\mathrm{Jac}(R)_A A_S = \mathrm{Jac}(R) A_S \subseteq \mathrm{Jac}(A_S).$$

On the other hand, by Proposition 2.8 (ii) the factor ring

$$A_S/\mathrm{Jac}(R)A_S \cong \big((R/\mathrm{Jac}(R))[[t;\bar{\sigma}]]\big)_S =: \bar{B}$$

is semisimple. Alternatively, part (iii) also follows from [7, thm. 3.2.3(b)]. $\square$

Now let $I_\bullet$ be a filtration of R satisfying (I). Then the localisation A_S of A with respect to S is naturally endowed with the filtration $I_\bullet A_S = A_S I_\bullet$. We denote the corresponding completion simply by $(A_S)^\wedge$. We shall prove the following

Proposition 2.27. $B \cong (A_S)^\wedge$. *Furthermore, for every $k \geq 0$ there are canonical isomorphisms* $(A/I_kA)_S \cong B/I_kB \cong (R/I_k)((t;\overline{\sigma},\overline{\delta}))$.

It is remarkable that the completion of A_S arises from A just by 'inverting one element: t.'

Proof. The statement follows from (1.23) and the following isomorphisms

$$B_k \cong (A_k)_{\mathcal{C}_{A_k}(\mathrm{Jac}(R_k)_{A_k})} \cong A_{\mathcal{C}_A(\mathrm{Jac}(R)_A)}/I_k A_{\mathcal{C}_A(\mathrm{Jac}(R)_A)},$$

where we put $R_k := R/I_k$, $A_k := R_k[[t;\bar{\sigma},\bar{\delta}]]$ and $B_k := R_k((t;\bar{\sigma},\bar{\delta})) = (A_k)_T$. $\square$

Since $A \subseteq B$ the natural map $A_S \longrightarrow B$ given by the above proposition must be injective. In particular, the filtration $I_\bullet A_S$ is separated.

Proposition 2.28. *Under the assumptions of Proposition 1.16 B is a flat A_S-module.*

Proof. By Proposition 1.16 B is a flat A-module. But on A_S-modules we have the natural isomorphism of functors $B \otimes_{A_S} - = B \otimes_A -$. $\square$

3. On the K-theory of A_S

In the situation of section 2 we assume that A is Noetherian and we write A_S for the localisation in theorem 2.25. By $\mathcal{M}_R(A)$ we denote the full subcategory of the category of left A-modules consisting of those modules M which are finitely generated over R. Since the condition $PC_l(R \subseteq A)$ is satisfied by Lemmata 2.23 and 2.21, Proposition 2.3 implies that the objects in $\mathcal{M}_R(A)$ are precisely those finitely generated A-modules M which are S-torsion, i.e. which satisfy $A_S \otimes_A M = 0$, or in other words $\mathcal{M}_R(A)$ is the 'kernel' of the functor

$$F(-) := A_S \otimes_A - : A\text{−mod} \to A_S\text{−mod}$$

from the category A−mod of finitely generated (left) A-modules to the corresponding category over A_S. It is well known that F induces an equivalence of categories

$$A\text{−mod}/\mathcal{M}_R(A) \xrightarrow{\sim} A_S\text{−mod}.$$

Thus, due to the isomorphisms $G_i(A\text{−mod}/\mathcal{M}_R(A)) \cong G_i(A_S)$ we obtain from the long exact localisation sequence of Quillen G-theory the following exact sequence

$$\cdots \longrightarrow K_1(\mathcal{M}_R(A)) \longrightarrow G_1(A) \longrightarrow G_1(A_S) \xrightarrow{\partial} K_0(\mathcal{M}_R(A)) \longrightarrow G_0(A) \longrightarrow G_0(A_S) \longrightarrow 0 \ ,$$

where $K_i(\mathcal{M}_R(A))$ denotes the ith Quillen K-group of the category $\mathcal{M}_R(A)$.

We assume from now on that σ is given by conjugation $\sigma(\cdot) = \gamma \cdot \gamma^{-1}$ with some unit $\gamma \in A^\times$. Then, by [11, rem. 2.3] we obtain an exact sequence

$$0 \longrightarrow A \otimes_R M \xrightarrow{\kappa(M)} A \otimes_R M \longrightarrow M \longrightarrow 0$$

of A-modules for any $M \in \mathcal{M}_R(A)$. Hence applying $A_S \otimes_A -$ induces an automorphism

$$\kappa_S(M) : \ A_S \otimes_R M \xrightarrow{\cong} A_S \otimes_R M$$

of the A_S-module $A_S \otimes_R M$.

We denote by $G_1^{\mathrm{Bass}}(A_S)$ Bass' first K-group of the category of finitely generated A_S-modules, which is generated by symbols $< \alpha | M_S >$ where M_S is a finitely generated A_S-module and α is an automorphism of M_S, cf. [13, thm. 16.11]. In [3, section VIII.5] a connecting homomorphism $\partial^{\mathrm{Bass}} : G_1^{\mathrm{Bass}}(A_S) \to K_0(\mathcal{M}_R(A))$ is described. Also Sherman constructs a natural homomorphism $\psi : G_1^{\mathrm{Bass}}(A_S) \to G_1(A_S)$ and shows that $\partial \circ \psi = \partial^{\mathrm{Bass}}$ holds, cf. [12, prop. 2.4].

It is easy to check that

$$[M] \mapsto \psi(< \kappa_S(M) | M_S >)$$

defines a homomorphism of groups

$$\mathrm{char} : K_0(\mathcal{M}_R(A)) \to G_1(A_S).$$

Proposition 3.1. *Assume that $A = R[[t; \sigma, \delta]]$ is a Noetherian ring such that*

(a) *$\sigma(\cdot) = \gamma \cdot \gamma^{-1}$ for some unit $\gamma \in A^\times$ and*
(b) *$\delta(R) \subseteq \mathrm{Jac}(R)$.*

Then char *is a splitting for the connecting homomorphism of the localisation exact sequence above. In particular, there are isomorphisms*

(i) *$G_0(A_S) \cong G_0(A)$ and*

(ii) $G_1(A_S) \cong \mathrm{im}G_1(A) \oplus K_0(\mathcal{M}_R(A))$, *where* $\mathrm{im}G_1(A)$ *denotes the image of* $G_1(A)$ *in* $G_1(A_S)$.

Proof. By the compatibility of ∂ and ∂^{Bass} and the explicit formula of the latter (see either the proof of [12, prop. 2.4] or section VIII.5 in [3]) we have

$$\partial \circ \mathrm{char}([M]) = \partial^{\mathrm{Bass}}(< \kappa_S(M)|M_S >) = [\mathrm{coker}(\kappa(M))] = [M].$$

□

Now assume that A, and hence A_S, is regular. Then, by the resolution principle of K-theory there are canonical isomorphisms $K_1(A) \cong G_1(A)$ and $K_1(A_S) \cong G_1(A_S)$, which we take as identifications.

Corollary 3.2. *Assume that* $A = R[[t;\sigma,\delta]]$ *is a Noetherian ring such that*

(a) $\sigma(\cdot) = \gamma \cdot \gamma^{-1}$ *for some unit* $\gamma \in A^\times$,
(b) $\delta(R) \subseteq \mathrm{Jac}(R)$ *and*
(c) A *is regular.*

Then char *is a splitting for the connecting homomorphism of the localisation exact sequence of* K*-theory*

$$K_1(A) \longrightarrow K_1(A_S) \xrightarrow{\partial} K_0(\mathcal{M}_R(A)) \longrightarrow K_0(A) \longrightarrow K_0(A_S) \longrightarrow 0\ .$$

In particular, there are isomorphisms

(i) $K_0(A_S) \cong K_0(A)$ *and*
(ii) $K_1(A_S) \cong \mathrm{im}K_1(A) \oplus K_0(\mathcal{M}_R(A))$, *where* $\mathrm{im}K_1(A)$ *denotes the image of* $K_1(A)$ *in* $K_1(A_S)$.

This proposition holds also in the case of Iwasawa algebras as discussed in the next section; in this situation it was first observed by David Burns.

4. Iwasawa algebras

We fix a prime p, let G be a compact p-adic Lie group, $\mathcal{O}$ the ring of integers of any fixed finite extension of $\mathbb{Q}_p$, and κ its residue field. We write $\Lambda(G)$ for the Iwasawa algebra $\Lambda(G)$ of G, i.e. the completed group algebra of G, with coefficients $\mathcal{O}$, while $\Omega(G)$ denotes the completed group algebra of G with coefficients in κ; both rings are well known to be Noetherian. Henceforth we assume that G has a closed normal subgroup H such that G/H is isomorphic to $\mathbb{Z}_p$. Recall from [11] that

$$A := \Lambda(G)$$

is isomorphic to a skew power series ring $R[[t;\sigma,\delta]]$ over the Iwasawa algebra $R := \Lambda(H)$ of H. For this one picks once and for all a topological generator γ of a subgroup of G which maps isomorphically onto $G/H \cong \mathbb{Z}_p$ and one defines $t := \gamma - 1$, $\sigma(r) := \gamma r \gamma^{-1}$ for $r \in R$, and $\delta := \sigma - \mathrm{id}$. As a consequence of [11, lem. 1.6] the σ-derivation δ is topologically nilpotent and hence σ-nilpotent. In particular, for any $k \geq 1$ we find an $m \geq 1$ such that $\delta^m(R) \subseteq \mathrm{Jac}(R)^k$. Clearly the ideals $\mathrm{Jac}(R)^k$ for $k \geq 1$ are σ-, σ'-, and δ-stable. Hence the assumption (I) holds and the topological ring

$$B := R{\ll}t;\sigma,\delta]]$$

exists by Proposition 1.10.

The literature we refer to here and later in this section usually assumes $\mathcal{O} = \mathbb{Z}_p$. But in every case it is easily checked that the cited results also hold in our slightly more general situation, for the statements in [4] this is already remarked on the bottom of page 203 (loc. cit.).

Remark 4.1. The ring B is independent, up to natural isomorphism, of the choice of the element γ.

Proof. Since this also follows indirectly from the subsequent results we only indicate a direct argument. First of all we note that $\mathcal{O}\ll t; \mathrm{id}, 0]]$ is a discrete valuation ring with residue class field $\kappa((t))$. It follows that with $t = \gamma - 1$ all $\gamma^\epsilon - 1$, for $\epsilon \in \mathbb{Z}_p \setminus \{0\}$, are units in $\mathcal{O}\ll t; \mathrm{id}, 0]]$ and a fortiori in B. Let now $\tilde{\gamma}$ be a second choice. It suffices to show that $\tilde{\gamma}^{p^m} - 1$ is a unit in B for some $m \geq 0$. Write $\tilde{\gamma}^{p^m} = h\gamma^\epsilon$ for some $h \in H$ and $\epsilon = \epsilon(m) \in \mathbb{Z}_p \setminus \{0\}$. Using $\tilde{\gamma}^{p^m} - 1 = h\gamma^\epsilon - 1 = h(\gamma^\epsilon - 1) + (h-1)$ one checks that with $\gamma^\epsilon - 1$ also $\tilde{\gamma}^{p^m} - 1$ is invertible in B provided the powers $(h-1)^i$ tend to zero in R with $i \to \infty$. In order to now make our choice of the exponent m let N be an open subgroup in H which is normal in G and which is pro-p. We choose m large enough so that $\tilde{\gamma}^{p^m}$ centralises the finite group H/N. Then $\tilde{\gamma}^{p^m}$ and γ^ϵ commute modulo N. Hence hN generates a p-group in H/N. It follows that h topologically generates a pro-p-subgroup $G' \subseteq G$. Since $h - 1$ lies in the Jacobson radical of $\Lambda(G')$ its powers tend to zero. $\square$

We fix a closed normal subgroup N of G which is contained in H as an open subgroup and which is a pro-p-group. By $\mathcal{N}(H)$ we denote the preimage of the prime radical $\mathcal{N}(\Omega(G/N))$ of $\Omega(G/N)$ under the canonical projection

$$\Lambda(G) \longrightarrow \Omega(G/N),$$

the kernel of which we denote by $\mathfrak{m}(N)$. The definition of $\mathcal{N}(H)$ is independent of the choice of N by [4, Lem. 2.5]. Then the set

$$S := \mathcal{C}_A(\mathcal{N}(H)) = \mathcal{C}_A(\mathfrak{m}(N))$$

is a (left and right) Ore set consisting of regular elements of A by [4, prop. 2.6, thm. 2.4] or [2, thm. G]. The localisation A_S of A at S is semi-local [4, prop. 4.2].

Lemma 4.2. (i) $\mathcal{N}(H) = \mathrm{Jac}(R)A$*; in particular,* $S = \mathcal{C}_A(\mathrm{Jac}(R)_A)$.

(ii) $\mathrm{Jac}(A_S) = \mathcal{N}(H)A_S = \mathrm{Jac}(R)A_S$.

Proof. (i) This follows from [1, prop. 6.3] by a simple lifting argument. (ii) The first identity follows by the same standard argument ([7, thm. 3.2.3(b)]) as in the proof of Proposition 2.26. $\square$

The additional assumption $\delta(R) \subseteq \mathrm{Jac}(R)$ which we needed in the previous section does not seem to be satisfied in this generality. But we do have the following.

Lemma 4.3. (i) *If H is a pro-p-group then the assumption* (SI_0) *is satisfied for the* $\mathrm{Jac}(R)$*-adic filtration. In particular B is pseudocompact and is isomorphic to the* $\mathrm{Jac}(R)A_S$*-adic completion of A_S.*

(ii) *If G is a powerful pro-p-group then the assumption* (SI) *is satisfied for the* $\mathrm{Jac}(R)$*-adic filtration. In particular σ induces the identity on the associated graded ring.*

(iii) *If H is an extra-powerful pro-p-group then the associated graded ring for the* $\mathrm{Jac}(R)$*-adic filtration is commutative and finitely generated over κ.*

(iv) *If G is powerful and H is extra-powerful then B is Noetherian and flat over A_S and A.*

Proof. (i) In this case $\mathrm{Jac}(R)$ is the unique maximal ideal in R and $R/\mathrm{Jac}(R) = \kappa$. Hence σ induces the identity on $R/\mathrm{Jac}(R)$ which implies that $\delta(R) \subseteq \mathrm{Jac}(R)$. Therefore Propositions 1.10 and 2.27 apply. (Note that in this situation we have $\mathcal{N}(H) = \mathfrak{m}(H) = \mathrm{Jac}(R)A$. It then follows directly from Theorem 2.25 that S is a left and right Ore set consisting of regular elements.)

(ii) The assumption about G means that $[G, G] \subseteq G^{p^\epsilon}$ with $\epsilon = 1$ and $= 2$ for p odd and $p = 2$, respectively. We in particular have

$$[\gamma, H] \subseteq G^{p^\epsilon} \cap H = H^{p^\epsilon}$$

where the latter identity comes from the fact that $G/H \cong \mathbb{Z}_p$ is torsionfree. For any $h \in H$ we therefore have $[\gamma, h] = g^p$ for some $g \in H$. We now compute

$$\begin{aligned} \delta(h-1) &= \sigma(h-1) - (h-1) = [\gamma, h]h - h \\ &= (g^p - 1)h = ((1 + (g-1))^p - 1)h \\ &= p(g-1)h + \big(\sum_{i \geq 2} \binom{p}{i} (g-1)^i\big)h \in \mathrm{Jac}(R)^2. \end{aligned}$$

This shows $\delta(\mathrm{Jac}(R)) \subseteq \mathrm{Jac}(R)^2$ and hence that (SI) is satisfied and that σ induces the identity on the associated graded ring.

(iii) The quotient $\mathrm{Jac}(R)/\mathrm{Jac}(R)^2$ as a κ-vector space is generated by the cosets of an uniformising element π of $\mathcal{O}$ and $h_1 - 1, \ldots, h_r - 1$ where $h_1, \ldots, h_r$ is a minimal system of topological generators of H. The assumption on H means that $[H, H] \subseteq H^{p^2}$. For any two h_i, h_j we therefore may write $[h_i, h_j] = h^{p^2}$ for some $h \in H$. We compute

$$\begin{aligned} &(h_i - 1)(h_j - 1) - (h_j - 1)(h_i - 1) = h_i h_j - h_j h_i = ([h_i, h_j] - 1)h_j h_i \\ &\quad = (h^{p^2} - 1)h_j h_i = ((1 + (h-1))^{p^2} - 1)h_j h_i \\ &\quad = p^2(h-1)h_j h_i + \frac{p^2(p^2-1)}{2}(h-1)^2 h_j h_i + \big(\sum_{k \geq 3} \binom{p^2}{k} (h-1)^k\big)h_j h_i \in \mathrm{Jac}(R)^3. \end{aligned}$$

This means that the corresponding cosets commute in the graded ring.

(iv) By (ii) and (iii) the assumptions in Proposition 1.16 are satisfied. $\square$

To establish the same facts about B also for general G and H we use the following descent technique. Let $G' \subseteq G$ be an open normal subgroup and put $H' := H \cap G'$, $R' := \Lambda(H')$, and $A' := \Lambda(G')$. We pick an element $\gamma' \in G'$ which topologically generates a pro-p-subgroup of G' and whose image in $G'/H' \cong \mathbb{Z}_p$ is a topological generator, and (suspending earlier notation from §1) we define $t' := \gamma' - 1$, $\sigma'(r') := \gamma' r' \gamma'^{-1}$ for $r' \in R'$, and $\delta' := \sigma' - \mathrm{id}$. Then $A' = R'[[t'; \sigma', \delta']]$ and we may define $B' := R' \ll t'; \sigma', \delta']]$. We introduce the Ore set $S' := \mathcal{C}_{A'}(\mathrm{Jac}(R')_{A'})$. As a piece of general notation we denote in the following by $\widehat{C}$ the $\mathrm{Jac}(C)$-adic completion of any given ring C.

Lemma 4.4. *The $\mathrm{Jac}(A_S)$-adic and the $\mathfrak{m}(N)A_S$-adic filtrations of A_S are equivalent.*

Proof. Since $\mathcal{N}(\Omega(G/N)) \subseteq \Omega(G/N)$ is nilpotent by [9, thm. 2.3.7] we obtain

$$\mathcal{N}(H)^n \subseteq \mathfrak{m}(N) \subseteq \mathcal{N}(H)$$

and thus (observing [9, prop. 2.1.16(vi)])

$$\mathrm{Jac}(A_S)^n \subseteq \mathfrak{m}(N)A_S \subseteq \mathrm{Jac}(A_S)$$

for some $n \in \mathbb{N}$. The claim follows. $\square$

We thank the referee for suggesting an alternative for our earlier, but erroneous proof of part (iii) in the next Proposition.

Proposition 4.5. *We have the natural identifications as bimodules:*
(i) $A_S = A'_{S'} \otimes_{A'} A$.
(ii) $\widehat{A_S} = \widehat{A'_{S'}} \otimes_{A'} A$.
(iii) $B = B' \otimes_{A'} A$.

Proof. (i) Obviously we have a crossed product representation

$$A = A' * G/G'.$$

Since regularity of ring elements is preserved under ring automorphisms S' is clearly G-stable, thus by [10, lem. 37.7] S' is also an Ore set in A consisting of regular elements and it holds that

$$A_{S'} = A'_{S'} \otimes_{A'} A$$

as bimodules. But by the subsequent Lemma 4.6 we have $A'_{S'} = A_S$. This establishes (i).

(ii) We choose our N in such a way that it is contained in H'. Let $\mathfrak{m}'(N)$ be the kernel of the projection $\Lambda(G') \to \Omega(G'/N)$, and note that

$$\mathfrak{m}(N)^n = A\mathfrak{m}'(N)^n = \mathfrak{m}'(N)^n A = \mathfrak{m}'(N)^n \otimes_{A'} A$$

for any $n \geq 1$. Using again the subsequent Lemma 4.6 we deduce that

$$\mathfrak{m}(N)^n_S = \mathfrak{m}(N)^n_{S'} = \mathfrak{m}'(N)^n_{S'} \otimes_{A'_{S'}} A_{S'} = \mathfrak{m}'(N)^n_{S'} \otimes_{A'_{S'}} A_S$$

and

$$A_S/\mathfrak{m}(N)^n_S = (A'_{S'}/\mathfrak{m}'(N)^n_{S'}) \otimes_{A'_{S'}} A_S = (A'_{S'}/\mathfrak{m}'(N)^n_{S'}) \otimes_{A'} A.$$

By passing to the projective limit with respect to n and using Lemma 4.4 we obtain (ii).

(iii) With N as in (ii) we let $\mathfrak{M}(N)$ denote the maximal ideal of $\Lambda(N)$. We set $R_k := R/\mathfrak{M}(N)^k R$, $R'_k := R'/\mathfrak{M}(N)^k R'$ as well as $A_k := R_k[[t;\bar\sigma,\bar\delta]]$, $A'_k := R'_k[[t';\bar\sigma',\bar\delta']]$, where $\bar\sigma,\bar\delta$ and $\bar\sigma',\bar\delta'$ are induced by σ,δ and σ',δ', respectively. Finally we put $B_k := (A_k)_T$ and $B'_k := (A'_k)_{T'}$ with $T = \{1,t,t^2,\ldots\}$ and $T' = \{1,t',t'^2,\ldots\}$. Note that we have $\mathfrak{M}(N)^k A = \mathfrak{m}(N)^k$ and $\mathfrak{M}(N)^k A' = \mathfrak{m}'(N)^k$. Using Lemma 1.12 it follows that

$$A_k = A/\mathfrak{m}(N)^k = A/\mathfrak{m}'(N)^k A \qquad \text{and} \qquad A'_k = A'/\mathfrak{m}'(N)^k$$

and hence, as above, that

$$A_k \cong A'_k \otimes_{A'} A$$

as bimodules and thus

$$(A_k)_{T'} \cong (A'_k)_{T'} \otimes_{A'} A = B'_k \otimes_{A'} A$$

as $B'_k - A$-bimodules by Proposition 1.2 (we only know that T' is an Ore set of A'_k). We claim that we have a natural isomorphism

$$B'_k \otimes_{A'} A \cong (A_k)_{T'} \cong (A_k)_T = B_k. \tag{4.1}$$

To this end, we first show that both B_k and B'_k are Artinian rings. Note that this does not immediately follow from the results in Section 2, because condition $(\mathrm{SI})_0$ need not be satisfied

for either A_k or A'_k, since it is entirely possible for A'_k to equal A_k : this happens for example when $G' = G$.
Now A_k is isomorphic to $\Lambda(G)/\mathfrak{M}(N)^k\Lambda(G)$ and hence pseudocompact, and $\bar{\sigma}$ and $\bar{\delta}$ commute. Therefore $\bar{\delta}$ is topologically nilpotent, by the remark following [11, lem. 1.6]. Because R_k is Artinian by construction (being a finite module over the Artinian ring $\Lambda(N)/\mathfrak{M}(N)^k$), we see that the "standard filtration" $I_\bullet$ on R_k is separated. Therefore B_k is Artinian by Proposition 1.6. The same argument applied to B'_k shows that this ring is also Artinian.
Since T' consists of regular elements in A'_k, it consists of regular elements in A_k because the latter ring is a free module over the former. So T' consists of regular elements in B_k. Since B_k is Artinian, T' consists of units in B_k by [9, prop. 3.1.1].
On the other hand, B'_k is the classical Artinian quotient ring of A'_k and $A_k = A'_k * (G/G')$ is a crossed product, meaning that $B'_k \otimes_{A'} A$ is the classical Artinian quotient ring of A_k by [10, lem. 37.7]. As T consists of regular elements in A_k, it consists of units in $B'_k \otimes_{A'} A$, again by [9, prop. 3.1.1].

Putting everything together we have established compatible isomorphisms

$$B_k \cong B'_k \otimes_{A'} A$$

of B'_k-modules; one easily checks that this isomorphism also respects the right A-module structures. Taking the inverse limit with respect to the canonical projections and using Remark 1.11 we obtain (iii). $\square$

We learned the following Lemma, which we have used in the above proof, from R. Sujatha. It also follows from [2, lem. 5.1] as was pointed out to us by the referee.

Lemma 4.6. $A_S = A_{S'}$.

Proof. The claim follows from Lemma 2.6 (i): (2.2) applied to $R' \hookrightarrow A' \hookrightarrow A$ says that all elements of S are already invertible in $A_{S'}$; thus the latter is also the localisation of A with respect to S. $\square$

Theorem 4.7. *We have:*

(i) *B is pseudocompact and Noetherian.*
(ii) *$B \cong \widehat{A_S}$.*
(iii) *B is flat over A_S and A.*

Proof. We choose an open normal subgroup $G' \subseteq G$ which is an extra-powerful pro-p-group ([17, prop. 8.5.2]). As G'/H' is torsionfree H' then is extra-powerful as well. The analogous assertions of our Theorem for B' and $\widehat{A'_{S'}}$ were already established in Lemma 4.3. Using Proposition 4.5 the fact that B is Noetherian and is flat over A_S then follows immediately by base extension.

The topology on B, by Lemma 1.12 (i), is given by the filtration $\mathrm{Jac}(R)^nB$. Since the pseudocompact ring R is finite free over the pseudocompact ring R' the same topology also is given by the filtration $\mathrm{Jac}(R')^nB$. This shows that the topology on B coincides with the natural topology of B as a finite free module (by Proposition 4.5 (iii)) over the pseudocompact ring B'. Hence B is pseudocompact.

By Proposition 4.5 (i) we have a ring homomorphism $B' \longrightarrow B$ and using Lemma 4.6 then the commutative diagram of solid arrows

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A'_{S'} \arrow[r] \arrow[d] & B' \arrow[d] \\
A \arrow[r] & A_S = A_{S'} \arrow[r, dashed] & B.
\end{tikzcd}$$

The dashed arrow then exists by the universal property of localisation. It is continuous for the $\mathrm{Jac}(A_S)$-adic topology on A_S since $\mathrm{Jac}(A_S) = \mathrm{Jac}(R)A_S$ by Lemma 4.2 (ii). But B is complete. Hence this arrow extends to a ring homomorphism

$$\widehat{A_S} \longrightarrow B.$$

As a homomorphism of bimodules this is, by Proposition 4.5, of course the base extension to A of the corresponding homomorphism $\widehat{A'_{S'}} \longrightarrow B'$. The latter is an isomorphism by Lemma 4.3 (i). Hence the former is an isomorphism as well. $\square$

5. Overconvergent skew Laurent series

In this section R denotes a Noetherian pseudocompact ring. We fix a ring norm $|\ |$ on R which defines the pseudocompact topology. To avoid confusion we recall that the function $|\ | : R \longrightarrow \mathbb{R}_{\geq 0}$ satisfies the axioms

(i) $|a - b| \leq \max(|a|, |b|)$,
(ii) $|a| = 0 \Longleftrightarrow a = 0$,
(iii) $|ab| \leq |a||b|$,
(iv) $|1| = 1$

for any $a, b \in R$. We also suppose that

(v) $|a| \leq 1$ for any $a \in R$.

As before σ is a topological ring automorphism of R and δ is a continuous left σ-derivation. We assume that

(vi) $|\sigma(a)| = |a|$ for any $a \in R$.

A standard example for a ring norm on R satisfying (i) – (vi) is given by

$$|a|_{\mathrm{Jac}} := 2^{-k} \qquad \text{if } a \in \mathrm{Jac}(R)^k \setminus \mathrm{Jac}(R)^{k+1}.$$

We also assume that there is a constant $0 < D < 1$ such that

(vii) $|\delta(a)| \leq D|a|$ for any $a \in R$.

In particular, δ is σ-nilpotent. It also follows that

$$|M_{k,l}(\delta, \sigma)(a)| \leq D^k|a| \qquad \text{for any } a \in R.$$

If $\delta(R) \subseteq \mathrm{Jac}(R)$ and $\delta(\mathrm{Jac}(R)) \subseteq \mathrm{Jac}(R)^2$ then $|\ |_{\mathrm{Jac}}$ satisfies this condition (vii) with $D := 2^{-1}$.

For any real constant $D < u < 1$ we now introduce the (left) R-submodule

$$B(|\ |; u) := \{\sum_{i\in\mathbb{Z}} a_i t^i \in B | \lim_{i\to-\infty} |a_i|u^i = 0\}$$

of B. It carries the norm $|\sum_{i\in\mathbb{Z}} a_i t^i|_u := \sup_{i\in\mathbb{Z}} |a_i|u^i$. By the proof of Proposition 1.10 the formula 1.9 gives a well-defined "multiplication" map $B \times B \longrightarrow B$.

Proposition 5.1. *This multiplication map restricts to a map*

$$B(|\ |;u) \times B(|\ |;u) \longrightarrow B(|\ |;u).$$

Proof. Let $x = \sum_{i\in\mathbb{Z}} a_i t^i$ and $y = \sum_{i\in\mathbb{Z}} b_i t^i$ be two arbitrary elements in $B(|\ |;u)$ and put $xy = \sum_{m\in\mathbb{Z}} c_m t^m$ in B with $c_m = c_m^+ + c_m^-$ as in (1.9) - (1.12). We have

$$|c_m^+|u^m \le \max_{j\ge n\ge 0} |a_j| \cdot |b_{m-n}|u^{m-n} \cdot u^n \le \max_{n\ge 0} |b_{m-n}|u^{m-n} = \max_{i\le m} |b_i|u^i$$

and

$$|c_m^-|u^m \le \max_{n\le j<0} |a_j|u^j \cdot |b_{m-n}|u^{m-n} \cdot \big(\frac{D}{u}\big)^{j-n}.$$

The first inequality obviously implies that $\lim_{m\to-\infty} |c_m^+|u^m = 0$. On the other hand, given any constant $\epsilon > 0$, we find natural numbers $N_0, N_1, N_2 > 0$ such that for $n \le j$ we have $|a_j|u^j \cdot |b_{m-n}|u^{m-n} \cdot \big(\frac{D}{u}\big)^{j-n} \le$

$$\begin{cases} |x|_u|y|_u \cdot \big(\frac{D}{u}\big)^{j-n} \le \epsilon & \text{for } j-n \ge N_1 \quad (\text{since } \frac{D}{u} < 1); \\ |a_j|u^j \cdot |y|_u \le \epsilon & \text{for } j \le -N_0 \quad (\text{since } \lim_{j\to-\infty} |a_j|u^j = 0); \\ |x|_u \cdot |b_{m-n}|u^{m-n} \le \epsilon & \text{for } m-n \le -N_2 \quad (\text{since } \lim_{k\to-\infty} |b_k|u^k = 0). \end{cases}$$

But $j-n < N_1, j > -N_0, m-n > -N_2$ together imply $m > -N_0 - N_1 - N_2$. It follows that $|c_m^-|u^m \le \epsilon$ for $m \le -N_0 - N_1 - N_2$. □

Corollary 5.2. *If the assumption* (I) *is satisfied then* $B(|\ |;u)$, *for any* $D < u < 1$, *as well as* $B^\dagger(|\ |) := \bigcup_u B(|\ |;u)$ *are subrings of* B.

Remark 5.3. (i) In general the ring $B^\dagger(|\ |)$ depends on the choice of the norm, but if one restricts to 'ideal norms' the ring $B^\dagger(|\ |)$ is independent of the particular choice of $|\ |$. More precisely let J_1 and J_2 be two ideals in R such that the J_i-adic filtrations define the given topology on R for $i = 1, 2$. Consider the norms

$$|a|_i := \rho_i^m \quad \text{if} \quad a \in J_i^m \setminus J_i^{m+1},$$

where $0 < \rho_i < 1$ are two real constants. Assuming that $|\ |_1$ and $|\ |_2$ also satisfy the conditions (vi) and (vii) above we have

$$B^\dagger(|\ |_1) = B^\dagger(|\ |_2).$$

(ii) The pseudocompact topology of R can always be defined by an 'ideal norm'.

Proof. In order to prove (i) we write $\rho_2 = \rho_1^\nu$ for some $\nu > 0$ and note that there exist natural numbers n_i, $i = 1, 2$, such that

$$J_1^{n_1} \subseteq J_2 \quad \text{and} \quad J_2^{n_2} \subseteq J_1. \tag{5.1}$$

holds. We shall show the following inequality

$$\rho_2^{n_2} |\ |_1^{n_2\nu} \le |\ |_2 \le \rho_2^{-1} |\ |_1^{\frac{\nu}{n_1}}, \tag{5.2}$$

which easily implies that $B^\dagger(|\ |_1) = B^\dagger(|\ |_2)$.

Let $a \in R$ be arbitrary and assume that $|a|_1 = \rho_1^n$ and $|a|_2 = \rho_2^m$ for some natural numbers n and m. Then from (5.1) we obtain $n < n_1(m+1)$, which is equivalent to $m > \frac{n}{n_1} - 1$, hence

$$|a|_2 = \rho_2^m < \rho_2^{\frac{n}{n_1}} \rho_2^{-1} = \rho_1^{\frac{\nu n}{n_1}} \rho_2^{-1} = |a|_1^{\frac{\nu}{n_1}} \rho_2^{-1},$$

which proves the second inequality, the first one is proven similarly.

For (ii) let $|\ |$ be any ring norm on R (satisfying (i) - (v)) which defines the topology of R. Then $J := \{a \in R : |a| < 1\}$ is an open ideal in R. Since R is Noetherian J is generated by finitely many elements $a_1, \ldots, a_s$. We put

$$\rho := \max\{|a| : a \in J\} = \max(|a_1|, \ldots, |a_s|)$$

and introduce the ideal norm

$$|a|' := \rho^m \qquad \text{if } a \in J^m \setminus J^{m+1}.$$

For a given $0 < \epsilon < 1$ choose an $m \in \mathbb{N}$ such that $\rho^m \leq \epsilon$. Then

$$J^m \subseteq \{a \in R : |a| \leq \rho^m\} \subseteq \{a \in R : |a| \leq \epsilon\}.$$

It follows that the J-adic topology is finer than the given topology. In fact we have

$$|\ | \leq |\ |'.$$

On the other hand each J^n/J^{n+1} is finitely generated over the Artinian (and Noetherian) ring R/J. It follows inductively that R/J^m is an R-module of finite length. Using [15, cor. 3.13] we see that J^m is open in R. Hence the J-adic topology coincides with the given topology of R.

□

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