Subgroup theorems for free profinite products with amalgamation

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Introduction

When Binz, Neukirch and Wenzel [1] proved the Kurosh subgroup theorem for free profinite products they had arithmetic applications in mind. For example, consider the Galois group $G(\mathbb{Q}(p)/\mathbb{Q}_\infty)$ of the maximal p-extension $\mathbb{Q}(p)$ of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$. It is known that a number theoretical analogue of Riemann's Existence Theorem holds (cf. [9]): $G(\mathbb{Q}(p)/\mathbb{Q}_\infty)$ is the free pro-p-product of certain decomposition groups. Therefore, in order to determine the structure of open subgroups of $G(\mathbb{Q}(p)/\mathbb{Q}_\infty)$ one can apply the pro-p version of Kurosh subgroup theorem.

But also the more general amalgamated free pro-$\mathcal{C}$-product occurs in algebraic number theory or arithmetic geometry, for example in the classical Seifert-van-Kampen theorem concerning topological fundamental groups or in the theorem about Galois groups of real function fields in one variable with restricted ramification, which we will consider below. Therefore the natural question arises, whether an analogous subgroup theorem for such amalgamations exists.

With regard to abstract groups Hanna Neumann showed in the 50th that in general subgroups of amalgamated products are not amalgamated products

2000 *Mathematical Subject Classification Codes.* Primary 20E18, 20E06; Secondary 05C25, 20E07.

any more, but"generalized free products". At the end of the 60th A. Karrass and D. Solitar described such subgroups by means of tree products and HNN-constructions ("HNN" stands for Highmann-Neumann-Neumann). But a satisfactory description was only given by the Bass-Serre theory, in which groups are acting on graphs in order to utilize geometric intuition: The fundamental group of a graph of groups generalizes both amalgamated products, HNN-constructions and tree products. From the structure theorem of this theory a subgroup theorem for such fundamental groups can be deduced, which contains H. Neumann, A. Karrass and D. Solitar' s results as well as the classical Kurosh subgroup theorem.

Various group theorists as Gildenhuys, Mel'nikov, Ribes or Zalesskii developed an analogous profinite Bass-Serre theory admitting only certain profinite topological spaces as graphs. Thereby they could derive important results concerning pro-$\mathcal{C}$-groups. Unfortunately, such a strong and general structure theorem does not exist within the profinite theory (but compare the introduction and §4 of [16]).

Therefore for the aim of this article it's more convenient to pursue Neukirch's method in order to obtain results on profinite groups by "carrying over" assertions about abstract groups to the profinite case by taking completions and projective limits.

In this article abstract graphs of pro-$\mathcal{C}$-groups $(\mathcal{G}, \Gamma)$ and their formal fundamental pro-$\mathcal{C}$-group $\pi_1(\mathcal{G}, \Gamma, T)$ are defined in such a way that we can transfer those results of the classical Bass-Serre theory which concern our question. The basic result of this paper is the following subgroup theorem for fundamental pro-$\mathcal{C}$-groups of graphs of groups:

Theorem 1 *If the canonical maps of the fundamental pro-$\mathcal{C}$group $\pi_1(\mathcal{G}, \Gamma, T)$ are injective, then each open subgroup $\mathcal{H} \subseteq \pi_1(\mathcal{G}, \Gamma, T)$ is again the fundamental group of an appropriate graph of groups:*

$$\mathcal{H} = \pi_1(\mathcal{H}, \Lambda, T')$$

Applying this result to amalgamated pro-$\mathcal{C}$-products in two special cases one gets the following corollaries:

Theorem 2 *If the amalgamated product* $\mathcal{G} = \mathop{\ast_{\mathcal{M}}}\limits_{i \epsilon I} \mathcal{G}_i$ *of pro-*$\mathcal{C}$*-groups* $\mathcal{G}_i$ *with common subgroup* $\mathcal{M}$ *exists, then each open subgroup* $\mathcal{H}$*, whose conjugates do not meet the amalgamation, is the free product of a free group* $\mathcal{F}$ *with subgroups of the form* ${\mathcal{G}_i}^g \ \cap \ \mathcal{H}$ *:*

$$\mathcal{H} = \mathop{\ast}\limits_{i \epsilon I, g \epsilon R_i} (\mathcal{G}_i^g \cap \mathcal{H}) \ast \mathcal{F}.$$

Theorem 3 *If the amalgamated product* $\mathcal{G} = \mathop{\ast_{\mathcal{N}}}\limits_{i \epsilon I} \mathcal{G}_i$ *of pro-*$\mathcal{C}$*-groups* $\mathcal{G}_i$ *with common normal subgroup* $\mathcal{N}$ *exists, then each open subgroup* $\mathcal{H}$ *has got the form*

$$\mathcal{H} = \mathop{\ast_{\mathcal{M}}}\limits_{i \epsilon I, g \epsilon R_i} (\mathcal{G}_i^g \cap \mathcal{H}) \ast_{\mathcal{M}} \mathcal{F}\mathcal{M},$$

where $\mathcal{F}\mathcal{M}$ *means the semi-direct product of* $\mathcal{M} = \mathcal{H} \cap \mathcal{N}$ *by a free pro-*$\mathcal{C}$*-group* $\mathcal{F}$ *(for details see section 1.4).*

We conclude with an illustration of theorem 2: The Riemann existence and uniqueness theorem makes it possible to deduce field theoretic and arithmetic conclusions about a function field K of transcendence degree 1 over $\mathbb{C}$ from the analytic, geometric properties of the compact Riemann surface associated to K.

If S denotes a finite set of primes of $\mathbb{R}(t)$ and $\bar{S}$ the set of places of $\mathbb{C}(t)$ lying above S, the Galois group G_S of the maximal extension $\mathbb{C}(t)_S$ of $\mathbb{C}(t)$ which is unramified outside S is isomorphic to the profinite completion of the topological fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \bar{S}$. By this principle, Krull and Neukirch were able not only to show that

$$G_S = \mathop{\ast}\limits_{\mathfrak{p} \epsilon \bar{S} \setminus \{\mathfrak{p}_0\}} \mathcal{G}_{\mathfrak{p}}$$

is the free product of decomposition groups of suitable extensions of $\mathfrak{p} \ \epsilon \ \bar{S} \setminus \{\mathfrak{p}_0\}$, $\mathfrak{p}_0 \ \epsilon \ \bar{S}$ a fixed place, but also to represent the Galois group $\mathcal{G}_S$ of the maximal extension $\mathbb{R}(t)_S$ of $\mathbb{R}(t)$ unramified outside S by generators and relations. As the group extensions

$$1 \longrightarrow G_S \longrightarrow \mathcal{G}_S \longrightarrow G(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

splits, it's sufficient to determine the action of the complex conjugation on the decomposition groups $\mathcal{G}_{\mathfrak{p}}$.

By a slight modification, Wingberg found a representation of $\mathcal{G}_S$ as profinite product with amalgamated subgroup Γ - the image of $G(\mathbb{C}/\mathbb{R})$ with respect to the section:

$$\mathcal{G}_S = \mathop{\ast_\Gamma}_{i\epsilon\{1,\ldots,r\}} \mathcal{G}_{r,i} \ast_\Gamma \mathop{\ast_\Gamma}_{j\epsilon\{1,\ldots,c\}} \mathcal{G}_{c,j}$$

(cf. section 2.1 where the groups $\mathcal{G}_{r,i}$, $\mathcal{G}_{c,j}$ are defined). The application of theorem 2 yields:

Theorem 4 *Let K be a normal extension of $\mathbb{R}(t)$ unramified outside S that is not contained in the fixed field $(\mathbb{R}(t)_S)^\Gamma$ of $\mathbb{R}(t)_S$ under Γ. Then the Galois group $\mathcal{H} = G(\mathbb{R}(t)_S/K)$ is the free product of certain subgroups $\mathcal{G}^g_{r,i} \cap \mathcal{H}$, $\mathcal{G}^g_{c,j} \cap \mathcal{H}$ and a free group $\mathcal{F}$:*

$$\mathcal{H} = \mathop{\ast}_{i\epsilon\{1,\ldots,r\}, g\epsilon R_{r,i}} (\mathcal{G}^g_{r,i} \cap \mathcal{H}) \ast \mathop{\ast}_{j\epsilon\{1,\ldots,c\}, g\epsilon R_{c,j}} (\mathcal{G}^g_{c,j} \cap \mathcal{H}) \ast \mathcal{F}$$

For an extension which is not Galois the theorem is true, if all conjugates gK are not contained in $(\mathbb{R}(t)_S)^\Gamma$.

♦ ♦ ♦

The author would like to express his thanks to L. Ribes for some discussions on this subject.

1 Fundamental pro-$\mathcal{C}$-groups of graphs of groups

During the whole paper let $\mathcal{C}$ be a class of finite groups, closed under the formation of subgroups, homomorphic images and group extensions. Then $\mathcal{PC}$ denotes the category of pro-$\mathcal{C}$-groups, i.e. projective limits of groups in $\mathcal{C}$, and all homomorphisms of pro-$\mathcal{C}$-groups are assumed to be continuous as well as subgroups to be closed.

1.1 Graphs of pro-$\mathcal{C}$-groups

A graph Γ consists of a non-empty set $V = V(\Gamma)$ of vertices, a set $E = E(\Gamma)$ of oriented edges, two maps $o, \tau : E \longrightarrow V$ that determine the origin $o(e)$ and the terminus $\tau(e)$ of the edge e, as well as of an inversion $^{-}: E \longrightarrow E$ with $o(\bar{e}) = \tau(e)$ and $\bar{e} \neq e$, $e \in E$, which maps the edge e to that of opposite orientation $\bar{e}$. A subset $O(\Gamma)$ is called orientation of Γ, if $E(\Gamma) = O(\Gamma) \dot{\cup} \overline{O(\Gamma)}$ is the disjoint union of $O(\Gamma)$ and its inverse. A graph is uniquely determined up to isomorphism by the sets $V(\Gamma)$ and $O(\Gamma)$. It is well known that each graph possesses a geometric realization (e.g. as quotient space of the topological sum $V(\Gamma) \coprod (O(\Gamma) \times [0,1])$ with respect to the smallest relation that implies $(e,0) \sim o(e)$ and $(e,1) \sim \tau(e)$). Γ is called a tree, if this geometric realization is simply connected.

Using Zorn' s lemma it can be shown that each connected graph Γ possesses a - in general not unique - maximal tree T as a subgraph, which fulfills $V(\Gamma) = V(T)$ (cf. [2], Chapter I.1, or [13] I.2.3).

Since in this article graphs are only used for combinatorial purposes and as a formal device in order to carry over results from the classical Bass-Serre theory, neither Boolean [4] nor profinite [15] graphs need to be considered. So the *abstract* graph of groups is defined as in the classical theory (cf. [2] and for finite graphs [15] §3):

Definition 1 (i) *An abstract graph of pro-$\mathcal{C}$-groups $(\mathcal{G}, \Gamma)$ is a functor*

$$\mathcal{G} : \Gamma \longrightarrow \mathcal{PC}$$

that assigns to

(a) *$P \in V(\Gamma)$ a pro-$\mathcal{C}$-group $\mathcal{G}_P$,*

(b) *$e \in E(\Gamma)$ a pro-$\mathcal{C}$-group $\mathcal{G}_e$ with $\mathcal{G}_e = \mathcal{G}_{\bar{e}}$ as well as an* injective *homomorphism $-^e : \mathcal{G}_e \longrightarrow \mathcal{G}_{\tau(e)}$.*

(ii) *A morphism $\psi : (\mathcal{G}, \Gamma) \longrightarrow (\mathcal{G}', \Gamma)$ of graphs of groups consists of two homomorphisms*

(a) *$\psi_P : \mathcal{G}_P \longrightarrow \mathcal{G}'_P$, $P \in V(\Gamma)$,*

(b) *$\psi_e : \mathcal{G}_e \longrightarrow \mathcal{G}'_e$ with $\psi_e = \psi_{\bar{e}}$, $e \in E(\Gamma)$,*

such that $\psi_e(g)^e = \psi_{\tau(e)}(g^e)$, $g \in \mathcal{G}_e$, $e \in E(\Gamma)$.

The idea of a fundamental group of an object X in an arbitrary category is to classify certain coverings over a fixed basis X - provided that all these concepts make sense at all. But in our context, in which we don't consider topological versions of graphs of groups, we cannot expect such a characterization. Nevertheless we know define the fundamental group of graphs just as *formal* analogue:

Definition 2 *Let Γ be a connected graph and let $(\mathcal{G}, \Gamma)$ be an abstract graph of pro-$\mathcal{C}$-groups with maximal tree $T \subseteq \Gamma$.*

(i) *A convergent T-specialization $(\mathcal{H}, \psi, t)$ of $(\mathcal{G}, \Gamma)$ into a pro-$\mathcal{C}$-group $\mathcal{H}$ with respect to T consists of a system of*

  (a) *homomorphisms $\psi_P : \mathcal{G}_P \longrightarrow \mathcal{H},\ P \in V(\Gamma)$,*

  (b) *elements $t_e \in \mathcal{H},\ e \in O(\Gamma)$,*

  with the following properties:

  (1) $t_e = 1$ *for* $e \in O(T)$,

  (2) $\psi_{o(e)}(g^{\bar{e}}) = t_e \psi_{\tau(e)}(g^e) t_e^{-1},\ e \in O(\Gamma),\ g \in \mathcal{G}_e$,

  (3) for all open subgroups $\mathcal{U} \subseteq \mathcal{H}$ is valid: $t_e \in \mathcal{U}$ and $\psi_P(\mathcal{G}_P) \subseteq \mathcal{U}$ for almost all $e \in O(\Gamma)$ and $P \in V(\Gamma)$.

(ii) *The restricted formal fundamental pro-$\mathcal{C}$-group of the graph of groups $(\mathcal{G}, \Gamma)$ with respect to T is the uniquely determined convergent T-specialization $(\pi_1(\mathcal{G}, \Gamma, T), \psi, t)$ with the following universal property:*
For every convergent T-specialization $(\mathcal{H}, \varphi, x)$ there is exactly one homomorphism $\omega : \pi_1(\mathcal{G}, \Gamma, T) \longrightarrow \mathcal{H}$, such that

  (a) $\omega(t_e) = x_e,\ e \in O(\Gamma)$,

  (b) $\omega \circ \psi_P = \varphi_P,\ P \in V(\Gamma)$.

The uniqueness is clear by the universal property and the existence is established by the following

Construction: (1) (cf. [15] (3.3)) Let $(\pi_1^d(\mathcal{G},\Gamma,T),\psi^d,t^d)$ denote the discrete fundamental group with the injective maps ψ_P^d ([2], cor. 2.7, p. 43). On $\pi_1^d := \pi_1^d(\mathcal{G},\Gamma,T)$ we can define a topology by the system $\mathcal{B}$ of normal subgroups $N \trianglelefteq \pi_1^d$ with

(i) $\pi_1^d/N \in C$,

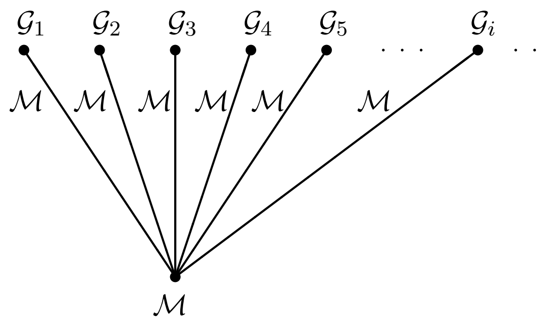
(ii) $N \cap \mathcal{G}_P \subseteq \mathcal{G}_P$ open, $P \in V(\Gamma)$, and

(iii) $\mathcal{G}_P \subseteq N$ $(t_e^d \in N)$ for almost all $P \in V(\Gamma)$ $(e \in O(\Gamma))$.

The completion $\pi_1(\mathcal{G},\Gamma,T)$ of π_1^d with respect to this topology results in the desired fundamental group together with the maps $\psi_P := \lambda \circ \psi_P^d$ and the elements $t_e := \lambda(t_e^d)$. Here $\lambda : \pi_1^d(\mathcal{G},\Gamma,T) \longrightarrow \pi_1(\mathcal{G},\Gamma,T)$ denotes the canonical map of the completion.

(2)(cf. [16] (2.2)) Alternatively we can build the restricted free product $\mathcal{W} = \mathop{\ast}\limits_{V(\Gamma)} \mathcal{G}_P \ast \mathcal{F}_{O(\Gamma)\setminus T}$ of the groups $\mathcal{G}_P$ and the freely on $O(\Gamma)\setminus T$ generated pro-$\mathcal{C}$-group $\mathcal{F}_{O(\Gamma)\setminus T}$. Its factor group $\mathcal{W}/\mathcal{N}$ by the smallest normal subgroup $\mathcal{N}$ containing all relations of the form $g^e = t_e g^{\bar{e}} t_e^{-1}$, $e \in O(\Gamma)$ has the desired properties.

Example 1 (cf. [13] I.4.4, Example (c)) The restricted pro-$\mathcal{C}$-push-out of a family $(\mathcal{G}_i)_{i \in I}$ of pro-$\mathcal{C}$-groups over a common subgroup $\mathcal{M}$ is an important example for such a fundamental group: Define via $V(\Gamma) := I \cup \{\star\}$, $E(\Gamma) := \{(\star,i) \mid i \in I\} \cup \{(i,\star) \mid i \in I\}$ with $o(\star,i) = \star$, $\tau(\star,i) = i$, $\overline{(\star,i)} = (i,\star)$ the tree Γ with vertex groups $\mathcal{G}_i$, $i \in I$, $\mathcal{G}_\star := \mathcal{M}$, and edge groups $\mathcal{G}_e := \mathcal{M}$, $e \in E(\Gamma)$, and choose as maps $-^{(\star,i)}$ the canonical embeddings $\mathcal{M} \longrightarrow \mathcal{G}_i$ and for $-^{(i,\star)}$ the identity $\mathcal{M} \longrightarrow \mathcal{M}$ respectively:



The fundamental group of this graph of groups is a generalized push-out in the category $\mathcal{PC}$ [11]. If the canonical maps $j_i : \mathcal{G}_i \longrightarrow \pi_1(\mathcal{G}, \Gamma, \Gamma)$ are injective, this group is called *free pro-$\mathcal{C}$-product*

$$\mathop{\ast_{\mathcal{M}}}_{i \epsilon I} \mathcal{G}_i$$

of the $\mathcal{G}_i$ with amalgamated subgroup $\mathcal{M}$. For the trivial subgroup $\mathcal{M} = \{1\}$ it is exactly the *free pro-$\mathcal{C}$-product*

$$\mathop{\ast}_{I} \mathcal{G}_i$$

in Neukirch's sense [8]. While the free product always exists, i.e. the canonical maps j_i are automatically injective, this is not true in the case of the amalgamation as was shown by Ribes in [11]. We should point out that for non-trivial $\mathcal{M}$ the free amalgamated product only exists for *finite* families $\mathcal{G}_i$ as can be seen easily using the "restriction-condition" (i) (3) in definition 2 and taking into consideration that the above graph is connected. Some criteria for its existence are listed below.

The adjective "restricted" should serve to tell our definition from other ones in the literature that do not demand the convergence of an T-specialization. But we are omitting this attribute in the following. Anyhow, for finite graphs or finite index sets it does not make any difference.

1.2 Classical Bass-Serre theory

As we are going to deduce the results in the next section from the classical Bass-Serre theory by completion, we recall some notations and the structure theorem at this place. All groups are *discrete* in this section!

Let G be a group that acts without inversion on an non-empty connected graph X, i.e. G acts on an orientation $O(X)$. Then we get a graph of groups (G, Γ) with quotient graph $\Gamma = G\backslash X$ in the following way: Choose a lifting $j : T \longrightarrow X$ of a maximal tree T of Γ to X and an orientation $O(\Gamma)$ of Γ. Further we assign to each $e \epsilon \ E(\Gamma) \setminus T$ an edge $je \epsilon \ E(X)$ above e with $o(je) \epsilon \ V(jT)$ and $x_e \epsilon \ G$, such that $\tau(je) = x_e j\tau(e)$ holds. Setting $G_P = G_{jP}$, $P \epsilon \ V(\Gamma)$, and $G_e = G_{je}$, $e \epsilon \ O(\Gamma)$, which are the isotropy groups of jP and je, respectively, and taking the homomorphisms $-^e : G_e \longrightarrow G_{\tau(e)}$, $g \mapsto x_e^{-1} g x_e$ and the inclusions $G_e \longrightarrow G_{o(e)}$, $e \epsilon \ O(\Gamma)$, for the maps $-^{\bar{e}}$, we get a graph of groups as well as a canonical map $\phi : \pi_1^d(G, \Gamma, T) \longrightarrow G$.

Structure theorem (Bass-Serre) *In this context the following assertions are equivalent:*

(i) X *is a tree.*

(ii) $\phi : \pi_1^d(G, \Gamma, T) \longrightarrow G$ *is a isomorphism.*

Proof: [13] I.5.4, theorem 13 or [2] I, theorems 6.1, 6.2 and 5.3. □

In addition, it is known that the fundamental group $\pi_1^d = \pi_1^d(G, \Gamma, T)$ of a graph of groups (G, Γ) with a non-empty connected graph Γ always acts without inversion on the standard tree $S = S(G, \Gamma, T)$. S consists of the vertices $V(S) = \coprod_{P \,\epsilon\, V(\Gamma)} \pi_1^d / G_P$ (disjoint union) and the orientation $O(S) = \coprod_{e \,\epsilon\, O(\Gamma)} \pi_1^d / G_e$ together with maps $o(gG_e) = gG_{o(e)}$ and $\tau(gG_e) = gt_e G_{\tau(e)}$. Here G_P and G_e are identified with its images with respect to φ_P, $P \epsilon \ V(\Gamma)$, and $\varphi_{o(e)}(-^{\bar{e}})$, $e \epsilon \ O(\Gamma)$, respectively.

1.3 A subgroup theorem

As before let Γ be a non-empty connected graph with maximal tree T such that $r_\Gamma := \#(O(\Gamma) \setminus T) < \infty$.

Theorem 1 *Let $(\pi_1(\mathcal{G},\Gamma,T),\psi,t)$ be the formal fundamental pro-$\mathcal{C}$-group of a graph of groups $(\mathcal{G},\Gamma)$ and let $\mathcal{H} \subseteq \pi_1(\mathcal{G},\Gamma,T)$ be an open subgroup.*
If all the ψ_P are injective, then $\mathcal{H}$ is the fundamental pro-$\mathcal{C}$-group of a graph of groups $(\mathcal{H},\Lambda)$

$$\mathcal{H} \cong \pi_1(\mathcal{H},\Lambda,T')$$

with the vertex groups $\mathcal{H}_{P,g} = \mathcal{H} \cap g\mathcal{G}_P g^{-1},\ g \in R_P,\ P \in V(\Gamma)$ and the edge groups $\mathcal{H}_{e,g} = \mathcal{H} \cap g\mathcal{G}_e g^{-1},\ g \in R_e,\ e \in E(\Gamma)$, while R_P and R_e denote suitable *systems of representatives for the double cosets $\mathcal{H}\backslash\pi_1/\mathcal{G}_P$ and $\mathcal{H}\backslash\pi_1/\mathcal{G}_e$.*
Furthermore, the canonical maps φ_P are injective again.

Remark 1 Before we give the proof of this statement, we would like to mention that the important invariant $r_{\mathcal{H}} := \#(O(\Lambda) \setminus T')$ can be calculated by the following formula:

$$(1) \qquad r_{\mathcal{H}} = \sum_{e \in O(\Gamma\cap T)} (\#R_e - \#R_{P_e}) - \#R_{P_0} + \sum_{e \in O(\Gamma)\setminus T} \#R_e + 1.$$

Here $P_e,\ e \in T,\ P_0$ are arbitrary chosen such that they fulfill $\{P_e \mid e \in T\} \cup \{P_0\} = V(\Gamma)$. Observe that $\#R_e = \#R_{P_e}$ for almost all $e \in O(\Gamma)$, since $\mathcal{G}_P \subseteq \mathcal{H}$ for almost all $P \in V(\Gamma)$, i.e. the sums are both finite ($O(\Gamma) \setminus T$ was required to be finite). This formula generalizes both that in [1] and that in [13] I.5.5, Exercise 2), p. 57. The simple proof is left to the interested reader.

Proof: (cf. the proofs of the Kurosh subgroup theorem in [1] and [3], Theorem 3.2). Consider the following diagram

$$\begin{array}{ccc} \pi_1^d = \pi_1^d(\mathcal{G},\Gamma,T) & \xrightarrow{\lambda} & \pi_1 = \pi_1(\mathcal{G},\Gamma,T) \\ \uparrow & & \uparrow \\ H := \lambda^{-1}(\mathcal{H}) & \xrightarrow{\lambda_{|H}} & \mathcal{H} \end{array}$$

H acts on the standard tree $S = (\mathcal{G},\Gamma,T)$ without inversion, since π_1^d does. Conferring the structure theorem we conclude that $H \cong \pi_1^d(H,\Lambda,T')$ with quotient graph $\Lambda = H\backslash S$ and a maximal tree T' of Λ. A section $j : T' \longrightarrow S$, which can be extended to $E(\Gamma)$ as in paragraph 1.2, defines *modulo* $\mathcal{G}_P$ systems of representatives R_P^d for the double cosets $H\backslash\pi_1^d/\mathcal{G}_P$ (respectively *modulo* $\mathcal{G}_e$ systems R_e^d for $H\backslash\pi_1^d/\mathcal{G}_e$) and it holds:

$$H_{Hg\mathcal{G}_P} = H \cap g\mathcal{G}_P g^{-1} \text{ for } g \in R_P^d, \text{i.e. } Hg\mathcal{G}_P \in V(\Lambda) = \coprod_{P \in V(\Gamma)} H\backslash\pi_1^d/\mathcal{G}_P$$

and

$$H_{Hg\mathcal{G}_e} = H \cap g\mathcal{G}_e g^{-1} \text{ for } g \in R_e^d, \text{i.e. } Hg\mathcal{G}_e \in O(\Lambda) = \coprod_{e \in O(\Gamma)} H\backslash\pi_1^d/\mathcal{G}_e.$$

The embeddings of the vertex groups into the edge groups were defined in section 1.2. As the canonical maps ψ_P are injective by assumption, we can identify the groups $H_{Hg\mathcal{G}_P}$ with the groups $\mathcal{H}_{P,\lambda(g)} := \mathcal{H} \cap \lambda(g)\mathcal{G}_P\lambda(g)^{-1}$ (respectively $H_{Hg\mathcal{G}_e}$ with $\mathcal{H}_{e,\lambda(g)} := \mathcal{H} \cap \lambda(g)\mathcal{G}_e\lambda(g)^{-1}$). It is easy to verify that λ induces a bijection between R_P^d and a system of representatives R_P for $\mathcal{H}\backslash\pi_1/\mathcal{G}_P$ (respectively between R_e^d and a system R_e for $\mathcal{H}\backslash\pi_1/\mathcal{G}_e$).

Let $\mathcal{T}_H$ denote the topology defined by the system $\mathcal{B}_H$ of normal subgroups $I \trianglelefteq H$ satisfying

(i) $H/I \in C$,

(ii) $I \cap (H \cap \mathcal{G}_P{}^g) \subseteq H \cap \mathcal{G}_P{}^g$ open, $(P,g) \in J := \bigcup_{P \in V(\Gamma)} \{P\} \times R_P^d$, and

(iii) $H \cap \mathcal{G}_P{}^g \subseteq I$ for almost all $(P,g) \in J$.

Looking back to the construction (1) of the fundamental group we see that our result follows if we can show $\mathcal{T}_H = \mathcal{T}_{\pi_1^d{}_{|H}}$, i.e. the topology that is induced by that of π_1^d. But the proof of this fact is completely analogous as in [1].[1] □

For finite graphs Zalesskii and Mel'nikov prove the same result using the profinite Bass-Serre theory they defined ([16], Cor. 4.5 of Prop. 4.4).

[1] It should be taken into account that there $H/\tilde{I}$ has to be corrected by $\tilde{H}/\tilde{I}$ in line 22 of page 167 in order to deduce $\tilde{H}/\tilde{I} \in \mathcal{C}$ directly.

1.4 Application to amalgamated products

One consequence of theorem 1 is the following modification of the Kurosh subgroup theorem including certain amalgamation:

Theorem 2 *Let* $\{\mathcal{M} \subseteq \mathcal{G}_i\}_{i \,\epsilon\, I}$ *be a family of pro-*$\mathcal{C}$*-groups* $\mathcal{G}_i$ *with common subgroup* $\mathcal{M}$ *such that the amalgamated product* $\mathcal{G} = \mathop{\ast_{\mathcal{M}}}\limits_{i \epsilon I} \mathcal{G}_i$ *exists. If* $\mathcal{H}$ *is an open subgroup of* $\mathcal{G}$ *with the property:*

$$(\star) \qquad \mathcal{H}^g \cap \mathcal{M} = \{1\} \quad \textit{for all } g \,\epsilon\, \mathcal{G},$$

then there is a free pro-$\mathcal{C}$*-group* $\mathcal{F} \subseteq \mathcal{H}$ *and suitable systems* R_i *of representatives for the double cosets* $\mathcal{H}\backslash\mathcal{G}/\mathcal{G}_i$, $i \,\epsilon\, I$, *such that* $\mathcal{H}$ *possesses the following representation:*

$$\mathcal{H} = \mathop{\ast}_{i \epsilon I, g \epsilon R_i} (\mathcal{G}_i^g \cap \mathcal{H}) \ast \mathcal{F},$$

where ${\mathcal{G}_i}^g = g\mathcal{G}_i g^{-1}$. *Furthermore* $\mathcal{F}$ *has finite rank*

$$r_{\mathcal{F}} = \sum_{i \,\epsilon\, I} (\#R_{\mathcal{M}} - \#R_i) - \#R_{\mathcal{M}} + 1,$$

in which $R_{\mathcal{M}}$ *denotes an arbitrary system of representatives for* $\mathcal{H}\backslash\mathcal{G}/\mathcal{M}$.

Proof: According to example (1) the group $\mathcal{G}$ is isomorphic to $\pi_1(\mathcal{G}, \Gamma, \Gamma)$ and the canonical maps ψ_P are injective by assumption. So theorem 1 implies

$$\mathcal{H} \cong \pi_1(\mathcal{H}, \Lambda, T')$$

with vertex groups $\mathcal{H}_{i,g} = \mathcal{H} \cap {\mathcal{G}_i}^g$, $g \,\epsilon\, R_i$ ($\mathcal{H}_{\star,g} = \mathcal{H} \cap \mathcal{M}^g = \{1\}$, $g \,\epsilon\, R_{\mathcal{M}}$, because of $(\star)$) and edge groups $\mathcal{H}_{(\star,i),g} = \mathcal{H} \cap \mathcal{M}^g = \{1\}$ due to $(\star)$. Since the edge groups are trivial the relations (2) in the fundamental group are also trivial: $t_e t_e^{-1} = 1$. Therefore the isomorphism $\pi_1(\mathcal{H}, \Lambda, T') \cong \mathop{\ast}\limits_{i \epsilon I, g \epsilon R_i} (\mathcal{G}_i^g \cap \mathcal{H}) \ast \mathcal{F}$ results from the universal properties of both objects. At this place $\mathcal{F}$ denotes the free pro-$\mathcal{C}$-group on generators $\{t_e \mid e \,\epsilon\, O(\Lambda) \setminus T'\}$, whence it has rank

$$r_{\mathcal{F}} = \sum_{i \,\epsilon\, I} (\#R_{\mathcal{M}} - \#R_i) - \#R_{\mathcal{M}} + 1,$$

according to the remark to theorem 1 (Observe that $O(\Gamma) \setminus \Gamma = \emptyset$). □

The following theorem is the pro-$\mathcal{C}$-analogue of a theorem by Hanna Neumann [10], Theorem 13.4, but which sometimes also is ascribed to H.W. Kuhn [5]:

Theorem 3 *Let* $\{\mathcal{N} \trianglelefteq \mathcal{G}_i\}_{i \,\epsilon\, I}$ *be a family of pro-*$\mathcal{C}$*-groups* $\mathcal{G}_i$ *with common normal subgroup* $\mathcal{N}$ *such that the amalgamated product* $\mathcal{G} = \mathop{\ast_{\mathcal{N}}}\limits_{i \epsilon I} \mathcal{G}_i$ *exists. If* $\mathcal{H}$ *is an open subgroup of* $\mathcal{G}$*, then there is*

(i) *a free pro-*$\mathcal{C}$*-group* $\mathcal{F} \subseteq \mathcal{H}$ *and*

(ii) *suitable systems* R_i *of representatives for the double cosets* $\mathcal{H}\backslash\mathcal{G}/\mathcal{G}_i$*,* $i \,\epsilon\, I$*,*

such that $\mathcal{H}$ *possesses the following representation:*

$$\mathcal{H} = \mathop{\ast_{\mathcal{M}}}\limits_{i \epsilon I, g \epsilon R_i} (\mathcal{G}_i^g \cap \mathcal{H}) \ast_{\mathcal{M}} \mathcal{F}\mathcal{M},$$

in which $\mathcal{M} = \mathcal{N} \cap \mathcal{H}$ *and* $\mathcal{F}\mathcal{M}$ *is the semi-direct product of* $\mathcal{M}$ *by* $\mathcal{F}$*.*

Furthermore, $\mathcal{F}$ *has finite rank*

$$r_{\mathcal{F}} = \sum_{i \,\epsilon\, I} (\#R_{\mathcal{N}} - \#R_i) - \#R_{\mathcal{N}} + 1,$$

in which $R_{\mathcal{N}}$ *denotes an arbitrary system of representatives for* $\mathcal{H}\backslash\mathcal{G}/\mathcal{N}$*.*

Proof: By means of the universal properties of all objects which are considered this theorem results similarly as theorem 2 from theorem 1. □

Remark 2 L. Ribes has shown several criteria that guarantee the existence of the amalgamated product $\mathop{\ast_{\mathcal{M}}}\limits_{i \epsilon I} \mathcal{G}_i$:

(i) Let $\mathcal{C}$ be the class of *all* finite groups and suppose that one of the following conditions holds:

  (a) $\#I < \infty$ and $\mathcal{M}$ is finite,

(b) $\#I = 2$ and $\mathcal{G}_1 = \mathcal{G}_2$,

(c) $\#I < \infty$ and $\mathcal{M} \subseteq center(\mathcal{G}_i)$ for all but one $i \in I$ or

(d) $\#I < \infty$, $\mathcal{M} \trianglelefteq \mathcal{G}_i,\ i \in I$ and $\mathcal{M}$ is (topologically) finitely generated.

(ii) Let $\mathcal{C}$ be the class of p-groups, p a fixed prime number, as well as $\#I < \infty$ and $\mathcal{M}$ is procyclic.

(iii) Let $\mathcal{P}$ be a set of primes and $\mathcal{C}$ the class of finite nilpotent groups, whose order is divisible only by the primes of $\mathcal{P}$, as well as $\#I < \infty$ and $\mathcal{M}$ is procyclic.

For the proofs and other, more general criteria see [11] and [12]: (i) (a) Corollary 1.3 in [11], (b) Theorem 2.1 in [11] (c) Theorem 2.3 in [11] respectively (d) Theorem 2.4 in [11] (ii) Theorem 3.2 in [11] and (iii) Theorem 3.4 in [12].

Ribes proved the existence in each case only for two factors. But as the push-out with a finite number of factors is built up inductively, the statements follow for $2 < \#I < \infty$ - except (i) (b), since generally $\mathcal{G} *_{\mathcal{M}} \mathcal{G} \neq \mathcal{G}$. □

1.5 An example: The Galois group $G(\mathbb{R}(t)_S/\mathbb{R}(t))$ of the maximal extension of $\mathbb{R}(t)$ unramified outside S and its open subgroups

In this section we want to illustrate the above result on amalgamated free products in the context of real function fields in one variable.

Let S be a finite set of places of $\mathbb{R}(t)$ inclusive of the infinite place $\mathfrak{p}_\infty$ (We make this convention only in order to fix notations). With $\bar{S}$ we shall mean the set of places of $\mathbb{C}(t)$ lying above S. As these places correspond uniquely to closed points in $\mathbb{P}^1(\mathbb{C})$ we can write

$$\bar{S} = \{a_1 < a_2 < \cdots < a_r,\ a_i \in \mathbb{R}\} \cup \{\alpha_j,\ \bar{\alpha}_j \in \mathbb{C} \setminus \mathbb{R},\ 1 \leq j \leq c\} \cup \{\infty\}$$

with real points a_i and pairs of complex places $\alpha_j,\ \bar{\alpha}_j$. After choosing a fixed base point $x_0 \in \mathbb{R} \setminus \bar{S}$, $x_0 < a_1$, it is possible to assign a path within the Gauss plane to each prime of $\mathbb{C}(t)$ (cf. [14]).

For the Galois group $G_S := G(\mathbb{C}(t)_{\bar{S}}/\mathbb{C}(t))$ of the maximal extension $\mathbb{C}(t)_S$ of $\mathbb{C}(t)$ which is unramified outside S and the Galois group $\mathcal{G}_S := G(\mathbb{R}(t)_S/\mathbb{R}(t))$ the following statements hold:

Theorem (Krull-Neukirch [6, Satz 1])

(i) *The exact sequence*

$$1 \longrightarrow G_S \longrightarrow \mathcal{G}_S \overset{s}{\underset{\longrightarrow}{\longleftarrow}} G(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

splits "canonically" by extending the complex conjugation to the "Spiegel"-automorphism γ of $\mathbb{R}(t)_S$.[2]

(ii) *The subgroups of G_S which are (topologically) generated by the homotopy classes associated to primes $\mathfrak{p} \in \bar{S}$ are free and they are decomposition groups with respect to this primes:*

$$G_{\mathfrak{p}} := \begin{cases} (\varrho_j) & \varrho_j \mathrel{\widehat{=}} \alpha_j \\ (\bar{\varrho}_j) \quad if & \bar{\varrho}_j \mathrel{\widehat{=}} \bar{\alpha}_j \\ (\tau_i) & \tau_i \mathrel{\widehat{=}} a_i \end{cases}$$

(iii) *The Galois group G_S is the free product of the decomposition groups $G_{\mathfrak{p}}$ of the finite places $\mathfrak{p}$ in $\bar{S}$:*

$$G_S = \mathop{\ast}_{\mathfrak{p} \in \bar{S} \setminus \{\infty\}} G_{\mathfrak{p}}$$

[2]The extension $\mathbb{R}(t)_S = \mathbb{C}(t)_S$ can be identified with the field of meromorphic functions $K(\bar{F})$ on the universal covering $\bar{F}$ of the Riemann surface $F = \mathbb{C} \setminus \bar{S}$ that emerges from the Gauss plane by removing the points $\bar{S}$. If this covering is realized as space of homotopy classes of continuous paths in F with base x_o the complex conjugation acts by reflecting the paths at the real axis. According to the Schwarz reflecting principle this action can be extended to the "Spiegel"-automorphism γ of $K(\bar{F})$: $f^\gamma(P) = \overline{f(\bar{P})}$ for $f \in \mathbb{R}(t)_S = K(\bar{F})$.

(iv) *If* $\mathfrak{p} \in S$ *is a prime of* $\mathbb{R}(t)$ *and* $\bar{\mathfrak{p}} \mathrel{\widehat{=}} a$ *a prime of* $\mathbb{C}(t)$ *lying above it, then one gets a decomposition group* $G_{\mathfrak{p}} \subseteq \mathcal{G}_S$ *associated to* $\mathfrak{p}$ *by*

$$\mathcal{G}_{\mathfrak{p}} = \begin{cases} (\varrho_j) & a = \alpha_j \\ (\bar{\varrho}_j) \qquad\qquad if & a = \bar{\alpha}_j \\ (\tau_i, \tau_{i-1} \cdots \tau_1 \gamma) & a = a_i, \end{cases}$$

as (closed) subgroup in $\mathcal{G}_S$*.*

Replacing the generators τ_i by the products $\tau_i \cdots \tau_1$ one gets new classes of paths (see [14]). With the convention $\tau^\gamma = \gamma\tau\gamma^{-1}$ we define

$$\mathcal{G}_{r,i} := \langle (\tau_i \cdots \tau_1), \gamma \mid (\tau_i \cdots \tau_1)^\gamma = (\tau_i \cdots \tau_1)^{-1},\ \gamma^2 = 1 \rangle,\ 1 \le i \le r,$$

for the real places,

$$\mathcal{G}_{c,i} := \langle \varrho_j, \bar{\varrho}_j, \gamma \mid \varrho_j^\gamma = \bar{\varrho}_j^{-1},\ \gamma^2 = 1 \rangle = \langle \varrho_j, \gamma \mid \gamma^2 = 1 \rangle,\ 1 \le j \le c,$$

for the pairs of complex places and

$$\Gamma := \langle \gamma \rangle$$

the subgroup of $\mathcal{G}_S$ which is generated by the "Spiegel"-automorphism. In this setting we recall the following theorem

Theorem (Wingberg)

(i) *The Galois group* $\mathcal{G}_S$ *is the free profinite product of the* $\mathcal{G}_{r,i},\ 1 \le i \le r$, *and* $\mathcal{G}_{c,j},\ 1 \le l \le c$, *with amalgamated subgroup* Γ*:*

$$\mathcal{G}_S = \mathop{\ast_\Gamma}_{i \in \{1,\dots,r\}} \mathcal{G}_{r,i} \ast_\Gamma \mathop{\ast_\Gamma}_{j \in \{1,\dots,c\}} \mathcal{G}_{c,j}$$

(ii) *Representing* G_S *as quotient of the free product of the decomposition groups above* all *primes* $\mathfrak{p} \in \bar{S}$*, one gets the following exact sequence:*

$$1 \longrightarrow R_S \longrightarrow \mathop{\ast}_{\mathfrak{p} \in \bar{S}} G_{\mathfrak{p}} \longrightarrow G_S \longrightarrow 1,$$

in which $R_S = (\sigma_\infty \varrho_c \cdots \tau_r \cdots \tau_1 \bar{\varrho}_1 \cdots \bar{\varrho}_c)_{NT}$ *(productformula) is a free profinite* G_S*-operator-group of rank 1.*[3]

[3] $()_{NT}$ means generated as normal subgroup.

Proof: [14] (2), (3), Theorem 1.1, Theorem 1.3. Since Γ is a finite subgroup of $\mathcal{G}_S$, this amalgamated product exists actually in our sense according to remark 2 (i)(a), chapter 1. $\square$

Now we are able to determine the structure of the Galois group $\mathcal{H} = G(\mathbb{R}(t)_S/K)$ of $\mathbb{R}(t)_S$ over an arbitrary finite extension K of $\mathbb{R}(t)$ unramified outside S. As open subgroup of the amalgamated product $\mathcal{G}_S = \mathop{\ast_\Gamma}\limits_{i \epsilon \{1,\ldots,r\}} \mathcal{G}_{r,i} \ast_\Gamma \mathop{\ast_\Gamma}\limits_{j\epsilon\{1,\ldots,c\}} \mathcal{G}_{c,j}$ the group $\mathcal{H}$ always can be represented as profinite fundamental group $\pi_1(\mathcal{H}, \Lambda, T)$ of a graph of groups $(\mathcal{H}, \Lambda)$ with vertex groups $\mathcal{H} \cap \mathcal{G}^g_{\cdot,i}$ and edge groups $\mathcal{H} \cap \Gamma^g$ in accordance with theorem 1. Unfortunately, this is not a very “easy” description in general. On the other hand we cannot expect a too simple description, when we look at the explicit generators and defining relations for $\mathcal{H}$ in [7] in the case of a regular real function field K. In that article the structure of $\mathcal{H}$ is determined by means of Krull and Neukirch's method (cf. section 2.1), i.e. by making use of the *topological* fundamental group of the Klein surface associated to K.

But one may ask in which field theoretic circumstances the condition $(\star)$, $\mathcal{H}^g \cap \Gamma = \{1\} \quad \text{for all } g \ \epsilon \ \mathcal{G}_S$, of theorem 2 holds. In this case $\mathcal{H}$ is the free profinite product of the $\mathcal{H} \cap \mathcal{G}^g_{\cdot,i}$. It turns out that this condition is equivalent to the following field theoretic one:

Definition 3 *A finite outside S unramified field extension K of $\mathbb{R}(t)$ fulfills by definition the property (A), if for all embeddings $\varphi : K \longrightarrow \mathbb{R}(t)_S$:*

$$\varphi(K) \not\subseteq (\mathbb{R}(t)_S)^\Gamma.$$

Theorem 4 *Let K be a finite extension of $\mathbb{R}(t)$,which is unramified outside S and satisfies the property (A). Then the Galois group $\mathcal{H}$ of the maximal extension $K_S = \mathbb{R}(t)_S$ above K unramified outside S is a free profinite product:*

$$\mathcal{H} = \mathop{\ast}\limits_{i\epsilon\{1,\ldots,r\}, g\epsilon R_{r,i}} (\mathcal{G}^g_{r,i} \cap \mathcal{H}) \ast \mathop{\ast}\limits_{j\epsilon\{1,\ldots,c\}, g\epsilon R_{c,j}} (\mathcal{G}^g_{c,j} \cap \mathcal{H}) \ast \mathcal{F}.$$

In this expression

(i) *$\mathcal{F}$ is a free profinite group of rank*

$$r_{\mathcal{F}} = \sum_{i \epsilon \{1,\ldots,r\}} (\#R_\Gamma - \#R_{r,i}) + \sum_{j \epsilon \{1,\ldots,c\}} (\#R_\Gamma - \#R_{c,j}) - \#R_\Gamma + 1,$$

(ii) *$R_{\cdot,i}$ and R_Γ are suitable systems of representatives for the double cosets $\mathcal{H}\backslash\mathcal{G}_S/\mathcal{G}_{\cdot,i}$ and $\mathcal{H}\backslash\mathcal{G}_S/\Gamma$, respectively.*

Proof: The property $(\star)$, $\mathcal{H}^g \cap \Gamma = \{1\}$ for all $g \,\epsilon\, \mathcal{G}_S$, is equivalent to the assertion that $(\mathbb{R}(t)_S)^\Gamma gK = \mathbb{R}(t)_S$ holds for all $g \,\epsilon\, \mathcal{G}_S$. Since the extension $\mathbb{R}(t)_S/(\mathbb{R}(t)_S)^\Gamma$ has degree two the latter is valid exactly if $gK \not\subseteq (\mathbb{R}(t)_S)^\Gamma$ holds for all $g \,\epsilon\, \mathcal{G}_S$. Therefore the statement results from theorem 2. □

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