Characteristic Elements in Noncommutative Iwasawa Theory

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Introduction

Let p be a prime number, which, for simplicity, we shall always assume odd. The goal of non-commutative Iwasawa theory is to extend Iwasawa theory over $\mathbb{Z}_p$-extensions to the case of p-adic Lie extensions. Thus it might be useful to recall briefly some main aspects of the classical theory: Let k be a finite extension of $\mathbb{Q}$, and write k_{cyc} for the cyclotomic $\mathbb{Z}_p$-extension of k. We put $\Gamma = G(k_{cyc}/k)$. The main conjecture for an elliptic curve E over k relates the Γ-module structure of the Pontryagin dual

$$X(k_{cyc}) = X_f(E/k_{cyc}) = (\mathrm{Sel}_{p^\infty}(E/k_{cyc}))^\vee$$

of the Selmer group $\mathrm{Sel}_{p^\infty}(E/k_{cyc})$ of E over k_{cyc} to the p-adic L-function of E which in turn interpolates the values at 1 of the complex Hasse-Weil L-function of E twisted by Artin characters. More precisely, assuming that E has good ordinary reduction at all places S_p above p Mazur's conjecture predicts that $X(k_{cyc})$ is a $\Lambda(\Gamma)$-torsion module. Here we write $\Lambda(G) = \mathbb{Z}_p[\![G]\!]$ for the Iwasawa algebra of a p-adic Lie group G and we recall that $\Lambda(\Gamma)$ can be identified with the formal power series ring $\mathbb{Z}_p[[X]]$ depending on the choice of a topological generator of Γ. Using the structure theory of finitely generated modules over an integrally closed domain, one assigns to $X(k_{cyc})$ its characteristic ideal

$$char(X(k_{cyc})) = (f_{X(k_{cyc})})$$

which is generated by a unique polynomial $f_{X(k_{cyc})}$ in $\Lambda(\Gamma) = \mathbb{Z}_p[[X]]$, which is a power of p times a distinguished polynomial. The existence of a p-adic L-function $\mathcal{L}_E$ of E is conjectured in general, but it has only been proven in special cases, e.g. if E is already defined over $\mathbb{Q}$ (with good ordinary reduction at p) and k is a finite abelian extension of $\mathbb{Q}$ [37] or if E is an elliptic curve, which is defined over its field of complex multiplication K and k is any abelian extension of K. In any case, we view the p-adic L-function as a p-adic measure and thus as an element of $\Lambda(\Gamma)$.

The main conjecture in the ordinary case asserts that, up to a unit in $\Lambda(\Gamma)$, the p-adic L-function $\mathcal{L}_E$ in $\Lambda(\Gamma)$ agrees with $f_{X(k_{cyc})}$, i.e.

$$\mathcal{L}_E\Lambda(\Gamma) = f_{X(k_{cyc})}\Lambda(\Gamma).$$

The reason for studying the main conjecture is that it is essentially the only known general procedure for studying the exact formulae of number theory (for example, the Birch and Swinnerton-Dyer conjecture for E over k in this case). One has to evaluate $\mathcal{L}_E$ on the analytic side while, on the Iwasawa module side, one has to consider Galois descent. More precisely, evaluation of $f_{X(k_{cyc})}$ corresponds to calculating the Γ-Euler characteristic

$$\chi(\Gamma, X(k_{cyc})) := \prod_{i\geq 0}(\#\mathrm{H}_i(\Gamma, X(k_{cyc})))^{(-1)^i}$$

of $X(k_{cyc})$ and one obtains for the p-adic absolute values

$$|\mathcal{L}_E(0)|_p^{-1} = |f_{X(k_{cyc})}(0)|_p^{-1} = \chi(\Gamma, X(k_{cyc}))$$

if $f_{X(k_{cyc})}(0)$ is non-zero or equivalently if $\chi(\Gamma, X(k_{cyc}))$ is defined, i.e. if $H_i(\Gamma, X(k_{cyc}))$ is finite for $i = 0$ and $i = 1$. Thus, if the value $L(E, 1)$ at 1 of the Hasse-Weil L-function of E is non-zero, one obtains as a corollary from the main conjecture a special case of the conjecture of Birch and Swinnerton-Dyer using the explicit determination of

$$\chi(\Gamma, X(k_{cyc})) = \rho_p(E/k)$$

by Perrin-Riou [44] and Schneider [47], as is explained in more detail in [53, p.592ff]. Here $\rho_p(E/k)$ denotes the p-Birch and Swinnerton-Dyer constant, see section 9.3.

Let us now replace k_{cyc} by some p-adic Lie extension k_∞ of k, i.e. a Galois extension such that its Galois group $G = G(k_\infty/k)$ is a p-adic Lie group. Moreover, in order to simplify our discussion, we assume that G is a pro-p group and has no elements of order p. The first main example arises by adjoining the coordinates of all p-power division points $E(p)$ on E (but note the technically nasty fact that we may then have to replace k by a finite extension to ensure that G is pro-p). If E does not admit complex multiplication, G is an open subgroup of $GL_2(\mathbb{Z}_p)$ via the representation on the Tate module of E by a celebrated result of Serre [50]. As a second important example we shall also discuss the "false Tate curve" where k_∞ arises as trivializing extension of a p-adic representation which is analogous to that of the local Galois representation associated with a Tate elliptic curve. More precisely, we assume that k contains the group μ_p of p^{th} roots of unity and then k_∞ is obtained by adjoining to k_{cyc} the p-power roots of an element in $k^\times$ which is not a root of unity. By Kummer theory, its Galois group is isomorphic to the semidirect product $G(k_\infty/k) \cong \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$ where the action is given by the cyclotomic character.

Again we consider the Selmer group of an elliptic curve over k with good ordinary reduction at S_p as above. Conjecturally, the dual

$$X(k_\infty) = X_f(E/k_\infty)$$

of the Selmer group $\mathrm{Sel}_{p^\infty}(E/k_\infty)$ of E over k_∞ is again a $\Lambda(G)$-torsion module [8] and in the above cases this can be shown under certain conditions ([9],[22]) using Kato's deep theorem on Euler systems.

Now the crucial algebraic question needed to be answered to formulate a main conjecture in general is the following:

> *Can one assign to any finitely generated $\Lambda(G)$-torsion module M, a characteristic element, F_M say, lying in the field of fractions of $\Lambda(G)$ having the following property: if M has finite G-Euler characteristic, then $F_M(0)$ exists and is equal, up to a p-adic unit, to $\chi(G, M)$, and similarly for all twists of M by Artin characters of G?*

We should mention that on the p-adic analytic part concerning p-adic L-functions adapted to this setting practically nothing is known and thus we will mainly discuss what one might expect only on the algebraic side of the picture that has been sketched. In the following we describe several attempts and problems in the quest of such characteristic elements.

A major achievement in non-commutative Iwasawa theory is certainly the structure theorem on torsion Iwasawa modules proved by J. Coates, P. Schneider and R. Sujatha [10]. It says roughly that every finitely generated $\Lambda(G)$-torsion module M decomposes - up to pseudo-isomorphism - into the direct sum of cyclic modules $\Lambda(G)/L_i$, $i = 1, \ldots, r$, where L_i denote reflexive left ideals of $\Lambda(G)$

$$M \equiv \bigoplus_{i=1}^{r} \Lambda(G)/L_i \quad \text{(modulo pseudo-null)}$$

(see 3.6 for the precise statement and the needed conditions on G). It was astonishing to obtain this strong (non-commutative) result almost totally parallel to the above mentioned structure theory over integrally closed (commutative) domains. Unfortunately, it turned out that the Euler characteristic is not invariant under pseudo-isomorphisms in general (see 3.7, 8.8) and reflexive left ideals need not be principal by an example of D. Vogel (see 5.13). Due to this dilemma it is still not clear which role this structure result plays in the above picture that is expected.

Going back to commutative Iwasawa theory once again we recall that the characteristic polynomial f_M of a $\Lambda(\Gamma)$-torsion module M can also be interpreted as an element in the relative K-group

$$K_0(\Lambda(\Gamma), Q(\Gamma)) \cong Q(\Gamma)^\times / \Lambda(\Gamma)^\times$$

associated to the ring homomorphism $\Lambda(\Gamma) \to Q(\Gamma)$, where $Q(\Gamma)$ denotes the field of fractions of $\Lambda(\Gamma)$. Here, for any ring R, we write $R^\times$ for the group of units. As observed by Kato, this interpretation can be described nicely using the determinant functor of Knudsen and Mumford [32]. We should remark that this relative K-group is naturally isomorphic to the Grothendieck group $K_0(\Lambda(\Gamma)\text{-mod}_{\text{tor}})$ of the (abelian) category $\Lambda(\Gamma)\text{-mod}_{\text{tor}}$ of finitely generated $\Lambda(\Gamma)$-torsion modules.

In the general situation of p-adic Lie groups (with our general assumption) there exists also a skew field of fractions $Q(G)$ of $\Lambda(G)$ and from the long exact sequence of G- respectively K-theory one obtains again an identification

$$K_0(\Lambda(G)\text{-mod}_{\text{tor}}) \cong K_0(\Lambda(G), Q(G)) \cong Q(G)^\times/[Q(G)^\times, Q(G)^\times]\Lambda(G)^\times.$$

Also the determinant functor generalizes to this situation in terms of Deligne's virtual objects ([14],[4]) and is used by Huber-Kings [27] for a view of (non-commutative) Iwasawa theory in the spirit of the equivariant Tamagawa number conjecture á la Bloch-Kato ([4],[30],[31]). Thus one can assign to any $\Lambda(G)$-torsion module M an element in $Q(G)^\times$ unique modulo $[Q(G)^\times, Q(G)^\times]\Lambda(G)^\times$. Unfortunately, the class of M in the group $K_0(\Lambda(G)\text{-mod}_{\text{tor}})$ is not able to recognize the G-Euler characteristic of M in general. Indeed, in [9] there is an example of a pseudo-null module N whose class in $K_0(\Lambda(G)\text{-mod}_{\text{tor}})$ is zero and whose Euler characteristic $\chi(G, M)$ is defined but non-trivial, see 4.2.

Let us now assume that k_∞ contains k_{cyc} (as it is the case in our two main examples) and we set $H := G(k_\infty/k_{cyc})$ and denote by Γ as before the Galois group $G(k_{cyc}/k) = G/H$. Then G is the semi-direct product $G \cong H \rtimes \Gamma$. In this situation we want to propose a way to circumvent the above problem. The main idea of our approach consists of firstly using the good knowledge we have on the cyclotomic level (we know characteristic elements and how to evaluate them there) and secondly trying to relate the k_∞-level to the prior one. To this end we search for a good subcategory $\mathcal{C}(G) \subseteq \Lambda(G)\text{-mod}_{\text{tor}}$ which contains "enough" torsion modules (in particular those which arise naturally in our arithmetic applications) and whose K-theory "descends" to the cyclotomic level. By this we mean the existence of a natural projection map

$$K_0(\mathcal{C}(G)) \to K_0(\Lambda(\Gamma)\text{-mod}_{\text{tor}}).$$

If, moreover, the Grothendieck group $K_0(\mathcal{C}(G))$ could again be identified with the relative K-group $K_0(\Lambda(G), R)$ associated to a (injective) ring homomorphism $\Lambda(G) \to R$ of $\Lambda(G)$ into an appropriate local regular Noetherian ring R, we would even obtain a natural candidate for a characteristic element in

$$K_0(\Lambda(G), R) \cong R^\times/[R^\times, R^\times]\Lambda(G)^\times.$$

In fact, such a ring exists - at least under certain conditions - and can be constructed as follows: Consider the canonical surjective ring homomorphism

$$\psi_H : \Lambda(G) \twoheadrightarrow \mathbb{F}_p[\![\Gamma]\!].$$

Writing $\mathfrak{m}(H)$ for $\ker(\psi_H)$ we obtain a multiplicatively closed subset

$$\mathcal{T} := \Lambda \setminus \mathfrak{m}(H)$$

of $\Lambda = \Lambda(G)$. In contrast to commutative ring theory it is a very delicate question whether one can localize a ring at a given multiplicative set. More precisely, $\mathcal{T}$ must satisfy the Ore-condition which roughly speaking says that any left fraction with denominator in $\mathcal{T}$ can also be written as right fraction with denominator in $\mathcal{T}$, and vice versa. Thus, from a technical point of view the following theorem is a main result of this paper

Theorem (Theorem 6.2). *Let G be the semi-direct product $H \rtimes \Gamma$ with $H \cong \mathbb{Z}_p$ or the direct product $H \times \Gamma$ with H a uniform group. Then the multiplicative set $\mathcal{T}$ satisfies the left and right Ore condition for $\Lambda(G)$. In particular, the localisation $\Lambda_{\mathcal{T}}$ exists and is a Noetherian regular local ring with global dimension* $\mathrm{gl}(\Lambda_{\mathcal{T}}) < \mathrm{gl}(\Lambda)$.

The proof relies on filtered ring techniques and makes decisive use of Lazard's fundamental work [35]. A further case in which $\mathcal{T}$ is an Ore set was kindly pointed out to us by J. Coates and R. Sujatha and is stated in Theorem 6.10. As mentioned before, the existence of $\Lambda_{\mathcal{T}}$, which we will assume always henceforth, would be useless if we couldn't link the relative K-group $K_0(\Lambda, \Lambda_{\mathcal{T}})$ with a nice category of torsion modules. By

$$\Lambda(G)\text{-mod}^H$$

we denote the full subcategory of $\Lambda(G)$-mod consisting of those $\Lambda(G)$-modules which are finitely generated over the subalgebra $\Lambda(H)$ of $\Lambda = \Lambda(G)$. We will see below that such modules play an important role in our arithmetic applications. Thus the significance of $\Lambda_{\mathcal{T}}$ results partly from the next

Proposition (Proposition 6.4). *There are a canonical isomorphisms of groups*

$$K_0(\Lambda(G)\text{-}mod^H) \cong K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}}) \cong \Lambda_{\mathcal{T}}^{\times}/[\Lambda_{\mathcal{T}}^{\times}, \Lambda_{\mathcal{T}}^{\times}]\Lambda^{\times}.$$

This identification enables us to define characteristic elements for modules M in $\Lambda(G)$-mod^H : We say that an element in $\Lambda_{\mathcal{T}}^{\times}$ is a characteristic element of M, and denote if by F_M, if it corresponds via the above isomorphism to the class $[M] \in K_0(\Lambda(G)\text{-mod}^H)$. Note that F_M is unique only modulo $[\Lambda_{\mathcal{T}}^{\times}, \Lambda_{\mathcal{T}}^{\times}]\Lambda^{\times}$.

From the definition of $\mathcal{T}$ it is clear that the projection $\pi_H : \Lambda(G) \to \Lambda(\Gamma)$ extends to a ring homomorphism

$$\pi_H : \Lambda(G)_{\mathcal{T}} \to Q(\Gamma)$$

and thus we obtain a commutative "descent" diagram of K-groups with exact rows

$$\begin{array}{ccccccc}
K_1(\Lambda) & \longrightarrow & K_1(\Lambda_{\mathcal{T}}) & \longrightarrow & K_0(\Lambda, \Lambda_{\mathcal{T}}) & \longrightarrow & 0 \\
\downarrow (\pi_H)_* & & \downarrow (\pi_H)_* & & \downarrow (\pi_H)_* & & \\
K_1(\Lambda(\Gamma)) & \longrightarrow & K_1(Q(\Gamma)) & \longrightarrow & K_0(\Lambda(\Gamma), Q(\Gamma)) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\Lambda(\Gamma)^{\times} & \longrightarrow & Q(\Gamma)^{\times} & \longrightarrow & Q(\Gamma)^{\times}/\Lambda(\Gamma)^{\times} & \longrightarrow & 1.
\end{array}$$

This can be used to define the evaluation of F_M at " 0 " or more generally at certain p-adic representations, see section 7.2 for details. Thus let $\rho : G \to GL(V)$ be a continuous linear representation on a finite dimensional vector space V of dimension m over a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We choose a G-invariant $\mathcal{O}$-lattice $T \subseteq V$ and define a generalized G-Euler characteristic as follows

$$\chi(G, V, M) := \prod_i (\#\mathrm{Tor}_i^{\Lambda(G)}(T, M))^{(-1)^i}$$

provided that all Tor-groups are finite. One checks easily that this is independent of the choice of T.

Since the class of M in $K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}})$ can also be described using virtual objects which should be considered as universal Euler characteristic of M and which behave well under change of rings one obtains immediately the second property we wished for the characteristic element.

Theorem (Theorem 8.6)**.** *Assume that M belongs to $\Lambda(G)$-mod^H. Then, if $\chi(G, V, M)$ is finite, $F_M(\rho)$ is defined and we have*

$$\chi(G, V, M) = |F_M(\rho)|_p^{-[K:\mathbb{Q}_p]}$$

for any choice of F_M. In particular, $F_M(\rho)$ is non-zero.

Having settled these purely algebraic properties we want to verify its usefulness in the study of Selmer groups over p-adic Lie extensions as considered above. Assume that we are in the false-Tate-curve- or GL_2-case. Our strongest confirmation relies on the fact that the characteristic element $F_{X(k_\infty)}$ of the dual of the Selmer group over k_∞ specializes - up to some local Euler factors - to the characteristic polynomial $f_{X(k_{cyc})}$ of the dual of the Selmer group over k_{cyc}.

Theorem (Theorems 9.4, 9.8)**.** *Assume that in the false-Tate-curve- or GL_2-case $X(k_{cyc})$ is a torsion $\Lambda(\Gamma)$-module with vanishing μ-invariant. Then $X(k_\infty)$ is finitely generated over $\Lambda(H)$ and we have*

$$\pi_H(F_{X(k_\infty)}) \equiv f_{X(k_{cyc})} \cdot \prod_{\nu \,\in\, \mathfrak{M}} \mathcal{P}_\nu(E(p)/k) \mod \Lambda(\Gamma)^\times,$$

where $\mathcal{P}_\nu(E(p)/k)$ are local Euler factors, see section 9.1, while $\mathfrak{M}$ denotes a certain finite set of primes of k which ramify in the extension k_∞/k, see subsections 9.3 and 9.4.

Beneath the techniques developed above this result relies heavily on the vanishing of higher H-homology groups of $X(k_\infty)$ and descent calculations done by J. Coates, P. Schneider and R. Sujatha [9] in the GL_2-case and by Y. Hachimori and the author [22] in the false Tate curve case.

By evaluation of the above formula at the trivial representation we reobtain under the above conditions the well known determination of the G-Euler characteristic which can now be interpreted as the value at " 0 " of the characteristic element of the dual of the Selmer group.

Theorem (Corollary 9.5). *In the situation of the theorem and assuming that the G-Euler characteristic $\chi(G, X(k_\infty))$ of $X(k_\infty)$ is finite let $F_{X(k_\infty)} \in \Lambda_{\mathcal{T}}$ be a characteristic element of $X(k_\infty)$. Then $F_{X(k_\infty)}(0)$ is defined (and non-zero) and it holds that*

$$\chi(G, X(k_\infty)) = |F_{X(k_\infty)}(0)|_p^{-1} = \rho_p(E/k) \times \prod_{v \in \mathfrak{M}} |L_v(E,1)|_p.$$

Here, $\mathfrak{M}$ denotes the same set of primes as above; for the definition of the p-Birch-Swinnerton-Dyer constant $\rho_p(E/k)$ see subsection 9.3.

These results encourage us to suggest a possible shape of a main conjecture in section 10 involving our definition of characteristic element for the dual of the Selmer group. For a precise conjecture one needs a good guess for the ϵ-factors, Deligne period, etc. and since there is very little empirical material available at the moment this seems a very delicate and subtle point. But a similar recipe as in [7] for the conjectural p-adic L-functions of motives over $\mathbb{Q}$ with respect to the cyclotomic $\mathbb{Z}_p$-extension should generalize to our situation.

We conclude the introduction giving a brief outline of the paper:

In sections 1 and 2 we review Lazard's work on p-adic Lie groups as well as on their Iwasawa algebras. Since twisting with characters might require base extension from $\mathbb{Z}_p$ to the ring of integers $\mathcal{O}_K$ of some (finite) extension K of $\mathbb{Q}_p$ we consider completed group algebras over more general coefficient rings than $\mathbb{Z}_p$. Lazard's filtrations on Iwasawa algebras will turn out to be crucial for the proof that the above mentioned set $\mathcal{T}$ satisfies the Ore condition.
In section 3 we collect basic facts on modules over the Iwasawa algebra and then we go on discussing the structure theory of torsion modules up to pseudo-isomorphism.
The purpose of section 4 consists of recalling some requisites from K-theory, which will be needed to define characteristic elements, in particular Swan's relative K-groups and Deligne's virtual objects.
From section 5 on we concentrate on the 'relative situation' where G has a fixed quotient isomorphic to $\mathbb{Z}_p$. We interpret $\Lambda(G)$ as a skew power series ring and apply an analogue of the Weierstrass preparation theorem (subsection 5.2) to get more insights in the module theory, e.g. we recall examples of faithful modules (subsection 5.3).
Technically, the heart of this work is section 6 where we prove that the set $\mathcal{T}$ associated with a certain group extension satisfies the Ore-condition. The proof splits into the case of a direct product and the case of a semi-direct product dealt with in different subsections. In the third subsection we discus the extension of the direct product case to the case of certain pro-p open subgroups of $GL_n(\mathbb{Z}_p)$

which was kindly explained to us by J. Coates and R. Sujatha.
In section 7 we study the effect and functoriality of twisting Iwasawa modules by p-adic representations. This is used to define the evaluation of characteristic elements at such representations in subsection 7.2. In subsection 7.3 we introduce an equivariant and twisted Euler characteristic.
The descent (of K-groups as well as of cohomology groups) from the level over k_∞ to that over k_{cyc} is described in section 8. In particular, we discuss the Akashi-series, an invariant introduced in [9, §4], and relate it to the characteristic element which we introduce in this section. The main result is the relation between the twisted Euler characteristic and evaluation of the characteristic element at a p-adic representation. Finally, we discuss some examples of Iwasawa modules for which we calculate the characteristic element, the Akashi-series and the Euler characteristic.
The application of the above theory to arithmetic geometry is described in section 9 where we first discuss local Euler factors over cyclotomic $\mathbb{Z}_p$-extensions (subsection 9.1) before we obtain the above mentioned descent result concerning the characteristic element of the dual of the Selmer group of an elliptic curve defined over the base field.
In section 10 we discuss the possible shape of a main conjecture involving the Selmer group of the elliptic curve and a (conjecturally existing) distribution interpolating the values at 1 of the Hasse-Weil L-function twisted by Artin characters.

The appendix A contains a technical lemma concerning filtered rings needed in section 2 while in appendix B we collect some facts about induction.

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General Notation and Conventions

(i) In this paper, a ring R is always associative and with a unit element. When we are talking about properties related to R like being "Noetherian", an "ideal", a "unit", of a certain "global dimension", etc. we always mean the left *and* right property if not otherwise stated. But by an R-module we usually mean left R-module (not a bi-module).

(ii) By $R^\times$ we denote the group of units, i.e. of right and left invertible elements.

(iii) By $\mathrm{pd}_R(M)$ we denote the projective dimension of a R-module M while $\mathrm{gl}(\Lambda)$ denotes the global dimension of Λ.

(iv) By a *local* ring R we mean a ring in which the non-units form a proper ideal, which is then automatically maximal as left, right and two-sided ideal. Equivalently, R has both a unique left and a unique right maximal ideal, which amounts to the same as the quotient $R/J(R)$ of R by its Jacobson radical being a skewfield.

(v) A filtration $F_\bullet R := \{F_n R | n \in \mathbb{Z}\}$ of a ring we shall always assume is indexed by $\mathbb{Z}$, increasing, exhaustive and separated. We write $\mathrm{gr}R = \bigoplus_{n \in \mathbb{Z}} F_n R/F_{n-1}R$ for its associated graded ring. A similar notation and convention is used for filtered left R-modules. For further notation on filtered rings and modules see appendix A.

(vi) For a discrete (resp. compact) $\mathbb{Z}_p$-module N with continuous action by some profinite group G,

$$N^\vee = \mathrm{Hom}_{\mathbb{Z}_p,cont}(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the compact (resp. discrete) Pontryagin dual of N with its natural G-action. If N is p-divisible,

$$T_p(N) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, N) = \varprojlim_i {}_{p^i}N$$

denotes the Tate module of N, where ${}_{p^i}N$ denotes the kernel of the multiplication by p^i. For $G = G_k$ the absolute Galois group of a number field or a local field k, we define the rth Tate twist of N by $N(r) := N \otimes_{\mathbb{Z}_p} T_p(\mu)^{\otimes r}$ for $r \in \mathbb{N}$ and $N(r) := N \otimes_{\mathbb{Z}_p} \mathrm{Hom}(T_p(\mu)^{\otimes r}, \mathbb{Z}_p)$ for $-r \in \mathbb{N}$, where μ denotes the G_k-module of all roots of unity and by convention $T_p(\mu)^{\otimes 0} = \mathbb{Z}_p$ with trivial G-action. Finally, we set

$$N^* := \varinjlim_i \mathrm{Hom}({}_{p^i}N, \mu_{p^\infty}) = T_p(N)^\vee(1).$$

(vii) Let G be a profinite group and H a closed subgroup of G. For a $\Lambda(H)$-module M, we define $\mathrm{Ind}_H^G M := M \widehat{\otimes}_{\Lambda(H)} \Lambda(G)$ (compact or completed induction), where $\widehat{\otimes}$ denotes completed tensor product, and $\mathrm{Coind}_H^G M := \mathrm{Hom}_{\Lambda(H)}(\Lambda(G), M)$ (co-induction).

(viii) If G is any profinite group, by $G(p)$ and G^{ab} we denote the maximal pro-p quotient and the maximal abelian quotient $G/[G,G]$ of G, respectively. For an abelian group A we also denote by $A(p)$ its p-primary component.

(ix) Let k be a field. For a G_k-module A, we write $A(k) := H^0(G_k, A)$.

1. p-adic Lie groups

Since the towers of number fields studied in non-commutative Iwasawa theory form a p-adic Lie extension, a good knowledge on (compact) p-adic analytic groups, i.e. the group objects in the category of p-adic analytic manifolds over $\mathbb{Q}_p$, is fundamental. Thus, in this subsection we recall basic facts about these groups which are also called p-adic Lie groups. The reader who is familiar with this topic may skip this section or only glance at it for notational reasons.

There is a famous characterization of p-adic analytic groups due to Lazard [35] (see also [15] 9.36):

A topological group G is a compact p-adic Lie group if and only if G contains a normal open uniformly powerful pro-p-subgroup of finite index.

Let us briefly recall the definitions: A pro-p-group G is called *powerful*, if $[G,G] \subseteq G^p$ for odd p, respectively $[G,G] \subseteq G^4$ for $p = 2$, holds. Here for any prime p and natural number n we write G^{p^n} for the subgroup of G which is generated by all elements of the form g^{p^n}, $g \in G$. A (topologically) finitely generated powerful pro-p-group G is *uniform* if the p-power map induces isomorphisms

$$P_i(G)/P_{i+1}(G) \xrightarrow{\cdot p} P_{i+1}(G)/P_{i+2}(G), \quad i \geq 1,$$

where $P_i(G)$ denotes the lower central p-series given by

$$P_1(G) = G,\ P_{i+1}(G) = P_i(G)^p[P_i(G), G],\ i \geq 1,$$

(for finitely generated pro-p-groups). It can be shown that for a uniform group G the sets $\{g^{p^i} |\ g \in G\}$ are equal to G^{p^i}, thus form subgroups and in fact $G^{p^i} = P_{i+1}(G),\ i \geq 0$. For example, all the congruence kernels of $GL_n(\mathbb{Z}_p), SL_n(\mathbb{Z}_p)$ or $PGL_n(\mathbb{Z}_p)$ are uniform pro-p-groups for $p \neq 2$, in particular the lower central p-series of the first congruence kernel corresponds precisely to the higher congruence kernels. We should mention also the following basic result (see [15], p. 62):

A pro-p powerful group is uniform if and only if it has no element of order p.

For instance, for $p \geq n+2$, the group $Gl_n(\mathbb{Z}_p)$ has no elements of order p, in particular, $GL_2(\mathbb{Z}_p)$ contains no elements of finite p-power order if $p \geq 5$ (see [24] 4.7) .

It is a remarkable fact that the analytic structure of a p-adic Lie group is already determined by its topological group structure. Also, the category of p-adic analytic groups is stable under the formation of closed subgroups, quotients and group extensions (See [15], chapter 10, for these facts). The following cohomological property is often very useful (for the definition of Poincaré groups see [39]).

A p-adic Lie group of dimension d (as p-adic analytic manifold) without p-torsion is a Poincaré group at p of dimension d.

We should mention that Lazard himself did not use the notation of powerful or uniform groups. Instead, he formulated the above characterization of p-adic analytic groups using the notation of *p-valuable* groups, i.e. complete p-valued groups of finite rank, see [35, thm. III.3.1.7]. We recall that a group G is called *p-valued* [35, III, Def. 2.1.2] if it possesses a p-valuation, i.e. a function $\omega : G \to (0, \infty]$ satisfying the following axioms for all g and h in G :

(i) $\omega(1) = \infty$, and $1/(p-1) < \omega(g) < \infty$ for $g \neq 1$,
(ii) $\omega(gh^{-1}) \geq \min\{\omega(g), \omega(h)\}$,
(iii) $\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h)$ and
(iv) $\omega(g^p) = \omega(g) + 1$.

In particular, it follows that $G_\nu := \{g \,\epsilon\, G | \omega(g) \geq -\nu\}$ and $G_{\nu^+} := \{g \,\epsilon\, G | \omega(g) > -\nu\}$ are normal subgroups for each ν in $\mathbb{R}$. A p-valued group (G, ω) is said to be *complete* if $G = \varprojlim_{\nu} G/G_\nu$. Putting

$$\mathrm{gr}\, G := \bigoplus_{\nu \,\epsilon\, \mathbb{R}} G_\nu / G_{\nu^+}$$

we obtain a graded Lie algebra (the Lie bracket being induced by the group commutator) over the graded ring $\mathbb{F}_p[\pi_0] = \mathrm{gr}\, \mathbb{Z}_p$ with π_0 in degree -1 (the action of π_0 on $\mathrm{gr}\, G$ corresponds to taking the p^{th} power of an element of G, see [35, pp. 464-465]). In fact, $\mathrm{gr}\, G$ is free as a $\mathrm{gr}\, \mathbb{Z}_p$-module and its rank is called the *rank* of (G, ω) ([35, III 2.1.3]). In particular, a p-valued group has no element of order p. The class of p-valuable groups is closed under taking closed subgroups and forming finite products. We should remark that by [35, III 3.1.11] a p-valuable group always admits a p-valuation which has rational values, more precisely in a discrete subset of $\mathbb{Q}$. Henceforth we assume that ω is of the latter sort. If a compact p-adic group G is p-valued it is automatically pro-p ([35, III 3.1.7].

As mentioned above, Lazard's original characterization now reads as

A topological group is a compact p-adic Lie group (of finite dimension d) if and only if it contains a normal open p-valuable subgroup of finite rank d ([35, III 3.1.3/7/9, 3.4.5]*). In this case the dimension* $\dim G$ *(of the underlying p-adic manifold) and the rank of G coincide (loc. cit.).*

As explained in the remark after [48, lem. 4.3] the relation between p-valuable and uniform pro-p-groups (for $p \neq 2$ for simplicity) is as follows (see also the Notes at the end of chapter 4 in [15]): A p-valuable group G is called *p-saturated* [35, III.2.1.6] if G has a p-valuation ω with the property that any $g \,\epsilon\, G$ with $\omega(g) > p/(p-1)$ is a p^{th} power. If moreover, G has a minimal system of (topological) generators $g_1, \ldots, g_d$ such that $\omega(g_i) + \omega(g_j) > p/(p-1)$ for any $1 \leq i \neq j \leq d$, we say that G is *strongly p-saturated* as this property implies that every commutator in G is a p^{th} power. In particular, every strongly p-saturated p-valued group

G is powerful, and thus uniform since it does not have an element of order p. Conversely, uniform groups even allow a p-valuation ω with $\omega(g_i) = 1$ for every g_i of an arbitrary minimal system of generators $g_1, \ldots, g_d$. Since any element in G_2 is a p^{th} power [15, le. 4.10] one has:

For $p \neq 2$, a p-valuable group G is uniform if and only if it is strongly p-saturated.

2. The Iwasawa algebra - a review of Lazard's work

From a technical point of view the theory of Iwasawa modules, i.e. modules under the Iwasawa algebra, forms the core of the purely algebraic part of Iwasawa theory. The purpose of this and the next section is to give a survey on that subject.

Throughout this section we make the following assumption: Let $\mathcal{O}$ be a commutative Noetherian local ring which is complete in its $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal. We assume that $\kappa = \mathcal{O}/\mathfrak{m}$ is a finite field of characteristic p, in particular $\mathcal{O}$ is compact.

In our applications $\mathcal{O}$ is usually the ring of integers in a finite extension field of $\mathbb{Q}_p$ or a finite field.

We denote by $\Lambda = \Lambda(G)$ the *Iwasawa algebra* of a compact p-adic Lie group G, i.e. the completed group algebra of G over $\mathcal{O}$

$$\Lambda(G) = \mathcal{O}[\![G]\!] = \varprojlim_{U} \mathcal{O}[G/U],$$

where U runs through the open normal subgroups of G. For a good treatment of basic properties of Λ, some of which we recall below, we refer the reader to [39, V§2]. First we should mention that

Λ is a semi-local ring, it is local if and only if G is pro-p.

The global dimension $\mathrm{gl}\Lambda(G)$ equals $\mathrm{cd}_p G + \mathrm{gl}\mathcal{O}$ where cd_p denotes p-cohomological dimension [3]. By a result of Serre [49] $\mathrm{cd}_p G$ is finite if and only if G does not contain an element of order p.

The whole deeper structure theory of Λ relies on the following observation which is essentially due to Lazard [35].

Theorem 2.1. (Lazard) *Assume that G is a p-valuable (hence compact p-adic Lie) group and, in addition to our general assumptions, that $\mathcal{O}$ is a finite field or a discrete valuation ring (DVR). Then $\mathcal{O}[\![G]\!]$ possesses a complete separated and exhaustive (increasing) filtration $F_\bullet\Lambda$ such that*

$$\mathrm{gr}\,\mathcal{O}[\![G]\!] \cong \begin{cases} \kappa[X_0, X_1, \ldots, X_d] & \text{if } \mathcal{O} \text{ is a DVR} \\ \kappa[X_1, \ldots, X_d] & \text{if } \mathcal{O} = \kappa \text{ is a finite field} \end{cases}$$

are isomorphisms of graded rings. Here $d = \dim G$*, and the grading on the polynomial ring is given by assigning to each variable a certain strictly negative integer degree.*

For $\mathcal{O} = \mathbb{Z}_p$ this reformulation of Lazard's result is [10, prop. 7.2]. Before we extend it to a bigger class of rings $\mathcal{O}$ we restate Lazard's original results more precisely. First note that any valuation ω of G extends to a filtration also called ω of the Iwasawa algebra $\mathbb{Z}_p[\![G]\!]$ and with respect to this filtration Lazard [35, Ch. III 2.3.3/4] has established a canonical isomorphism

$$\mathrm{gr}_\omega \mathbb{Z}_p[\![G]\!] \cong U(\mathrm{gr}_\omega G).$$

Using this results he shows that one can describe the elements of $\mathbb{Z}_p[\![G]\!]$ as certain power series in non-commuting variables. For this purpose we fix an *ordered basis* of G, i.e. a sequence of elements $g_1, \ldots, g_d \in G\backslash\{1\}$ such that the elements $g_i G_{\omega g_i^+}$ form a basis of $\mathrm{gr}\, G$ as an $F_p[\pi_0]$-module. Then any $\lambda \in \mathbb{Z}_p[\![G]\!]$ has a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha (g_1 - 1)^{\alpha_1} \cdot \ldots \cdot (g_d - 1)^{\alpha_d},$$

with $\lambda_\alpha \in \mathbb{Z}_p$ for all α and conversely all such series converge in $\mathbb{Z}_p[\![G]\!]$ ([35, III 2.3.8], see also [48, §4]). In the language of Lazard this means that Λ is complete-free ([35, I 2.1.17]) with topological basis

$$\mathbf{b}_\alpha := (g_1 - 1)^{\alpha_1} \cdot \ldots \cdot (g_d - 1)^{\alpha_d}, \alpha \in \mathbb{N}^d.$$

In other words there is an isomorphism of filtered (in particular topological) $\mathbb{Z}_p$-modules

$$\mathbb{Z}_p[\![G]\!] \cong \prod_\alpha \mathbb{Z}_p \mathbf{b}_\alpha,$$

where the filtration of the left hand side is the product filtration with $\mathbb{Z}_p\mathbf{b}_\alpha$ being isomorphic to the filtered $\mathbb{Z}_p$-module $\mathbb{Z}_p(r_\alpha)$ which is isomorphic to $\mathbb{Z}_p$ but with shifted filtration by $r_\alpha = \sum_{i=1}^d \omega(g_i)$.

Now let R be a compact (possibly non-commutative) $\mathbb{Z}_p$-algebra endowed with a filtration $F_\bullet R$ consisting of closed $\mathbb{Z}_p$-modules and such that the structure map $\mathbb{Z}_p \to R$ is a homomorphism of filtered rings. We do not assume that $F_\bullet R$ induces the given topology but a possibly finer one. The following two cases are of particular interest:

(A) $\mathrm{gr}\, R$ *is finitely generated as* $\mathrm{gr}\, \mathbb{Z}_p$*-module,*

(B) $\mathrm{gr}\, R$ *is* $\mathrm{gr}\, \mathbb{Z}_p$ *torsionfree.*

Example 2.2. (i) Let $\mathcal{O}$ be as at the beginning of this section but assume in addition that $\mathcal{O}$ is a DVR in mixed characteristic. Then the rings $\mathcal{O}$ and $\mathcal{O}/\mathfrak{m}^n$, $n \geq 1$, (endowed with the $\mathfrak{m}$-adic filtration) are examples for case (A). But note that one possibly has to rescale the $\mathfrak{m}$-adic filtration in order to obtain an injective map

$$\mathbb{F}_p[\pi_0] \cong \mathrm{gr}\, \mathbb{Z}_p \to \mathrm{gr}\, \mathcal{O}$$

(assign to π_0 the degree $-1/e$ where e denotes the ramification index, i.e. $(p) = \mathfrak{m}^e$).

(ii) Examples for case (B) arise by taking for R the power series ring $\mathbb{Z}_p[[X]]$ (with its $\mathfrak{m}$-adic filtration) ore more general the Iwasawa algebra $R = \mathbb{Z}_p[\![G]\!]$ of a compact p-valued p-adic Lie group with the filtration induced by ω, see [35, Ch. III 2.3.4]. If we take the p-adic filtration instead we obtain examples where the filtration does not induce the compact topology.

Proposition 2.3. *Let (G, ω) be a p-valued compact p-adic Lie group.*

(i) *If $\mathcal{O}$ is of type (A), then there is a canonical isomorphism of topological $\mathcal{O}$-algebras*

$$\mathcal{O}[\![G]\!] \cong \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!].$$

Thus the tensor product filtration on the right hand side induces a filtration on $\mathcal{O}[\![G]\!]$.

(ii) *If $\mathcal{O}$ is of type (B), then there is a canonical isomorphism of topological $\mathcal{O}$-algebras*

$$\mathcal{O}[\![G]\!] \cong \mathcal{O}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!],$$

where the right hand side can be considered both as completed tensor product of topological modules and as completion of the tensor product $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!]$ with respect to the tensor product filtration. The latter induces again a filtration on $\mathcal{O}[\![G]\!]$.

In both cases there is an isomorphism of filtered (thus in particular topological) $\mathcal{O}$-modules

$$\mathcal{O}[\![G]\!] \cong \prod_{\alpha} \mathcal{O}\mathbf{b}_\alpha$$

and an isomorphism of graded gr $\mathcal{O}$*-algebras*

$$\mathrm{gr}\, \mathcal{O}[\![G]\!] \cong U(\mathrm{gr}\, \mathcal{O} \otimes_{\mathrm{gr}\, \mathbb{Z}_p} \mathrm{gr}\, G),$$

where $U(\mathrm{gr}\mathcal{O}\otimes_{\mathrm{gr}\,\mathbb{Z}_p}\mathrm{gr}G)$ denotes the enveloping algebra of the Lie algebra $\mathrm{gr}\mathcal{O}\otimes_{\mathrm{gr}\,\mathbb{Z}_p}$ $\mathrm{gr}\, G$. Furthermore, it is always possible to replace the valuation ω by a valuation ω' with values in $\mathbb{Q}$ such that (G, ω') is also p-valued and such that $\mathrm{gr}_{\omega'}G$ is an abelian Lie algebra. In this case, we obtain an isomorphism

$$\mathrm{gr}_{\omega'}\mathcal{O}[\![G]\!] \cong (\mathrm{gr}\, \mathcal{O})[X_1, \ldots, X_d]$$

where d is the dimension of G.

If $\mathcal{O}$ is a DVR of mixed characteristic, then gr $\mathcal{O} \cong \kappa[\pi]$, where π is induced by an uniformiser of $\mathcal{O}$, and we obtain the isomorphisms

$$(\mathrm{gr}\,\mathcal{O})[\![G]\!] \cong \kappa[\pi][X_1, \ldots, X_d] \cong \kappa[X_0, \ldots, X_d].$$

If $\mathcal{O} = \kappa$ is a finite field then clearly gr $\mathcal{O} = \kappa$ which again implies the claimed isomorphism. Finally, if $\mathcal{O}$ is a DVR of equal characteristic, then one can apply the analogue of Proposition 2.3 in case (B) for the pair $\kappa \to \mathcal{O}$ (instead of $\mathbb{Z}_p \to R$) making use of the previous case of a finite field. This proves Theorem 2.1.

Before proving the proposition we state an obvious corollary. By $\mathfrak{m}_G$ we denote the maximal ideal of $\mathcal{O}[\![G]\!]$.

Corollary 2.4. *Let G be a uniform group and assume that $\mathcal{O}$ is of type (A) or (B). Then, with respect to the $\mathfrak{m}_G$-adic filtration there is an isomorphism of (possibly non-commutative) graded* gr $\mathcal{O}$*-algebras*

$$\mathrm{gr}_{\mathfrak{m}_G}\,\mathcal{O}[\![G]\!] \cong U(\mathrm{gr}\,G \otimes_{\mathrm{gr}\,\mathbb{Z}_p} \mathrm{gr}\,\mathcal{O}).$$

Proof of the proposition. Since, in case (A), $\mathcal{O}$ is a finitely generated $\mathbb{Z}_p$-module we have $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!] \cong \mathcal{O}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]$ by [39, Ch. V §2 ex. 1] and the latter can be calculated as follows

$$\begin{aligned}
\mathcal{O}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!] &\cong \varprojlim_U \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G/U] \\
&\cong \varprojlim_U \mathcal{O}[G/U] \\
&= \mathcal{O}[\![G]\!],
\end{aligned}$$

where U runs through all open normal subgroups of G and where we used again [39, Ch. V §2 ex. 1/2]. This proves (i) and also (ii) where we use [35, I 3.2.10] to see that both interpretations of the completed tensor product coincide under our assumptions. The isomorphism of filtered $\mathcal{O}$-modules is proved in the appendix A.1 based on Lazard's above result in the case $\mathcal{O} = \mathbb{Z}_p$. In the graded situation there is always a canonical surjective homomorphism of graded gr $\mathbb{Z}_p$-modules

$$\mathrm{gr}\,\mathcal{O} \otimes_{\mathrm{gr}\,\mathbb{Z}_p} \mathrm{gr}\,\mathbb{Z}_p[\![G]\!] \twoheadrightarrow \mathrm{gr}\,(\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![G]\!])$$

by [35, I 2.3.10.1] which is easily checked to be a morphism of graded gr $\mathcal{O}$-algebras. Thus we only have to verify that this is an isomorphism of gr$\mathcal{O}$-modules which is achieved by lemma A.1 in the appendix. Finally we use the above result of Lazard for $\mathcal{O} = \mathbb{Z}_p$ and observe that the enveloping algebra behaves well under scalar extension of the (free) Lie algebra. The possible choice of ω' with the desired properties is explained in [10, proof of 7.2/3]. □

Remark 2.5. Lazard uses filtrations indexed by the positive real numbers in general. But in [10, proof of 7.2/3] it is explained that in our situation one always can rescale the filtration on $\mathbb{Z}_p[\![G]\!]$ to get one indexed by $\mathbb{Z}$. Of course, this extends to general $\mathcal{O}[\![G]\!]$.

Remark 2.6. In fact we have proved more. If R denotes a compact $\mathbb{Z}_p$-algebra (with additional filtration F as above) of type (A) or (B) and if we consider the complete tensor product of filtered rings $R\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]$, then we obtain an isomorphism of graded gr $\mathbb{Z}_p$-algebras

$$\mathrm{gr}_F R \otimes_{\mathrm{gr}\,\mathbb{Z}_p} \mathrm{gr}\,\mathbb{Z}_p[\![G]\!] \cong \mathrm{gr}\,(R\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]).$$

Note that due to compactness of R there is an isomorphism of abstract $\mathbb{Z}_p$-modules between the filtered tensor product $R\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![G]\!]$ and the complete tensor product of topological $\mathbb{Z}_p$-modules (linearly topologized), which need not be an homoeomorphism as we do not require that $F_\bullet R$ induces the original topology on R. An important example is the following: If $G = H\times\Gamma$ is a direct product of two p-adic Lie groups H and Γ, then it is easy to see that we have an isomorphism of topological $\mathbb{Z}_p$-algebras

$$\mathbb{Z}_p[\![H]\!]\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[\![G]\!]$$

and similarly with coefficient ring $\mathcal{O}$. If we now assume that, in addition, H and Γ are p-valued and if the Iwasawa algebras of both are endowed with Lazard's induced filtration, then we obtain an isomorphism of graded algebras

$$\mathrm{gr}\,\mathcal{O}[\![H]\!] \otimes_{\mathrm{gr}\,\mathcal{O}} \mathrm{gr}\,\mathcal{O}[\![\Gamma]\!] \cong \mathrm{gr}\,\mathcal{O}[\![G]\!],$$

which corresponds to the canonical isomorphism of enveloping algebras

$$U(\mathrm{gr}\,\mathcal{O}\otimes_{\mathrm{gr}\,\mathbb{Z}_p}\mathrm{gr}\,H)\otimes_{\mathrm{gr}\,\mathcal{O}} U(\mathrm{gr}\,\mathcal{O}\otimes_{\mathrm{gr}\,\mathbb{Z}_p}\mathrm{gr}\,\Gamma) \cong U(\mathrm{gr}\,\mathcal{O}\otimes_{\mathrm{gr}\,\mathbb{Z}_p}\mathrm{gr}\,G).$$

But if we change the filtration e.g. on $\mathcal{O}[\![\Gamma]\!]$ taking for instance just the $\mathfrak{m}$-adic filtration with respect to the maximal ideal of $\mathcal{O}$, then we obtain an isomorphism

$$\begin{aligned}\mathrm{gr}_{\omega_H,\mathfrak{m}}\mathcal{O}[\![G]\!] &\cong \mathrm{gr}_{\omega_H}\mathcal{O}[\![H]\!]\otimes_{\mathrm{gr}\,\mathcal{O}}\mathrm{gr}_{\mathfrak{m}}\mathcal{O}[\![\Gamma]\!]\\ &\cong U(\mathrm{gr}\,\mathcal{O}\otimes_{\mathrm{gr}\,\mathbb{Z}_p}\mathrm{gr}\,H)\otimes_{\mathrm{gr}\,\mathcal{O}}\kappa[\![\Gamma]\!][\pi],\end{aligned}$$

where for the last isomorphism we assume that $\mathcal{O}$ is a DVR with uniformizer π.

Now, the strategy is to use techniques from the theory of filtered rings and modules to fully exploit Lazard's Theorem and to derive further properties of Λ and of its module category. Most of these methods can be found in the book [36]. Some of these techniques or results or very tricky and sophisticated (e.g. Theorem 3.6) while others - like the following - are straight forward.

Corollary 2.7. *Assume that $\mathcal{O}$ is finitely generated as a $\mathbb{Z}_p$-module and let G be a compact p-adic Lie group. Then $\mathcal{O}[\![G]\!]$ is Noetherian.*

Proof. Using Lazard's characterization of p-adic Lie groups and the fact that for an open subgroup H of G the module $\Lambda(G)$ is finitely generated over $\Lambda(H)$ we may assume that G itself is p-valued. Now the result follows since $\mathrm{gr}\,\Lambda(G) \cong (\mathrm{gr}\,\mathcal{O})[X_1,\ldots,X_d]$ is Noetherian and $\Lambda(G)$ is complete ([36, II§1 prop. 3]). □

Using similar techniques one shows that for a p-valued compact p-adic Lie group the ring $\mathcal{O}[\![G]\!]$ has no zero divisor if $\mathcal{O}$ is either a finite field or a DVR:

Corollary 2.8. *In the situation of the Theorem $\mathcal{O}[\![G]\!]$ is an integral domain.*

In the mixed characteristic case A. Neumann [40] shows this for all torsionfree pro-p p-adic Lie groups (in the case $\mathcal{O} = \mathbb{Z}_p$) by different means, which unfortunately do not generalize e.g. to the case of a finite field $\mathcal{O} = \kappa$.

Proposition 2.9. (Neumann) *Assume that $\mathcal{O}$ is regular with mixed characteristic and let G be a pro-p-group without any element of order p and such $\mathcal{O}[\![G]\!]$ is Noetherian (e.g. if G is a pro-p p-adic Lie group). Then $\mathcal{O}[\![G]\!]$ has no zero divisor.*

The proof is completely analogous to [40, thm 1] using a theorem of Walker since the finiteness of the global dimension is well known by Brumer's result.

If $G \cong {\mathbb{Z}_p}^d$ and $\mathcal{O}$ is an integrally closed domain, then $\mathcal{O}[\![G]\!]$ is also integrally closed. The generalization of this property to non-commutative rings is the following: Assume that R is a Noetherian ring without zero divisors and Q its skewfield of fractions (see [18]). Then R is called *maximal order* if, for any intermediate ring S with $R \subseteq S \subseteq Q$ such that there exist elements $u, v \in Q^\times$ with $uSv \subseteq R$, we always have $R = S$.

One approach for the structure theory of modules up to pseudo-isomorphism which we will discuss in the next section depends crucially on the next

Theorem 2.10 ([10, prop. 7.2])**.** *Let G be a p-valuable group and $\mathcal{O}$ a finite field or a DVR of mixed characteristic. Then $\mathcal{O}[\![G]\!]$ is a maximal order.*

This is again a corollary of Theorem 2.1 using filtered ring techniques, see e.g. [10, lem. 3.6].

3. Iwasawa-modules

In this section we make the general assumption that

$\mathcal{O}$ is a finite field or a DVR in mixed characteristic and that G is a p-valuable group. Then $\Lambda = \mathcal{O}[\![G]\!]$ is Noetherian and without zero-divisors and thus has a skew field $Q(G)$ of fractions (see [18]).

We start defining the rank of a Λ-module:

Definition 3.1. The rank $\mathrm{rk}_\Lambda M$ is defined to be the dimension of $M \otimes_\Lambda Q(G)$ as a left vector space over $Q(G)$

$$\mathrm{rk}_\Lambda M = \dim_{Q(G)}(M \otimes_\Lambda Q(G)).$$

Obviously, the rank is finite for any M in the category Λ-mod of finitely generated Λ-modules. For the rest of this section, we assume that *all Λ-modules considered are finitely generated.*

We say that $M \epsilon \Lambda$-mod is *torsion* if its rank is zero. It is easy to see that this is equivalent to the vanishing of $\mathrm{Hom}_\Lambda(M, \Lambda)$. On the other hand, torsion-free Λ-modules, i.e. without any non-trivial torsion submodule, are precisely the submodules of free modules (see [56, before prop. 2.7] for details).

One possibility to determine the rank of a Λ-module M is given by calculating a certain Euler characteristic involving the $\mathcal{O}$-rank of the G-homology groups $\mathrm{H}_i(G, M)$ of M. This can be interpreted as a generalized strong Nakayama lemma.

Proposition 3.2. ([25, thm. 1.1, cor. 1.2]) *The rank of $M \epsilon \Lambda$-mod is given by*

$$\mathrm{rk}_\Lambda M = \sum_{i=0}^{\dim G} (-1)^i \mathrm{rk}_{\mathcal{O}} \mathrm{H}_i(G, M).$$

In particular, if the $\mathrm{H}_i(G, M)$ are finite for all $i \geq 0$, then M is Λ-torsion.

Proof. Since Λ is local and both the Λ-rank and the Euler characteristic formula are additive on short exact sequences, the claim can be verified on free modules for which the statement is obvious. □

For another, possibly more important criterion to check whether a Λ-module is torsion we refer the reader to Proposition 5.12.

For $M \epsilon \Lambda$-mod we define the Iwasawa adjoints of M to be

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_\Lambda(M, \Lambda), \quad i \geq 0,$$

which are a priori right Λ-modules by functoriality and the right Λ-structure of the bi-module Λ but will be considered as left modules via the involution of Λ. By convention we set $\mathrm{E}^i(M) = 0$ for $i < 0$. The Λ-dual $\mathrm{E}^0(M)$ will also be denoted by M^+.

A Λ-module M is called *reflexive* if the canonical map ϕ_M from M to its bi-dual is an isomorphism $M \cong M^{++}$. Note that for general $M \epsilon \Lambda$-mod the kernel of ϕ_M is just the maximal torsion submodule of M.

The following methods sometimes allow to determine the projective dimension of a Λ-module.

Proposition 3.3. ([56, cor. 6.3, cor. 7.2]) *Let M be a finitely generated Λ-module. Then*

$$\begin{aligned} \mathrm{pd}(M) &= \max\{i \mid \mathrm{Tor}_i^\Lambda(M, k) \neq 0\} \\ &= \max\{i \mid \mathrm{Ext}^i_\Lambda(M, k) \neq 0\} \\ &= \max\{i \mid \mathrm{E}^i(M) \neq 0\}. \end{aligned}$$

Above we saw that a module $M \in \Lambda$-mod being torsion can be characterized by the vanishing of $\mathrm{E}^0(M)$. Using higher Iwasawa adjoints one obtains in fact a whole (co-)dimension filtration as we will describe below. The starting point is again a corollary of Lazard's theorem 2.1, which grants that these higher Ext-groups and this (co-)dimension filtration behave well. We first recall that a Noetherian ring R is called *Auslander regular ring*, if R has finite global dimension, d say, and satisfies the Auslander condition: For any R-module M, any integer m and any submodule N of $\mathrm{E}^m(M)$, the grade of N satisfies $j(N) \geq m$. Recall that the *grade* $j(N)$ is the smallest number i such that $\mathrm{E}^i(N) \neq 0$ holds. If R is commutative, one can show that the first already implies the second condition.

Theorem 3.4 ([56])**.** *Λ is Auslander regular.*

For the proof use the fact that for a filtered ring R, if $\mathrm{gr}R$ is Auslander regular, so is R [36, chap. III, thm. 2.2.5].

Therefore there is a nice dimension theory for Λ-modules which we will recall briefly (for proofs and further references see [56]). A priori, any $M \in \Lambda$-mod comes equipped with a finite filtration

$$\mathrm{T}_0(M) \subseteq \mathrm{T}_1(M) \subseteq \cdots \subseteq \mathrm{T}_{d-1}(M) \subseteq \mathrm{T}_d(M) = M.$$

If we call the number $\delta := min\{i \mid \mathrm{T}_i(M) = M\}$ the *dimension* $\delta(M)$ then $\mathrm{T}_i(M)$ is just the maximal submodule of M with δ-dimension less or equal to i. We should mention that for abelian G the dimension $\delta(M)$ coincides with the Krull dimension of $supp_\Lambda(M)$.

The filtration is related to the Iwasawa adjoints via a spectral sequence, in particular we have

$$\mathrm{T}_i(M)/\mathrm{T}_{i-1}(M) \subseteq \mathrm{E}^{d-i}\mathrm{E}^{d-i}(M)$$

and either of these two terms is zero if and only the other is. Furthermore, the equality $\delta(M) + j(M) = d$ holds for any $M \neq 0$.

Note that M is a Λ-torsion module if and only if its codimension $codim(M) := d - \delta(M)$ is greater or equal to 1.

A Λ-module M is called *pseudo-null* if its codimension $codim(M)$ is greater or equal to 2. This is equivalent to the vanishing of $\mathrm{E}^i(M)$ for $i = 0, 1$. As in the commutative case we say that a homomorphism $\varphi : M \to N$ of Λ-modules is a *pseudo-isomorphism* if its kernel and cokernel are pseudo-null. A module M is by definition pseudo-isomorphic to a module N, denoted

$$M \sim N,$$

if and only if there exists a pseudo-isomorphism from M to N. In general, $\sim$ is not symmetric even in the $\mathbb{Z}_p$-case. While in the commutative case $\sim$ is symmetric at least for torsion modules, we do not know whether this property still holds in the general case.

If we want to reverse pseudo-isomorphisms, we have to consider the quotient category Λ-mod$/\mathcal{PN}$ with respect to subcategory $\mathcal{PN}$ of pseudo-null Λ-modules, which is a Serre subcategory, i.e. closed under subobjects, quotients and extensions. By definition, this quotient category is the localization $(\mathcal{PI})^{-1}\Lambda$-mod of Λ-mod with respect to the multiplicative system $\mathcal{PI}$ consisting of all pseudo-isomorphisms. Since Λ-mod is well-powered, i.e. the family of submodules of any module $M \,\epsilon\, \Lambda$-mod forms a set, these localization exists, is an abelian category and the universal functor $q : \Lambda\text{-mod} \to \Lambda\text{-mod}/\mathcal{PN}$ is exact. Furthermore, $q(M) = 0$ in Λ-mod$/\mathcal{PN}$ if and only if $M \,\epsilon\, \mathcal{PN}$. Recall that a morphism $h : q(M) \to q(N)$ in the quotient category can be represented for instance by two Λ-homomorphisms $f : M' \to M$ and $g : M' \to N$ where f is a pseudo-isomorphism and such that $h \circ q(f) = q(g)$; it is an isomorphism if and only if g is a pseudo-isomorphism. If there exists an isomorphism between $q(M)$ and $q(N)$ in the quotient category we also write $M \equiv N \bmod \mathcal{PN}$.

The first kind of a weak structure theorem for Λ-torsion modules up to pseudo-isomorphism was proven in the author's thesis; it only concerns Λ-modules which are $\mathcal{O}$-torsion modules. Though it can now easily obtained as a consequence of the general structure theorem by Coates-Schneider-Sujatha (using their characteristic ideal defined for bounded (see below) objects) we state it separately because it allows a conceptual definition of the μ-invariant. For this we assume that $\mathcal{O}$ is a DVR in mixed characteristic with uniformizer π.

We shall write Λ-mod(p) for the plain subcategory of Λ-mod consisting of $\mathcal{O}$-torsion modules while by $\mathcal{PN}(\pi)$" $= \mathcal{PN} \cap \Lambda$-mod$(\pi)$" we denote the Serre subcategory of Λ-mod(π) the objects of which are pseudo-null Λ-modules. In other words M belongs to $\mathcal{PN}(\pi)$ if and only if it is a Λ/π^n-module for an appropriate n such that $\mathrm{E}^0_{\Lambda/\pi^n}(M) = 0$. Recall that there is a canonical exact functor $q : \Lambda\text{-mod}(\pi) \to \Lambda\text{-mod}(\pi)/\mathcal{PN}(\pi)$. Then, there is the following structure theorem on the $\mathbb{Z}_p$-torsion part of a finitely generated Λ-module:

Theorem 3.5 (cf. [56, Thm. 3.40])**.** *Assume that G is a compact p-adic Lie group without element of order p and such that $\kappa[\![G]\!]$ is an integral domain. Let M be in Λ-mod(π). Then there exist uniquely determined natural numbers $n_1, \ldots, n_r$ and an isomorphism in Λ-mod$(\pi)/\mathcal{PN}(\pi)$*

$$M \equiv \bigoplus_{1 \le i \le r} \Lambda/\pi^{n_i} \bmod \mathcal{PN}(\pi).$$

This result has been generalized and sharpened by S. Howson to the class of central torsion modules, which we will not define here, see [26].

We define the μ-invariant of a Λ-module M as

$$\mu(M) = \sum_i n_i(\mathrm{tor}_{\mathbb{Z}_p} M),$$

where the $n_i = n_i(\mathrm{tor}_{\mathbb{Z}_p}M)$ are determined uniquely by the structure theorem applied to $\mathrm{tor}_{\mathbb{Z}_p}M$. This invariant is additive on short exact sequences of Λ-torsion modules and stable under pseudo-isomorphisms. Alternatively, it can be described as

$$\mu(M) = \mathrm{rk}_{\mathbb{F}_p[\![G]\!]} \bigoplus_{i\geq 0} {}_{\pi^{i+1}}M/{}_{\pi^i}M = rk_{\mathbb{F}_p[\![G]\!]} \bigoplus_{i\geq 0} \pi^i \mathrm{tor}_{\mathbb{Z}_p}M/\pi^{i+1}\mathrm{tor}_{\mathbb{Z}_p}M.$$

Finally, J. Coates, R.Sujatha and P. Schneider [10] have found a general structure theorem for Λ-torsion modules. They prove that any finitely generated $\Lambda(G)$-torsion module decomposes into the direct sum of cyclic modules up to pseudo-isomorphism, i.e. in the quotient category Λ-mod/$\mathcal{PN}$.

Theorem 3.6 (Coates-Schneider-Sujatha)**.** *Let G be a p-valued compact p-adic analytic group. Then, for any finitely generated $\Lambda(G)$-torsion module M there exist finitely many reflexive left ideals $J_1, \ldots, J_r$ and an injective $\Lambda(G)$-homomorphism*

$$\bigoplus_{1\leq i\leq r} \Lambda/J_i \hookrightarrow M/M_{ps}$$

with pseudo-null cokernel, where $M_{ps} = T_{\dim(G)-2}(M)$ denotes the maximal pseudo-null submodule of M. In particular, it holds

$$M \equiv \bigoplus_{1\leq i\leq r} \Lambda/J_i \mod \mathcal{PN}.$$

There are two approaches to prove this result. The first uses the fact that Λ is a maximal order (Theorem 2.10)[1] and Chamarie's general work on modules over these kind of rings, see [10, §3] for details. The second approach uses the graded ring of Λ in order to lift certain Ore-sets which permit roughly speaking to imitate Serre's proof of the structure theorem for commutative integrally closed Noetherian domains (loc. cit. §4).

We remind the reader that over non-commutative rings there exist torsion modules M whose annihilator ideal $Ann(M) := \{\lambda \,\epsilon\, \Lambda | \lambda M = 0\}$ is zero and such modules are called *faithful*. This notation can be extended to the quotient category:
The annihilator ideal of an object $\mathcal{M}$ of Λ-mod/$\mathcal{PN}$ is defined by $Ann(\mathcal{M}) := \sum_{q(M)\cong\mathcal{M}} Ann_\Lambda(M)$, where q denotes again the natural functor

$$q : \Lambda\text{-mod} \to \Lambda\text{-mod}/\mathcal{PN}.$$

Note that by [45, lem 2.5] in conjunction with [5, cor of thm. 2.5] $Ann(q(M)) = Ann_\Lambda(M/M_{ps})$ where M_{ps} denotes as before the maximal pseudo-null submodule

[1] To apply Chamarie's general result for maximal orders one has to verify that every finitely generated torsion A-module induces an object of finite length in the quotient category Λ-mod/$\mathcal{PN}$.

of M. Recall that the object $\mathcal{M}$ is called *completely faithful* if $Ann(\mathcal{N}) = 0$ for any non-zero subquotient $\mathcal{N}$ of $\mathcal{M}$. It is called *bounded* if $Ann(\mathcal{N}) \neq 0$ for any subobject $\mathcal{N} \subseteq \mathcal{M}$ (In particular, the image $q(M)$ of any pseudo-null module M is both bounded and completely faithful, by definition).

Now it holds for a Λ-torsion module M, that its image $\mathcal{M} := q(M)$ decomposes uniquely into a direct sum

$$\mathcal{M} \cong \mathcal{M}_{cf} \oplus \mathcal{M}_b$$

where $\mathcal{M}_{cf}$ is completely faithful and $\mathcal{M}_b$ bounded, cf. [10, prop. 4.1]. Moreover, $\mathcal{M}_{cf}$ or more general all completely faithful objects of finite length $\mathcal{N}$ of $q(\Lambda\text{-mod})$ are cyclic, i.e.

$$\mathcal{N} \cong q(\Lambda/L)$$

for some non-zero left ideal $L \subseteq \Lambda$ (loc. cit. lemma 2.7 and prop. 4.1).

Trivial examples for bounded modules or objects in the quotient category arise by taking central elements $c \in \Lambda$ and forming the corresponding cyclic module $\Lambda/\Lambda c$. In general one would replace Λc by a bounded left ideal, i.e. an left ideal which contains a non-zero two-sided ideal. But the existence of (non-trivial) bounded ideals in Λ is a very delicate question. In section 5.3, Corollary 5.10, we will see examples for completely faithful objects, which surprisingly even occur in arithmetic situations, see Theorem 9.3. Thus the characteristic ideal defined in [10] for bounded objects is not sufficient to describe these arithmetic examples.

We want to point out that reflexive left ideals need not be principal for Iwasawa algebras, see Example 5.13. Another problem for applying the structure theorem directly to describe arithmetic situations relies on the fact that the G-Euler characteristic is not invariant under pseudo-isomorphisms, in general. The following example of a pseudo-null module with non-trivial Euler characteristic stems from [9]

Example 3.7. Let $G = H \times \Gamma$ be a p-valuable group with $\Gamma \cong \mathbb{Z}_p$, generated by γ say, and assume that H contains a subgroup which is a semi-direct product of the following form. Let H_1 and H_2 be two closed subgroups of H which are isomorphic to $\mathbb{Z}_p$, and which are such that $h_2 h_1 h_2^{-1} = h_1^{\phi(h_2)}$ for fixed topological generators h_i of H_i, where $\phi : H_2 \hookrightarrow \mathrm{Aut}(H_1) = \mathbb{Z}_p^\times$ is a continuous injective group homomorphism; in particular, the subgroup $H_1 H_2$ of H is non-abelian. For example, such subgroup exists for any H which is open in $SL_n(\mathbb{Z}_p)$ $(n \geq 2.)$ Putting

$$g = h_1 - 1,\ \omega := h_2 + p^r,$$

for any integer $r \geq 1$, and

$$u = h_2 \cdot \frac{h_1^{\phi(h_2)} - 1}{h_1 - 1}$$

one verifies that

$$g\omega = ug \tag{3.1}$$

holds. Using this relation one sees immediately that right multiplication by g on Λ induces a short exact sequence

$$0 \longrightarrow \Lambda/\Lambda(\gamma - u) \xrightarrow{\cdot g} \Lambda/\Lambda(\gamma - \omega) \longrightarrow M \longrightarrow 0 \tag{3.2}$$

defining the Λ-module M. Note that $\Lambda/\Lambda(\gamma-\omega)$ is isomorphic as $\Lambda(G)$-module to $\Lambda(H)$ on which Γ acts via right multiplication by ω, similar for $\Lambda/\Lambda(\gamma-u)$. Thus M is a pseudo-null $\Lambda(G)$-module because its $\Lambda(H)$-rank is zero, cf. Proposition 5.12. We postpone the calculation of the Euler characteristic to section 8 where we will have more techniques available, see Example 8.8 where we also construct a pseudo-null module with non-trivial Euler characteristic for the semi.direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_p$. Another aspect of the above example will be discussed in Remark 4.2.

4. Virtual objects and some requisites from K-theory

We begin recalling Swan's construction of relative K-groups. For any ring Λ we denote by $\mathcal{P}(\Lambda)$ the category of finitely generated projective Λ-modules. For any homomorphism of rings $\phi : \Lambda \to \Lambda'$, the relative K-group $K_0(\Lambda, \Lambda')$ is defined by generators and relations, as follows. Consider triples (M, N, f) with $M, N \,\epsilon\, \mathcal{P}(\Lambda)$, $f : \Lambda' \otimes_\Lambda M \cong \Lambda' \otimes N$. For brevity, let $M' = \Lambda' = \Lambda' \otimes_\Lambda M$, etc. A morphism

$$(\mu, \nu) : (M_1, N_1, f_1) \to (M_2, N_2, f_2)$$

consists of a pair of maps $\mu \,\epsilon\, \mathrm{Hom}_\Lambda(M_1, M_2),\ \nu \,\epsilon\, \mathrm{Hom}_\Lambda(N_1, N_2)$, such that

$$\nu' \circ f_1 = f_2 \circ \mu' : M_1' \to N_2'.$$

We write $(M_1, N_1, f_1) \cong (M_2, N_2, f_2)$ if both μ and ν are isomorphisms. A short exact sequence of triples is a sequence

$$0 \longrightarrow (M_1, N_1, f_1) \xrightarrow{(\mu_1,\nu_1)} (M_2, N_2, f_2) \xrightarrow{(\mu_2,\nu_2)} (M_3, N_3, f_3) \longrightarrow 0$$

such that each pair (μ_i, ν_i) is a morphism, and where the sequences of Λ-modules

$$0 \longrightarrow M_1 \xrightarrow{\mu_1} M_2 \xrightarrow{\mu_2} M_3 \longrightarrow 0$$

and similarly for N_i with ν_i are exact. Now $K_0(\Lambda, \Lambda')$ is defined as the free abelian group generated by all isomorphism classes of triples, modulo the relations

$$(L, N, gf) = (L, M, f) + (M, N, g)$$

and for each short exact sequence as above

$$(M_2, N_2, f_2) = (M_1, N_1, f_1) + (M_3, N_3, f_3).$$

This relative K-group fits into the following exact sequence of groups

$$K_1(\Lambda) \longrightarrow K_1(\Lambda') \xrightarrow{\delta} K_0(\Lambda, \Lambda') \xrightarrow{\lambda} K_0(\Lambda) \longrightarrow K_0(\Lambda'),$$

where the map δ is defined by $\delta(f) = [\Lambda^n, \Lambda^n, f]$ for $f \in GL_n(\Lambda')$, while the map λ is given by $\lambda([M, N, f]) = [M] - [N]$, and where the brackets denote classes of triples in $K_0(\Lambda, \Lambda')$ and $K_0(\Lambda)$, respectively.

Next we consider the special case where Λ' arises as localisation $\Lambda_{\mathcal{T}}$ of a Noetherian regular ring Λ without zero-divisors by an Ore set $\mathcal{T}$.

Recall that a multiplicative closed subset $\mathcal{T}$ of a ring R is said to satisfy the *right Ore condition* if, for each $r \in R$ and $s \in \mathcal{T}$, there exist $r' \in R$ and $s' \in \mathcal{T}$ such that $rs' = sr'$. If R is Noetherian, then the right Ore condition guarantees that the right localisation $R_{\mathcal{T}}$ of R at $\mathcal{T}$ exists. There is an analogous left version of this and we say that $\mathcal{T}$ is an *Ore set* if it satisfies both the left and right Ore condition. In this case the left and right localisation are canonically isomorphic and thus identified and called localisation of R at $\mathcal{T}$.

We say that a Λ-module M is $\mathcal{T}$-torsion if $\Lambda_{\mathcal{T}} \otimes_\Lambda M = 0$ and we denote by $\Lambda\text{-mod}_{\mathcal{T}-\mathrm{tor}}$ the full subcategory of Λ-mod consisting of all $\mathcal{T}$-torsion modules. Now the category $\Lambda_{\mathcal{T}}$-mod can be identified with the quotient category of Λ-mod with respect to the Serre subcategory $\Lambda\text{-mod}_{\mathcal{T}-\mathrm{tor}}$. Using that by regularity of the rings Λ and $\Lambda_{\mathcal{T}}$ their G- and K-theory coincide the localisation exact sequence of K-theory looks like

$$K_1(\Lambda) \longrightarrow K_1(\Lambda') \xrightarrow{\delta} K_0(\Lambda\text{-mod}_{\mathcal{T}-\mathrm{tor}}) \xrightarrow{\lambda} K_0(\Lambda) \longrightarrow K_0(\Lambda'),$$

where the map λ is induced by the inclusion of categories while the map δ is defined by $\delta(f) = [\mathrm{coker}(f)]$ for $f \in GL_n(\Lambda') \cap M_n(\Lambda)$ and noting that any element of $GL_n(\Lambda')$ is a product of the form fg^{-1} with $f, g \in GL_n(\Lambda') \cap M_n(\Lambda)$, see [1].

It is well known and follows from the 5-lemma that there is a canonical isomorphism of groups

$$K_0(\Lambda\text{-mod}_{\mathcal{T}-\mathrm{tor}}) \;\cong\; K_0(\Lambda, \Lambda_{\mathcal{T}}) \tag{4.3}$$

once we have established a map commuting with the δ's and λ's. Since in our applications both Λ and $\Lambda_{\mathcal{T}}$ are regular rings we only describe it in this situation, for simplicity. First note that under this assumptions the maps λ are both trivial because $[\Lambda] \in \mathbb{Z}[\Lambda] \cong K_0(\Lambda)$ is mapped to $0 \neq [\Lambda_T] \in \mathbb{Z}[\Lambda_{\mathcal{T}}] \cong K_0(\Lambda_{\mathcal{T}})$. Now let M be in $\Lambda\text{-mod}_{\mathcal{T}-\mathrm{tor}}$ and choose a (finite) projective, thus free resolution

$$F^\bullet = F^\bullet(M): \qquad \cdots \longrightarrow F^i \longrightarrow \cdots \longrightarrow F^2 \longrightarrow F^1 \longrightarrow F^0 \longrightarrow 0$$

of M. After tensoring with $\Lambda_{\mathcal{T}}$ this becomes an acyclic resolution $F^\bullet_{\mathcal{T}}$ of free $\Lambda_{\mathcal{T}}$-modules by assumption on M. By $F^+ = F^+(M)$ and $F^- = F^-(M)$ we denote the even and odd sum

$$\bigoplus_{i \text{ even}} F^i \quad \text{and} \quad \bigoplus_{i \text{ odd}} F^i,$$

respectively, and similarly for $F^\bullet_{\mathcal{T}}$. Now we choose successively sections in order to obtain a map $\phi : F^+_{\mathcal{T}} \to F^-_{\mathcal{T}}$ and we define the image of M in $K_0(\Lambda, \Lambda_{\mathcal{T}})$ as

$$M \mapsto [F^+, F^-, \phi].$$

We leave it to the reader to check that this is independent of the choice of the resolution $F^\bullet$ as well as of the map ϕ, for more details in a slightly different context see also [13, lem. 1.1.3,1.2.2,1.2.3, thm. 1.2.1]. One also has to check that our map is additive on short exact sequences in order to see that the above assignment really induces a map on classes $[M] \in K_0(\Lambda\text{-mod}_{\mathcal{T}-\text{tor}})$, cf. (thm. 1.2.7, loc. cit.). For an example of this construction, see 8.8 (ii).

Note that under the assumption that both Λ and $\Lambda_{\mathcal{T}}$ are local, i.e. $K_1(\Lambda_{\mathcal{T}}) \cong \Lambda^\times_{\mathcal{T}}/[\Lambda^\times_{\mathcal{T}}, \Lambda^\times_{\mathcal{T}}]$ and similarly for Λ by [52, ex. 1.6], the group $K_0(\Lambda, \Lambda_T)$ is generated by triples of the form $[\Lambda, \Lambda, f]$ with $f \in \Lambda^\times_T \cap \Lambda$. One sees immediately that $[\Lambda, \Lambda, f] \mapsto [\text{coker}(f)]$ induces the inverse map $K_0(\Lambda, \Lambda_{\mathcal{T}}) \cong K_0(\Lambda\text{-mod}_{\mathcal{T}-\text{tor}})$.

Now we shall describe the image of M in $K_0(\Lambda, \Lambda_{\mathcal{T}})$ using the concept of virtual objects due to Deligne [14] and further developed by Burns and Flach [4] and also used by Huber and Kings [27] in their approach to non-commutative Iwasawa theory.

Let R be a associative ring with unit. The category of *virtual objects* $V(R)$ is a Picard category, i.e. a groupoid (a category in which all morphisms are isomorphisms) equipped with a bifunctor $(L, M) \mapsto L \boxtimes M$ satisfying certain conditions, see [4, §2.1] for details, in particular it has a unit object $\mathbf{1}_{V(R)}$ unique up to unique isomorphism. Furthermore $V(R)$ comes equipped with a functor

$$[-] : (D^p(R), is) \to V(R),$$

where $D^p(R)$ denotes the category of perfect complexes (as full triangulated subcategory of the derived category $D^b(R)$ of the homotopy category of bounded complexes of R-modules) and $(D^p(R), is)$ denotes the subcategory of isomorphisms. For the construction of that functor and a list of its compatibility properties we refer the reader to (prop. 2.1, loc. cit.), here we only mention that $[-]$ commutes with the functors $R' \otimes_R - : D^p(R) \to D^p(R')$ and $R' \otimes_R - : V(R) \to V(R')$ induced by any ring extension $R \to R'$ (prop. 2.1 d), loc. cit.) and that for each exact triangle in $D^p(R)$

$$\Sigma = \Sigma(u, v, w) : X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

there is a nonempty set $[\Sigma]$ of isomorphisms $\phi : [Y] \cong [X] \boxtimes [Z]$ in $V(R)$; in case R is a regular ring $[\Sigma]$ consists of precisely one element, ϕ_Σ say (prop. 2.1, loc. cit.). Finally, one has for $X \in D^p(R)$ such that all $\mathrm{H}_i(X) \in D^p(R)$, too, a canonical isomorphism

$$[X] \cong \boxtimes_{i \in \mathbb{Z}} [\mathrm{H}_i(X)]^{(-1)^{i-1}}, \tag{4.4}$$

see (loc. cit. (9)).

For a commutative local ring the pair $(V(R), [-])$ is equivalent to the determinant functor in the sense of Knudsen and Mumford [32] taking values in the category of graded line bundles on $Spec(R)$ (this is the best way to think about virtual objects). Finally we should mention that the fundamental groups of the Picard category $V(R)$ ($\pi_0(V(R))$ is the group of isomorphism classes of objects of $V(R)$ while $\pi_1(V(R)) = \mathrm{Aut}_{V(R)}(\mathbf{1}_{V(R)})$) are canonically isomorphic to the K-groups $K_0(R)$ and $K_1(R)$ of R (§2.3, loc. cit.). Also the relative K-group $K_0(R, R')$ for a ring homomorphism $R \to R'$ can be realized as fundamental group of a Picard category: Let $\mathcal{P}$ be the Picard category with unique object $\mathbf{1}_{\mathcal{P}}$ and $\mathrm{Aut}_{\mathcal{P}}(\mathbf{1}_{\mathcal{P}}) = 0$. Following [4, (20)] we define $V(R, R')$ to be the fibre product category $V(R) \times_{V(R')} \mathcal{P}$. Thus objects of $V(R, R')$ consists of pairs (M, λ) with $M \in V(R)$ and $\lambda : R' \otimes_R M \to \mathbf{1}_{V(R')}$ an isomorphism in $V(R')$. In analogy with prop. 2.4 (loc. cit) we obtain an isomorphism

$$K_0(R, R') \cong \pi_0(V(R, R'))$$

where $[M, N, f]$ is mapped to $([M] \boxtimes [N]^{-1}, [f] \boxtimes \mathrm{id}_{[R' \otimes_R N]^{-1}})$.

Now we return to our previous situation and let M be in Λ-mod$_{\mathcal{T}-\mathrm{tor}}$. Then the associated complex $M[0]$ concentrated in degree 0 belongs to $D^p(\Lambda)$ and its image in $D^p(\Lambda_{\mathcal{T}})$ is given by an acyclic complex. Thus the isomorphism $\Lambda_{\mathcal{T}} \otimes_\Lambda M[0] \to 0$ in $D^p(\Lambda_{\mathcal{T}})$ induces an isomorphism

$$\lambda_M : \Lambda_{\mathcal{T}} \otimes_\Lambda [M[0]] = [\Lambda_{\mathcal{T}} \otimes_\Lambda M[0]] \to [0] = \mathbf{1}_{V(\Lambda_{\mathcal{T}})}$$

and hence the pair $([M[0]], \lambda_M)$ is an element in $V(\Lambda, \Lambda_{\mathcal{T}})$. We denote its class in $K_0(\Lambda, \Lambda_{\mathcal{T}})$ by $\mathrm{char}_\Lambda(M)$ and call it the *characteristic class* of M.

From the analog of [4, rem. in 2.7] we obtain the following

Proposition 4.1. *Under the above identifications* $K_0(\Lambda\text{-}mod_{\mathcal{T}-\mathrm{tor}}) = K_0(\Lambda, \Lambda_{\mathcal{T}}) = \pi_0(V(\Lambda, \Lambda_{\mathcal{T}}))$ *the different classes associated to* $M \in \Lambda\text{-}mod_{\mathcal{T}-\mathrm{tor}}$ *coincide*

$$[M] = [F(M)^+, F(M)^-, \phi] = \mathrm{char}_\Lambda(M).$$

Since K_1 of a local ring can be calculated via the Dieudonné determinant and the relative K-group $K_0(\Lambda, \Lambda_{\mathcal{T}})$ fits into the following short exact sequence

$$\begin{array}{ccccccc}
K_1(\Lambda) & \longrightarrow & K_1(\Lambda_{\mathcal{T}}) & \longrightarrow & K_0(\Lambda, \Lambda_{\mathcal{T}}) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\Lambda^\times/[\Lambda^\times, \Lambda^\times] & \longrightarrow & \Lambda_{\mathcal{T}}^\times/[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times] & \longrightarrow & \Lambda_{\mathcal{T}}^\times/[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times]\Lambda^\times & \longrightarrow & 1,
\end{array}$$

we can consider $\mathrm{char}_\Lambda(M)$ also as an element $F_M[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times]\Lambda^\times \in \Lambda_{\mathcal{T}}^\times/[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times]\Lambda^\times$. Then we call any such choice $F_M \in \Lambda_{\mathcal{T}}^\times$ a *characteristic* element of M.

Remark 4.2. All the above applies for example to the ring homomorphism $\Lambda(G) \to Q(G)$ and then the relative K-group $K_0(\Lambda(G), Q(G))$ describes nothing else than the Grothendieck group of the full torsion subcategory $\Lambda(G)\text{-mod}_{\text{tor}}$ of $\Lambda(G)$-mod. Thus one could define characteristic classes for $\Lambda(G)$-torsion modules inside this K-group. Unfortunately, in the non-commutative case there are examples M of (pseudo-null) $\Lambda(G)$-torsion modules whose class in $K_0(\Lambda(G)\text{-mod}_{\text{tor}})$ vanishes though the G-Euler-Poincaré characteristic of M does not (cf. [9, §4]), i.e. the latter K-group does not discover this characteristic and thus cannot bear the arithmetic content of e.g. the Selmer group of an elliptic curve (see section 9). In fact, the module in Example 3.7 has this property. Indeed, from the short exact sequence (3.2) one sees that the class of

$$(\gamma - \omega)(\gamma - u)^{-1}$$

in $Q(G)^\times/[Q(G)^\times, Q(G)^\times]\Lambda(G)^\times$ represents that module, and is zero. For the convenience of the reader we recall the short argument from [9, §4]: Since γ is in the center of G, by (3.1) we have

$$g(\gamma - \omega) = (\gamma - u)g$$

and thus we get in $Q(G)^\times/[Q(G)^\times, Q(G)^\times]\Lambda(G)^\times$

$$g(\gamma - \omega)(\gamma - u)^{-1} = (\gamma - u)g(\gamma - u)^{-1} \tag{4.5}$$
$$= (\gamma - u)(\gamma - u)^{-1}g \tag{4.6}$$
$$= g. \tag{4.7}$$

Since g is invertible in $Q(G)^\times/[Q(G)^\times, Q(G)^\times]\Lambda(G)^\times$ the result follows. But we should also mention that we do not know any example for a pseudo-null module with non-vanishing class in $K_0(\Lambda(G)\text{-mod})$.

Thus one could try to search for smaller subcategories of $\Lambda(G)\text{-mod}_{\text{tor}}$. But the characteristic class in the corresponding K-group can only be identified with an characteristic element if the suitable subcategory of $\Lambda\text{-mod}_{-\text{tor}}$ can be described by a pair of rings as for $\Lambda \to \Lambda_{\mathcal{T}}$ for a suitable Ore-set $\mathcal{T}$. Also, only in this case the formalism of virtual objects can be applied to "descent" as we will do in section 8. This is our motivation to study Ore-sets associated with certain group extensions in subsection 6.

5. The relative situation

In the context of arithmetic one often encounters a situation where G is isomorphic to the semi-direct $G = H \rtimes \Gamma$ or even direct product $G = H \times \Gamma$ of H by $\Gamma \cong \mathbb{Z}_p$, mostly Γ will correspond to the cyclotomic $\mathbb{Z}_p$-extension sitting at the bottom of a larger p-adic Lie extension. Then it turns out to be useful considering the Iwasawa algebra of G as a certain skew power series ring $\Lambda(H)[[X; \sigma, \delta]]$ in one variable X over the Iwasawa subalgebra $\Lambda(H)$; the meaning of σ and δ and further basics on such rings will be explained in the next subsection. For such rings one has an analogue of a Weierstrass preparation theorem which we will state in subsection 5.2. In subsection 5.3 we mention some applications of it, e.g. the existence of faithful modules or D. Vogel's example of non-principal reflexive left ideals. Subsection 6 forms the heart of this section (and technically of the whole paper) where we prove that certain multiplicative sets arising from group extensions are Ore-sets. The localisations of the Iwasawa algebra Λ at these Ore sets will give rise to a class of Λ-torsion modules for which we want to define characteristic elements.

5.1. **Non-commutative power series rings.** We shortly recall the definition and some basic facts about skew power series rings (see [38], [36], [11] and [57]).

Let R be a ring, $\sigma : R \to R$ a ring endomorphism and $\delta : R \to R$ a σ-derivation of R, i.e. a group homomorphism satisfying

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s) \text{ for all } r, s \in R.$$

Then the (formal) skew power series ring

$$R[[X; \sigma, \delta]]$$

is defined to be the ring whose underlying set consists of the usual formal power series $\sum_{n=0}^{\infty} r_n X^n$, with $r_n \in R$. However, the multiplication of two such power series is based on the right multiplication of X by elements of R which is defined by the formula

$$Xr = \sigma(r)X + \delta(r). \tag{5.8}$$

This clearly implies that, for all $n \geq 1$, we have

$$X^n r = \sum_{i=0}^{n} (X^n r)_i X^i$$

for certain elements $(X^n r)_i \in R$ and in general the multiplication is given by

$$\left(\sum r_i X^i\right)\left(\sum s_j X^j\right) \;=\; \sum_n c_n X^n$$

where

$$c_n = \sum_{j=0}^{n} \sum_{i=n-j}^{\infty} r_i (X^i s_j)_{n-j}. \tag{5.9}$$

To grant that the above series converges to an element $c_n \in R$ we have to impose further conditions: Either $\delta = 0$ (then $X^n r = \sigma^n(r) X^n$, i.e. the sum $c_n = \sum_{i+j=n} r_j \sigma^i(s_j)$ is finite) or we assume that R is a complete ring with respect to the I-adic topology, I some σ-invariant ideal, and that it holds

$$\delta(R) \subseteq I, \; \delta(I) \subseteq I^2.$$

This implies by induction that $\delta(I^k) \subseteq (I^{k+1})$ for all $k \geq 0$ ($I^0 = R$ by convention) and [57, lem. 2.1] shows that the multiplication law in $R[[X;\sigma,\delta]]$ is well defined. In this case we say that δ is *I-convergent*. But we remark that the convergence might be granted without this rather strong condition as it applies to the following

Example 5.1. Let $G = \mathrm{H} \rtimes \Gamma$ be the semidirect product of a pro-finite group H with $\Gamma \cong \mathbb{Z}_p$. Then the Iwasawa-algebra of G is isomorphic to

$$\Lambda(G) \cong \Lambda(H)[[Y,\sigma,\delta]],$$

where $Y := \gamma - 1$ for some generator γ of Γ, the ring automorphism σ is induced by $h \mapsto h^\gamma$ and $\delta = \sigma - \mathrm{id}$. Indeed, once we have verified the above isomorphism as topological $\mathcal{O}$-modules the assertion concerning the ring structure is immediate. But considered as $\mathcal{O}$-modules we have

$$\begin{aligned} \Lambda(G) &\cong \Lambda(H \times \Gamma) \\ &= \Lambda(H)\widehat{\otimes}_{\mathcal{O}}\Lambda(\Gamma) \\ &\cong \Lambda(H)\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[Y]] \\ &\cong \Lambda(H)[[Y]], \end{aligned}$$

where we use the standard identification $\Lambda(\Gamma) \cong \mathcal{O}[[Y]]$ (after choosing a generator of Γ) and the fact that the complete tensor product commutes with projective limits. The convergence of multiplication is now implicit in the above identification. If we take for instance $H = \mathbb{Z}_p^p$ and let Γ act on H via cyclic permutation of the standard basis of $\mathbb{Z}_p^p$ we cannot expect $\mathfrak{m}_H$-convergence of δ.

Putting more conditions on the semi-direct product we obtain examples satisfying also the I-convergence property for δ :

Example 5.2. Let G be the semi-direct product $G = \Gamma_1 \rtimes \Gamma_2$ of the pro-p-groups Γ_1 and Γ_2, both isomorphic to the additive group of p-adic integers $\mathbb{Z}_p$, where the action of Γ_2 on Γ_1 is given by a continuous group homomorphism $\rho : \Gamma_2 \to \mathrm{Aut}(\Gamma_1) \cong \mathbb{Z}_p^\times$. Then the completed group algebras $\mathbb{Z}_p[\![G]\!]$ and $\mathbb{F}_p[\![G]\!]$ of G with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$ respectively are isomorphic to the skew power series rings

$$\begin{aligned} \mathbb{Z}_p[\![G]\!] &\cong \mathbb{Z}_p[[X]][[Y;\sigma,\delta]] =: \mathbb{Z}_p[[X,Y;\sigma,\delta]], \\ \mathbb{F}_p[\![G]\!] &\cong \mathbb{F}_p[[X]][[Y;\sigma,\delta]] =: \mathbb{F}_p[[X,Y;\sigma,\delta]], \end{aligned}$$

where

(i) $X := \gamma_1 - 1,\ Y := \gamma_2 - 1$ for some generators γ_i of Γ_i, $i = 1, 2$.
(ii) Setting $R := \mathbb{Z}_p[[X]]$ ($\mathbb{F}_p[[X]]$) and $\epsilon := \rho(\gamma_2)$ the ring automorphism σ is induced by $X \mapsto (X+1)^\epsilon - 1$, while $\delta = \sigma - \mathrm{id}$.

Indeed, conferring Lazard 's work [35] (see also [15]) the ring $\mathbb{Z}_p[\![G]\!]$ and $\mathbb{F}_p[\![G]\!]$ has X^iY^j, $0 \leq i,j < \infty$, as a (complete) $\mathbb{Z}_p$- and $\mathbb{F}_p$-basis, respectively, and it is straight forward to verify that the relation between X and Y is described via σ

and δ. Furthermore, R is obviously complete with respect to the topology induced by its maximal ideal $\mathfrak{m}$ which is generated by X and p. Since σ is induced by choosing another generator of Γ_1 under the isomorphism $\mathbb{Z}_p[[X]] \cong \mathbb{Z}_p[\![\Gamma_1]\!]$ we see that $\sigma(\mathfrak{m}) = \mathfrak{m}$. Thus we only have to verify that $\delta(\mathfrak{m}^k) \subseteq \mathfrak{m}^{k+1}$ for $k = 0, 1$: Since δ and σ are $\mathbb{Z}_p$- linear the claim follows immediately from the special case $r = X$ observing also that $\delta(\mathbb{Z}_p) = 0$ (and analogously for $\mathbb{F}_p$):

$$\begin{aligned}
\delta X &= \sigma(X) - X \\
&= (X+1)^\epsilon - 1 - X \\
&= \sum_{i\geq 1} \binom{\epsilon}{i} X^i - X \\
&= (\epsilon - 1)X^k + \text{terms of higher degrees.}
\end{aligned}$$

Since $p|(\epsilon - 1)$ we conclude $\delta(X) \subseteq \mathfrak{m}^2$.

Example 5.3. The previous example generalizes immediately to the following situation: Let H be a uniform pro-p group on which the group $\Gamma \cong \mathbb{Z}_p$ acts via a continuous group homomorphism $\rho : \Gamma \to \mathrm{Aut}(H)$ and assume that the image of ρ is contained in $\Gamma(H^p) := \{\gamma \in \mathrm{Aut}(H) | [\gamma, H] \subseteq H^p\}$ where $[\gamma, h] = h^\gamma h^{-1}$ for $h \in H$. Then it is easy to see that the semidirect product $G = H \rtimes \Gamma$ is again an uniform pro-p group. Its Iwasawa algebras $\Lambda(G) := \mathbb{Z}_p[\![G]\!]$ and $\Omega(G) := \mathbb{F}_p[\![G]\!]$ of G with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$ are isomorphic to the skew power series rings over the Iwasawa algebras $\Lambda(H)$ and $\Omega(H)$, respectively,

$$\begin{aligned}
\Lambda(G) &\cong \Lambda(H)[[Y;\sigma,\delta]], \\
\Omega(G) &\cong \Omega(H)[[Y;\sigma,\delta]],
\end{aligned}$$

where $Y := \gamma - 1$ for some generators γ of Γ, the ring automorphism σ is induced by $h \mapsto h^\gamma$ and again $\delta = \sigma - \mathrm{id}$. Indeed, since for any $h \in H$

$$\begin{aligned}
\delta(h-1) &= \sigma(h-1) - (h-1) \\
&= h^\gamma - h \\
&= (g^p - 1)h \text{ (for some } g \in H \text{ by assumption)} \\
&\in \mathfrak{m}^2 \text{ (see the claim in the proof of [56, lemma 3.24]),}
\end{aligned}$$

$\delta(\mathfrak{m}) \in \mathfrak{m}^2$ and the conditions on σ and δ are fulfilled as above.

Example 5.4. A rather trivial special case of the previous example is $R[[X]]$ where R is any (possibly non-commutative) ring, $\sigma = \mathrm{id}$ and $\delta = 0$. In the Iwasawa theory of elliptic curves there occur naturally open subgroups $G \subseteq GL_2(\mathbb{Z}_p)$ such that $G = H \times \Gamma$ where $H = SL_2(\mathbb{Z}_p) \cap G$ and $\Gamma \cong \mathbb{Z}_p$ is the center of G. Then the Iwasawa algebra $\Lambda(G)$ can be identified with the power series ring $\Lambda(H)[[T]]$ in one (commuting) variable with coefficients in the sub Iwasawa algebra $\Lambda(H)$.

The advantage of identifying the Iwasawa algebra with a skew power series ring is based on the Weierstrass preparation theorem which turns out to be a quite powerful tool in dealing with modules over skew power series rings as it reduces questions concerning power series to equivalent ones concerning polynomials. This will be discussed in the next subsection.

5.2. **The Weierstrass preparation theorem.** Let R be a (not necessarily commutative) local ring with maximal (left) ideal $\mathfrak{m}$ and suppose that R is separated and complete with respect to its $\mathfrak{m}$-adic topology. As usual we denote the residue class skewfield $R/\mathfrak{m}$ by k. As before we assume that $\sigma : R \to R$ is a ring-automorphism satisfying $\sigma(\mathfrak{m}) = \mathfrak{m}$, i.e. σ induces also a ring automorphism $\sigma : k \to k$, while $\delta : R \to R$ is a σ-derivation such that $\delta(R) \subseteq \mathfrak{m},\ \delta(\mathfrak{m}) \subseteq \mathfrak{m}^2$. In particular, δ operates trivially on k. Therefore there is a canonical surjective reduction map:

$$\bar{\ }: R[[X;\sigma,\delta]] \to k[[X;\sigma]].$$

If $R[[X;\sigma,\delta]]$ exists without the $\mathfrak{m}$-convergence of δ, we have a similar reduction map taking values in $k[[X,\sigma,\delta]]$. In examples 5.1(for $\Lambda(H)$ local), 5.2 and 5.3 δ will always be trivial on k while σ will turn out to be the identity on k. For any $f = \sum a_i X^i \ \epsilon\ A := R[[X;\sigma,\delta]]$ the reduced order $\mathrm{ord}^{red}(f)$ of f is defined to be the order of the reduced power series $\overline{f}$, i.e.

$$\mathrm{ord}^{red}(f) = \min\{i | a_i \ \epsilon\ R^{\times}\}.$$

Theorem 5.5. *Assume that δ is $\mathfrak{m}$-convergent. Let $f \ \epsilon\ A$ be a power series with finite reduced order $s := ord^{red}(f) < \infty$. Then A decomposes into the direct sum of the R-modules*

$$A = Af \oplus \bigoplus_{i=0}^{s-1} RX^i.$$

In analogy with the commutative situation we call a monic polynomial $F = X^s + a_{s-1} + \cdots + a_1 X + a_0 \ \epsilon\ R[X;\sigma,\delta] \subseteq R[[X;\sigma,\delta]]$ *distinguished* or *Weierstrass polynomial* (of degree s) if $a_i \ \epsilon\ \mathfrak{m}$ for all $0 \leq i \leq s - i$. Here, $R[X;\sigma,\delta]$ denotes the skew polynomial ring in one variable over R with respect to σ and δ.

Corollary 5.6. *Under the assumptions of the theorem f can be expressed uniquely as the product of an unit ϵ of A and a distinguished polynomial $F \ \epsilon\ A$:*

$$f = \epsilon F.$$

Remark 5.7. (i) Let J be a left ideal of $A = R[[X;\sigma,\delta]]$. Then $M := A/J$ is finitely generated as R-module if and only if J contains a Weierstrass polynomial. Indeed, if all elements of J reduce to zero in $k[[X;\sigma]]$, then $M/\mathfrak{M}M \cong \kappa$ where $\mathfrak{M} := \ker(\bar{\ }: A \to k[[X;\sigma]])$. But since there is some surjection $R^n \to M$ and since $\mathfrak{m} \subseteq \mathfrak{M}$ the module $M/\mathfrak{M}M$ is a finitely generated k-module, a contradiction. The other implication is a direct consequence of theorem 5.5.

(ii) If, moreover, $J = Af$ is principal, then A/J is a finitely generated R-module if and only if $\mathrm{ord}^{red}(f)$ is finite, i.e. if and only if $Af = AF$ can also be generated by some Weierstrass polynomial F.

(iii) (i) and the first part of (ii) extend to the situation where $R[[X;\sigma,\delta]]$ exists without δ being $\mathfrak{m}$-convergent, if R is a local ring which is compact with respect to its $\mathfrak{m}$-adic topology. Indeed, note that in this case $\mathfrak{M} = R\mathfrak{m} = \mathfrak{m}R$. Thus replacing the condition that "J contains a Weierstrass polynomial" by "J contains an element of finite reduced order" we may replace theorem 5.5 in the second implication by Nakayama's lemma: Assume for simplicity that $J = Af$ with $\mathrm{ord}^{red}(f) < \infty$, i.e. we have a short exact sequence

$$A \xrightarrow{\cdot f} A \longrightarrow M \longrightarrow 0.$$

Tensoring with $k[[X;\sigma,\delta]] \cong A/\mathfrak{M}$ leads to

$$k[[X;\sigma,\delta]]4 \xrightarrow{\cdot \bar{f}} k[[X;\sigma,\delta]] \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

showing that $M/\mathfrak{m}M$ is a finite-dimensional k-module (note that $k[[X;\sigma,\delta]]\bar{f} = k[[X;\sigma,\delta]]X^{\mathrm{ord}^{red}(f)}$).

Of course, there exist right versions for all statements of this section. Indeed, since σ is assumed to be an automorphism, one gets a natural ring isomorphism

$$R[[X;\sigma,\delta] \cong [[X;\sigma',\delta']]R,$$

where $\sigma' = \sigma^{-1}$, $\delta' = -\delta \circ \sigma^{-1}$ and in the latter ring the coefficients are written on the right side of the variable X. Moreover, the reduced order is invariant under this isomorphism, i.e. the notion of an distinguished polynomial and its degree is independent of the representation as left or right power series.

5.3. **Faithful modules and non-principal reflexive ideals.** We apply the Weierstrass preparation theorem to show the existence of faithful modules and completely faithful objects in the quotient category up to pseudo-isomorphism. At the end we present D. Vogel's example of a non-principal reflexive left ideal.

We consider again example 5.2, i.e. $A = R[[Y;\sigma,\delta]]$ with $R = \mathbb{Z}_p[[X]]$ or $R = \mathbb{F}_p[[X]]$. Recall that in this case δ is just given as $\sigma - \mathrm{id}$. To exclude the case where G respectively A are commutative, we assume that $\epsilon \neq 1$. We first observe

Remark 5.8. (i) There is a canonical isomorphism

$$R[Y;\sigma,\delta] \cong R[Z;\sigma],\ r \mapsto r,\ Y \mapsto Z - 1$$

of skew polynomial rings, because

$$\begin{aligned} Zr &= (Y+1)r \\ &= Yr + r \\ &= \sigma(r)Y + \delta r + r \\ &= \sigma(r)Y + \sigma(r) - r + r \\ &= \sigma(r)(Y+1) = \sigma(r)Z. \end{aligned}$$

(ii) Any ideal I of A which contains a polynomial $0 \neq b = \sum_{i=0}^{s} b_i Z^i$ also contains a non-zero element $r \,\epsilon\, R$. First note that $Z^i\gamma = \sigma^i(\gamma)Z^i$ for $\gamma := X+1 \,\epsilon\, R^\times$, i.e. $\sigma^j(\gamma)Z^i - Z^i\gamma = (\sigma^j(\gamma) - \sigma^i(\gamma))Z^i$ the latter being zero if and only if $i = j$ because σ operates without fixpoint on Γ_1. Since I is two-sided it also contains the element

$$\begin{aligned} \sigma^s(\gamma)b - b\gamma &= \sum_{i=0}^{s} b_i(\sigma^s(\gamma)Z^i - Z^i\gamma) \\ &= \sum_{i=0}^{s-1} b_i(\sigma^s(\gamma) - \sigma^i(\gamma))Z^i \end{aligned}$$

which is nonzero whenever one of the $b_i, 0 \leq i \leq s-1$, is nonzero because R is integral. Proceeding recursively one concludes that I contains an element of the form rZ^i for some $0 \neq r \,\epsilon\, R$ and $i \,\epsilon\, \mathbb{N}$. But then it contains also r itself because $Z = Y + 1$ is a unit in A.

(iii) The same argument proves that the skew Laurent polynomial ring $Q[Z, Z^{-1}; \sigma]$ over the maximal ring of quotients Q of R is simple (note that the ring automorphism σ extends uniquely to Q.)

Proposition 5.9. (i) *Let $R = \mathbb{F}_p[[X]]$. If M is an A-module which is not torsion as R-module, then its annihilator ideal over A vanishes:*

$$Ann_A(M) = 0.$$

(ii) *Now let $R = \mathbb{Z}_p[[X]]$. If M is an A-module without p-torsion and such that the module M/p is not torsion as $R/p = \mathbb{F}_p[[X]]$-module (for example if M is a free R-module), then $Ann_A(M) = 0$.*

Recall that an A-module M is called *faithful* if the annihilator ideal $Ann_A(M)$ is zero and otherwise M is called *bounded*. Before proving the proposition we want to draw an immediate conclusion from it in the case $R = \mathbb{Z}_p[[X]]$. If M is an A-module which is finitely generated as R-module and of strictly positive R-rank $\mathrm{rk}_R M > 0$, then the conditions of (ii) in the proposition are satisfied for $N := M/\mathrm{tor}_{\mathbb{Z}_p} M$. Indeed, by [25, cor. 1.11] it holds $\mathrm{rk}_{R/p} N/pN = \mathrm{rk}_{R/p}({}_pN) +$

$\mathrm{rk}_R N = \mathrm{rk}_R N \neq 0$ in this case, where ${}_pN$ denotes the kernel of multiplication by p. Thus

Corollary 5.10. *Every A-module M which is finitely generated as R-module and of strictly positive R-rank $\mathrm{rk}_R M > 0$ is faithful. In particular, $q(M)$ is a completely faithful object in the quotient category Λ-mod$/\mathcal{PN}$.*

Further examples of faithful modules can be found at the end of section B in the Appendix.

Proof (of prop. 5.9). By the following lemma the second statement is a consequence of the first one, which can be proven as follows. First note that the R-torsion submodule $M_{R-\mathrm{tor}}$ of M is an A-module, because X is a normal element of A (see below). Thus M has a quotient N without R-torsion and since $Ann_A(M) \subseteq Ann_A(N)$ we may assume without loss of generality that M itself is a torsionfree R-module. Now, if $Ann_A(M)$ contains a non-trivial element $b \in A$ this can be written as $X^n\widetilde{b}$ with $\mathrm{ord}^{red}(\widetilde{b}) < \infty$ because R is a principal ideal domain all ideals of which are generated by some power of X. Due to the Weierstrass preparation theorem $\widetilde{b}$ is a Weierstrass polynomial in the variable Y up to a unit, i.e. we may assume that b is already a Weierstrass polynomial because neither the unit nor X^n can annihilate M by assumption. By the above remark then the annihilator ideal contains a nonzero element $r \in R$ which contradicts the assumption that M is a torsionfree R-module. The second statement is an immediate consequence, for a slightly different proof, see [57, thm. 6.3] □

Lemma 5.11. *Let $\Lambda := \mathbb{Z}_p[\![G]\!]$ and $\Omega := \mathbb{F}_p[\![G]\!] \cong \Lambda/p$ be the completed group algebras of some pro-finite group G with coefficients in $\mathbb{Z}_p$ and $\mathbb{F}_p$, respectively, and assume that both rings are Noetherian integral domains. If M is an A-module without p-torsion then the following holds*

$$Ann_\Lambda(M) \neq 0 \;\Rightarrow Ann_\Omega(M/p) \neq 0.$$

Proof. If f is any nonzero element in $Ann_\Lambda(M)$ and $f = p^n f'$ is the unique factorization such that f' is not divisible by p then f' is also an element in the annihilator of M because the latter module does not have any p-torsion by assumption. Thus the image of f' in Ω is obviously a non-zero element of $Ann_\Omega(M/p)$. □

Another consequence of the Weierstrass preparation theorem is (partly) the next proposition.

Proposition 5.12. ([57, prop. 5.4]) *Let M be in $\Lambda(G)$-mod^H. Then, M is $\Lambda(G)$-torsion and the following holds: M is pseudo-null if and only if $\mathrm{rk}_{\Lambda(H)} M = 0$, i.e. if and only if M is a torsion $\Lambda(H)$-module.*

Finally, we state D. Vogel's example of a non-principal reflexive left ideal in a Iwasawa algebra:

Example 5.13. ([57, Appendix A.3]) Consider the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![G]\!]$ of the semidirect product $G = \Gamma_1 \rtimes \Gamma_2$ of two copies Γ_1, Γ_2 of $\mathbb{Z}_p$ where p is any odd prime number. Let the action of Γ_2 on Γ_1 be given by the continuous group homomorphism

$$\rho : \Gamma_2 \to \mathrm{Aut}(\Gamma_1),\ \gamma_2 \mapsto (\gamma_1 \mapsto \gamma_1^{p+1})$$

for appropriate generators γ_i of Γ_i. As in 5.2 we identify Λ with the skew power series ring

$$\mathbb{Z}_p[\![G]\!] \cong \mathbb{Z}_p[[X, Y; \sigma, \delta]]$$

where $X := \gamma_1 - 1, Y := \gamma_2 - 1$ and recall that the ring automorphism σ of $\mathbb{Z}_p[[X]]$ is induced by $X \mapsto (X+1)^{p+1} - 1$ and δ is the σ-derivation given by $\delta = \sigma - \mathrm{id}$. Now let u be in $\mathbb{Z}_p$ with the following properties: (i) $u\pi + \sigma^2(\pi)$ is divisible by $\sigma(\pi)$ in R, (ii) $u \equiv 1 \bmod p$ and (iii) $\frac{u\pi+\sigma^2(\pi)}{\sigma(\pi)} \equiv 2 \bmod (p, X)$. Putting $\xi = X - p$ he defines skew polynomials

$$f := Y^2 + (2 - \frac{u\xi + \sigma^2(\xi)}{\sigma(\xi)})Y + (u - \frac{u\xi + \sigma^2(\xi)}{\sigma(\xi)} + 1)$$

and

$$k := \xi Y + (\xi - \sigma(\xi)).$$

He then shows that an element $u \in \mathbb{Z}_p$ with the above properties exists and that the left ideal $L := \Lambda f + \Lambda k$ generated by f and k is reflexive, but cannot be generated by a single element in Λ. By the Weierstrass preparation theorem the proof can be reduced to calculations with skew polynomials (instead of skew power series). He observed that an analogue of the Gauss lemma in the situation of a skew polynomial ring over a field does *not* hold and the reason for this failure is in turn used to conclude the desired properties of L. We want to point out that we do *not* know any example of a non-principal reflexive left ideal in the Iwasawa algebra of an open subgroup of $GL_n(\mathbb{Z}_p)$ or $SL_n(\mathbb{Z}_p)$.

6. Ore sets associated with group extensions

In section 4 we saw that for every Ore-set $\mathcal{T}$ the group $K_0(\Lambda, \Lambda_{\mathcal{T}})$ describes the Grothendieck group of $\mathcal{T}$-torsion Λ-modules. From classical Iwasawa theory over $\mathbb{Z}_p$-extensions we know that characteristic elements live in $K_0(\Lambda(\Gamma), Q(\Gamma))$. In order to make use of this information we are looking for a ring R with $\Lambda(G) \subseteq R \subseteq Q(G)$ such that the projection $\psi_H : \Lambda(G) \to \Lambda(\Gamma)$ extends to a commutative diagram

$$\begin{array}{ccc} \Lambda(G) & \twoheadrightarrow & \Lambda(\Gamma) \\ \downarrow & & \downarrow \\ R & \longrightarrow & Q(\Gamma). \end{array}$$

The first candidate for R would be $\Lambda_{\mathcal{T}'}$ in case $\mathcal{T}' := \Lambda(G) \setminus \ker(\psi_H)$ satisfies the Ore condition. But firstly it seems difficult to prove this for a general class of groups (only the case $G = \mathbb{Z}_p \rtimes \mathbb{Z}_p$ is known and straightforward) and secondly even if the existence is known the associated $\mathcal{T}'$-torsion category is not closed under the kind of twisting by representations (even for G abelian) that will be discussed in section 7. Thus we shall take a slightly smaller set $\mathcal{T}$ below.

Since these type of questions are of general interest and since there does not seem to exist any localisation result of this sort in the literature we treat this topic in greater generality than needed for our applications. But see [43, thm. 2.14/15] where a similar topic is discussed in the context of the (usual) group algebra of polycyclic-by-finite groups with coefficients in a field.

Let G be an extension of a torsionfree pro-p p-adic Lie group Γ by a p-adic Lie group H, i.e. we have an short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow \Gamma \longrightarrow 1.$$

The projection $G \twoheadrightarrow \Gamma$ induces a canonical surjective ring homomorphism

$$\psi_H : \mathcal{O}[\![G]\!] \twoheadrightarrow \kappa[\![\Gamma]\!],$$

where κ denotes the residue class field of $\mathcal{O}$ as before. Note that due to compactness $\mathfrak{m}(H) := \ker(\psi_H)$ equals $\mathcal{O}[\![G]\!]\mathfrak{m}_H = \mathfrak{m}_H\mathcal{O}[\![G]\!]$, where $\mathfrak{m}_H$ denotes the kernel of the canonical map

$$\mathcal{O}[\![H]\!] \twoheadrightarrow \kappa.$$

We put $\mathcal{T} := \Lambda \setminus \mathfrak{m}(H)$ and observe that, if $\Gamma \cong \mathbb{Z}_p$ and H is pro-p this set coincides with the set of elements of finite reduced order

$$\mathcal{T} = \{\lambda \in \Lambda(G) | \mathrm{ord}^{\mathrm{red}}(\lambda) < \infty\},$$

where we identify $\Lambda(G)$ with the skew power series ring $\Lambda(H)[[X; \sigma, \delta]]$ according to Example 5.1.

We dare to formulate the following

Conjecture 6.1. *Assume that H is a pro-p group. Then, the multiplicative closed set $\mathcal{T}$ is an Ore set of Λ.*

Recall that a multiplicative closed subset $\mathcal{T}$ of a ring R is said to satisfy the *right Ore condition* if, for each $r \in R$ and $s \in \mathcal{T}$, there exist $r' \in R$ and $s' \in \mathcal{T}$ such that $rs' = sr'$. If R is Noetherian, then the right Ore condition grants that the right localisation $R_{\mathcal{T}}$ of R at $\mathcal{T}$ exists. There is an analogous left version of this and we say that $\mathcal{T}$ is an *Ore set* if it satisfies both the left and right Ore condition. In this case the left and right localisation are canonically isomorphic and thus identified and called localisation of R at $\mathcal{T}$. Recall that an element $x \in R$ is *right regular* if $xr = 0$ implies $r = 0$ for $r \in R$. Similarly *left regular* is defined and *regular* means both right and left regular (and hence not a zero divisor). For an ideal I of R we define the multiplicatively closed set

$$\mathcal{C}_R(I) \quad := \quad \{s \in R | s + I \text{ is regular in } R/I\}.$$

A good reference for (classical) localisation is the book [38, Ch. 2].

Note that the above set $\mathcal{T}$ is nothing else than $\mathcal{C}(\mathfrak{m}(H))$ because $\kappa[\![\Gamma]\!] \cong \kappa[[X]]$ is an integral domain or in other words $\mathfrak{m}(H)$ is a completely prime ideal.

Theorem 6.2. *Let G the semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ or the direct product $H \times \Gamma$ of a uniform group H with a torsion-free pro-p p-adic Lie group Γ. Then the multiplicative set $\mathcal{T}$ satisfies the left and right Ore condition for $\Lambda = \mathcal{O}[\![G]\!]$ for $\mathcal{O}$ either a finite field or a DVR. In particular, the localisation $\Lambda_{\mathcal{T}}$ exists and is a Noetherian regular local ring with* $\mathrm{gl}(\Lambda_{\mathcal{T}}) < \mathrm{gl}(\Lambda)$.

Proof. We mention that since the uniformising element of $\mathcal{O}$ is central in Λ and contained in $\mathfrak{m}(H)$ it would suffice to prove the statement for a finite field κ by [51, lem. 4.2]. Now the strategy is to show that

(i) $\mathfrak{m}(H)$ satisfies the (left and right) Artin Rees property ([38, 4.2.2]) and
(ii) $\mathcal{C}_{\Lambda/\mathfrak{m}(H)^2}(\mathfrak{m}(H)/\mathfrak{m}(H)^2)$ is an Ore set of $\Lambda/\mathfrak{m}(H)^2$.

Then [51, cor. 4.7] (see also [38, 4.2.10]) implies that $\mathcal{T}$ is an Ore set of Λ. These properties will be investigated in the next subsections. The other statements follow from the following lemma. □

We hope that one can modify the above criteria (i) and (ii) replacing the $\mathfrak{m}(H)$-adic filtration by a filtration induced by Lazard's more general filtration associated with $\Lambda(H)$ for any p-valued group (H, ω) or using the stronger criterion of Smith [51, lem. 4.1, thm. 4.6] applied to the $\mathfrak{m}(H')$-adic filtration associated with an uniform normal subgroup H' of G contained in H. This hope is the reason for the above conjecture which could even extend to a larger class of not necessarily pro-p groups, e.g. to open subgroups of $GL_n(\mathbb{Z}_p)$. In fact, there is already joint work with J. Coates and R. Sujatha in progress in order to settle these cases.

Lemma 6.3. *Let Λ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{P} \neq \mathfrak{m}$ a completely prime ideal. Suppose that $\mathcal{T} := \mathcal{C}(\mathfrak{P}) = \Lambda \setminus \mathfrak{P}$ satisfies the Ore condition. Then $\Lambda_{\mathcal{T}}$ is again a local ring, i.e. its non-units form a maximal ideal, and the global dimension of $\Lambda_{\mathcal{T}}$ is (strictly) less than that of Λ. More precisely, the units are of the following form*

$$\Lambda_{\mathcal{T}}^{\times} = \{\lambda t^{-1} | \, \lambda, t \in \mathcal{T}\}.$$

Proof. Let $\mathcal{M} \subsetneqq \Lambda_{\mathcal{T}}$ be a maximal left ideal. By [38, Prop. 2.1.16] we conclude that $\mathcal{M} = \Lambda_{\mathcal{T}}(\mathcal{M} \cap \Lambda) \subseteq I(H)_{\mathcal{T}}$ as $\mathcal{M} \cap \Lambda \subseteq \mathfrak{P}$ by the properness of $\mathcal{M}$ (Note that this argument holds only for localisations at completely prime ideals). By symmetry we see that $\mathfrak{P}_{\mathcal{T}}$ is the unique left and the unique right maximal ideal of $\Lambda_{\mathcal{T}}$ which implies that $\Lambda_{\mathcal{T}}$ is local. The statement concerning the global dimension follows from [38, cor. 4.3] and theorem 4.4 (loc. cit.) applied to the unique simple Λ-module $\Lambda/\mathfrak{m}$. The description of the units is an immediate consequence from the well known fact that the maximal ideal of $\Lambda_{\mathcal{T}}$ is $\mathfrak{P}\Lambda_{\mathcal{T}} = \{pt^{-1} | p \in \mathfrak{P}, t \in \mathcal{T}\}$. □

For the next statement we assume that $\Gamma \cong \mathbb{Z}_p$ and we identify $\Lambda(G)$ with the skew power series ring $\Lambda(H)[[X;\sigma,\delta]]$. In the arithmetic applications we have in mind those $\Lambda(G)$-modules which are finitely generated over $\Lambda(H)$ play an important role, see Theorem 9.4. We write $\Lambda(G)\text{-mod}^H$ for the full subcategory of $\Lambda(G)$-mod consisting of such modules. Recall that a $\Lambda(G)$-module M is called $\mathcal{T}$-torsion, if $M \otimes_\Lambda \Lambda_{\mathcal{T}} = 0$. The significance of $\mathcal{T}$ being an Ore set results from the following observation.

Proposition 6.4. *Assume that $\mathcal{T}$ is an Ore-set.*

(i) *Let $g \in M_n(\Lambda(G)) \cap GL_n(\Lambda(G)_{\mathcal{T}})$ for some natural number n, Then the cokernel of g is finitely generated as $\Lambda(H)$-module while* $\ker g$ *is trivial.*
(ii) *The inclusion $\Lambda(G)\text{-}mod^H \subseteq \Lambda(G)\text{-}mod_{\mathcal{T}-\mathrm{tor}}$ is an identity of categories where the latter one consists of all those finitely generated $\Lambda(G)$-modules which are $\Lambda(G)_{\mathcal{T}}$-torsion.*
(iii) $K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}}) \cong K_0(\Lambda(G)\text{-}mod_{\mathcal{T}-\mathrm{tor}}) \cong K_0(\Lambda(G)\text{-}mod^H)$.

Before we give the proof consider the canonical ring homomorphism $\pi_H : \Lambda(G) \to \Lambda(\Gamma)$ which is induced by the group homomorphism $G \to \Gamma$. Since $\pi_H^{-1}((p)) = \mathfrak{m}(H)$ and thus $\pi_H(\mathcal{T}) \subseteq \Lambda(\Gamma) \setminus (p)$, one obtains a ring homomorphism

$$\Lambda(G)_{\mathcal{T}} \to \Lambda(\Gamma)_{(p)}.$$

Proof. Set $M := \mathrm{coker}(g)$. For all n the augmentation map π_H induces homomorphisms of groups

$$GL_n(\Lambda(G)_{\mathcal{T}}) \to GL_n(\Lambda(\Gamma)_{(p)}).$$

Thus reduction modulo $I(H)$ induces a short exact sequence

$$0 \longrightarrow \Lambda(\Gamma)^n \longrightarrow \Lambda(\Gamma)^n \longrightarrow M_H \longrightarrow 0,$$

where M_H is a finitely generated $\mathbb{Z}_p$-module because any element in M_H has an annihilator prime to p (this implies the left-exactness because M_H being $\Lambda(\Gamma)$-torsion forces the kernel to be torsion as well, but $\Lambda(\Gamma)^n$ is torsionfree). Then Nakayama's lemma implies (i) (for the kernel use the same argument as above). Next we prove that every module in $\Lambda(G)\text{-mod}^H$ is indeed $\Lambda(G)_{\mathcal{T}}$-torsion:

Let $m \in M$ arbitrary. By Remark 5.7 (iii) there is an element of finite reduced order $f \in \Lambda(H)[X]$ and a surjection $\Lambda/\Lambda f \twoheadrightarrow \Lambda m \subseteq M$. Since f obviously annihilates m it suffices to show that f belongs to $\mathcal{T}$: As f has finite reduced order its image in $\mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[[X]]$ is non-zero.

Now let $M \in \Lambda(G)\text{-mod}_{\mathcal{T}-\mathrm{tor}}$ be arbitrary and choose a finite set of $\Lambda(G)$-generators $m_i, i \in I$ of M. By assumption there exists for every $i \in I$ an element of finite reduced order f_i such that $f_i m_i = 0$ and thus M is the homomorphic image of the finitely generated $\Lambda(H)$-module $\bigoplus_{i \in I} \Lambda(G)/\Lambda(G) f_i$ by Remark 5.7 (iii) again. This proves (ii). The last item follows from (ii) and the standard exact localisation sequences of (relative) K-theory. □

6.1. **The direct product case.**

Theorem 6.5. *Let $G = H \times \Gamma$ be the direct product of a uniform group H and a torsion-free pro-p p-adic Lie group Γ. Then the Iwasawa algebra $\Lambda(G)$ is complete and separated with respect to its $\mathfrak{m}(H)$-adic filtration. Its associated graded ring* $\mathrm{gr}\Lambda(G)$ *is isomorphic to a generalized enveloping algebra*

$$\mathrm{gr}_{\mathfrak{m}(H)}\Lambda(G) \cong \begin{cases} U(\mathrm{gr}H \otimes_{\mathrm{gr}\mathbb{Z}_p} \kappa[\pi]) \otimes_{\kappa[\pi]} \kappa[\![\Gamma]\!][\pi] & \text{if } \mathcal{O} \text{ is a DVR,} \\ U(\mathrm{gr}H \otimes_{\mathrm{gr}\mathbb{Z}_p} \kappa) \otimes_{\kappa} \kappa[\![\Gamma]\!] & \text{if } \mathcal{O} = \kappa \text{ is a finite field.} \end{cases}$$

in particular, it is a Noetherian integral domain.

For $G = H \rtimes \Gamma$, the above isomorphism is still an isomorphism of grade $\kappa[\pi]$-modules. Analyzing the induced ring structure will hopefully lead to a proof of Conjecture 6.1 in this case.

Proof. Since $\mathfrak{m}(H)$ is contained in the maximal ideal of $\Lambda(G)$ the $\mathfrak{m}(H)$-adic filtration is separated. By compactness of $\Lambda(G)$ the canonical map

$$\Lambda(G) \to \varprojlim_n \Lambda(G)/\mathfrak{m}(H)^n$$

is thus an isomorphism. Now we calculate the graded ring, using flatness of $\Lambda(G)$ over $\Lambda(H)$ (Lemma B.1) and Corollary 2.4 we obtain the following isomorphisms of graded $\kappa[\pi]$-modules

$$\begin{aligned} \mathrm{gr}_{\mathfrak{m}(H)}\Lambda(G) &\cong \bigoplus_{n\geq 0}\mathfrak{m}(H)^n/\mathfrak{m}(H)^{n+1} \\ &\cong \bigoplus_{n\geq 0}\mathfrak{m}_H^n\Lambda(G)/\mathfrak{m}_H^{n+1}\Lambda(G) \\ &\cong \bigoplus_{n\geq 0}\Big((\mathfrak{m}_H^n/\mathfrak{m}_H^{n+1})\otimes_{\Lambda(H)}\Lambda(G)\Big) \\ &\cong \bigoplus_{n\geq 0}\Big((\mathfrak{m}_H^n/\mathfrak{m}_H^{n+1})\otimes_{\kappa}\kappa[\![\Gamma]\!]\Big) \\ &\cong \Big(\mathrm{gr}_{\mathfrak{m}_H}\Lambda(H)\Big)\otimes_\kappa \kappa[\![\Gamma]\!] \\ &\cong U(\mathrm{gr}H\otimes_{\mathrm{gr}\mathbb{Z}_p}\mathrm{gr}\mathcal{O})\otimes_\kappa\kappa[\![\Gamma]\!] \\ &\cong U(\mathrm{gr}H\otimes_{\mathrm{gr}\mathbb{Z}_p}\mathrm{gr}\mathcal{O})\otimes_{\kappa[\pi]}\kappa[\![\Gamma]\!][\pi]. \end{aligned}$$

Since $G = H\times\Gamma$ this is also a ring-isomorphism. By [38, 1.7.14] and 2.1 the latter ring is an Noetherian integral domain. Alternatively, one can use Remark 2.6 for the determination of $\mathrm{gr}_{\mathfrak{m}(H)}\Lambda(G)$. The above calculation has the advantage to hold (up to the ring structure) also for arbitrary group extensions of the form considered above. □

Recall that a filtered ring R with filtration $F_\bullet R$ is called a Zariski ring if its associated Rees ring

$$\widetilde{R} = \bigoplus_{n\,\epsilon\,\mathbb{Z}} F_nRt^n \subseteq R[t,t^{-1}]$$

is Noetherian and $F_{-1}R$ is contained in the Jacobson ideal of F_0R. Now, the equivalences of (1), (3) and (4) of [36, ch. II §2 thm. 2.1.2] imply the following

Corollary 6.6. *Under the assumptions of the theorem $\Lambda(G)$ endowed with its $\mathfrak{m}(H)$-adic filtration is a Zariski ring, in particular it satisfies the (left and right) Artin Rees property for the ideal $\mathfrak{m}(H)$.*

Lemma 6.7. *Under the conditions of the theorem it holds that*

$$\mathcal{C}_\Lambda(\mathfrak{m}(H)) \subseteq \mathcal{C}_\Lambda(\mathfrak{m}(H)^2).$$

Thus $\mathcal{C}_{\Lambda/\mathfrak{m}(H)^2}(\mathfrak{m}(H)/\mathfrak{m}(H)^2)$ is an Ore set of $\Lambda/\mathfrak{m}(H)^2$.

Proof. Let λ be in $\mathcal{C}_\Lambda(\mathfrak{m}(H)) = \Lambda\setminus\mathfrak{m}(H)$ and let λ' be an element in Λ such for which $\lambda'\lambda\,\epsilon\,\mathfrak{m}(H)^2$ holds. Note that λ' is in $\mathfrak{m}(H)$ because λ belongs to $\mathcal{C}_\Lambda(\mathfrak{m}(H))$. We have to prove that λ' is in $\mathfrak{m}(H)^2$. We assume the contrary, i.e. $\lambda'\,\epsilon\,\mathfrak{m}(H)\setminus\mathfrak{m}(H)^2$. But then we obtain that in the graded ring $\mathrm{gr}_{\mathfrak{m}(H)}\Lambda(G)$

$$(\lambda' + \mathfrak{m}(H)^2)\cdot(\lambda + \mathfrak{m}(H)) = \lambda'\lambda + \mathfrak{m}(H)^2 = 0,$$

which contradicts the integrality of that ring.

The implication that if $\lambda\lambda' \in \mathfrak{m}(H)$ then $\lambda' \in \mathfrak{m}(H)$ follows by symmetry and thus we have shown the first statement which in turn implies that

$$\mathcal{C}_{\Lambda/\mathfrak{m}(H)^2}(\mathfrak{m}(H)/\mathfrak{m}(H)^2) \subseteq \mathcal{C}_{\Lambda/\mathfrak{m}(H)^2}(0).$$

Now the second statement follows from Small's theorem [38, 4.1.3/4]. □

6.2. **The semi-direct product case.** In this subsection we restrict to the easiest semi-direct product case, viz $G = \mathbb{Z}_p \rtimes \mathbb{Z}_p$, though the methods certainly extend to a wider class of poly-cyclic pro-p-groups.

Again, we are first concerned with the Artin-Rees property.

Proposition 6.8. *Let $\mathcal{O} = \kappa$ be a finite field. Then $\mathfrak{m}(H)$ satisfies the Artin Rees property.*

Proof. We identify $\Lambda(G)$ with the skew power series ring $\kappa[[Y, X; \sigma, \delta]]$. Note that Y is a normal element of $\Lambda(G)$, i.e. $\Lambda(G)Y = Y\Lambda(G)$, which generates $\mathfrak{m}(H)$. Thus the statement follows from [38, thm. 4.2.7]. □

With the same technique and some calculations one easily shows that the proposition holds also if $\mathcal{O}$ is a DVR.

Lemma 6.9. *For $\mathcal{O}$ a DVR or a finite field and a uniform group $G = H \rtimes \Gamma$ which is the semidirect product of a normal uniform subgroup H and $\Gamma \cong \mathbb{Z}_p$ it holds that*

$$\mathcal{C}_\Lambda(\mathfrak{m}(H)) \subseteq \mathcal{C}_\Lambda(\mathfrak{m}(H)^2).$$

Thus $\mathcal{C}_{\Lambda/\mathfrak{m}(H)^2}(\mathfrak{m}(H)/\mathfrak{m}(H)^2)$ is an Ore set of $\Lambda/\mathfrak{m}(H)^2$.

Proof. Let λ be in $\mathcal{C}_\Lambda(\mathfrak{m}(H)) = \Lambda \setminus \mathfrak{m}(H)$ and let λ' be an element in Λ such that $\lambda'\lambda \in \mathfrak{m}(H)^2$ holds. Note that λ' is in $\mathfrak{m}(H)$ because λ belongs to $\mathcal{C}_\Lambda(\mathfrak{m}(H))$. We have to prove that λ' is in $\mathfrak{m}(H)^2$. We assume the contrary, i.e. $\lambda' \in \mathfrak{m}(H)\setminus\mathfrak{m}(H)^2$. We identify $\Lambda(G)$ with the skew power series ring $\Lambda(H)[[X; \sigma, \delta]]$ and expand λ and λ' as

$$\lambda = \sum_{i\geq 0} \lambda_i X^i \text{ and } \lambda' = \sum_{i\geq 0} \lambda'_i X^i,$$

where all $\lambda'_i \in \mathfrak{m}_H$ (note that for any $n \geq 0$ the ideal $\mathfrak{m}(H)^n$ consists precisely of those power series in the variable X whose coefficients all lie in $\mathfrak{m}_H^n$). By assumption there exist i_0 and j_0 such that λ_{i_0} and λ'_{j_0} are not in $\mathfrak{m}_H$ and $\mathfrak{m}_H^2$, respectively. Let us assume that these indices are chosen minimal with this property. We want to calculate the product $\lambda'\lambda$ in $\Lambda/\mathfrak{m}(H)^2$ and we observe that the latter ring is isomorphic to the skew power series ring $\Lambda(H)/\mathfrak{m}_H^2[[X; \overline{\sigma}]]$

where $\overline{\sigma}$ is induced by σ, while δ induces the zero derivation. For the (i_0+j_0)th coefficient one obtains

$$0 \equiv (\lambda'\lambda)_{i_0+j_0} \equiv \sum_{k+l=i_0+j_0} \lambda'_k \sigma^k(\lambda_l) \bmod \mathfrak{m}_H^2.$$

The products $\lambda'_k\sigma^k(\lambda_l)$ are in $\mathfrak{m}_H^2$ for $k < j_0$ by definition of j_0 and for $k > j_0$ because then λ'_k and $\sigma^k(\lambda_l)$ $(l < i_0)$ both belong to $\mathfrak{m}_H$ as $\mathfrak{m}_H$ is σ-invariant. Thus $\lambda'_{j_0}\sigma^{j_0}(\lambda_{i_0})$ belongs to $\mathfrak{m}_H^2$, which is a contradiction as $\sigma^{j_0}(\lambda_{i_0})$ is a unit of $\Lambda(H)$. The rest is identical as in the proof of lemma 6.7. □

6.3. **The GL_2 case.** An open pro-p subgroup G of $GL_n(\mathbb{Z}_p)$ is virtually, i.e. after going over to a possibly smaller open subgroup, of the form $H \times \Gamma$ with $H \subseteq SL_n(\mathbb{Z}_p)$ and $\Gamma \cong \mathbb{Z}_p$. For the latter type of subgroups Theorem 6.2 shows that $\mathcal{T}$ is an Ore set. I am very grateful to J. Coates and R. Sujatha for pointing out to me that the above result extends to a larger class of open pro-p subgroups of $GL_n(\mathbb{Z}_p)$ by using a slightly different criterion. Assume that $p \neq 2$ and that n is prime to p. Let G be a pro-p open subgroup of $GL_n(\mathbb{Z}_p)$, with no element of order p. Let $H = G \cap SL_n(\mathbb{Z}_p)$, so that H is the kernel of the determinant map $\det : G \to \mathbb{Z}_p^\times$ (note that $\Gamma := \det(G)$ must be isomorphic to $\mathbb{Z}_p$ as G is pro-p and $p \neq 2$). Assume that H is uniform. Let C be the center of G and set $\Gamma' = \det(C)$. Note that $C \cap H = \{1\}$ as G is pro-p and $p \neq 2$ and that Γ' is isomorphic to C because p does not divide n. We define $G' = C \times H$, so that G' is open in G. Recall that $\mathfrak{m}_H$ denotes the maximal ideal of $\Lambda(H)$ and define $\mathfrak{m}_G(H)$ via

$$\mathfrak{m}_G(H) = \ker(\Lambda(G) \to \kappa[\![\Gamma]\!])$$

and similarly $\mathfrak{m}_{G'}(H)$ via

$$\mathfrak{m}_{G'}(H) = \ker(\Lambda(G') \to \kappa[\![\Gamma']\!]).$$

Note that $\mathfrak{m}_G(H)^n = \Lambda(G)\mathfrak{m}_H^n = \mathfrak{m}_h^n\Lambda(G)$ and similarly for $\mathfrak{m}_{G'}(H)^n$ for all n as H is normal in both G and G'. As before we set $\mathcal{T} := \Lambda(G) \setminus \mathfrak{m}_G(H)$.

Theorem 6.10 (Coates-Sujatha)**.** *The multiplicative set $\mathcal{T}$ is an Ore set in $\Lambda(G)$.*

For the proof we need some preparation. For any Noetherian ring R and finitely generated (left or right) R-module M we denote by $\mathcal{K}_R(M)$ the Krull dimension of a M, see [38, § 6.2] for the precise definition. We recall that $\mathcal{K}_R(M) = 0$ if and only if M has finite length, that $\mathcal{K}_R(M) = 1$ if and only if M has finite length up to modules of Krull dimension less than 1, and so forth. For commutative rings this Krull dimension coincides with the classical one based on the support $\mathrm{supp}(M)$ of the module M contained in the ring spectrum $\mathrm{Spec}(R)$ of R. An ideal I of R is called *weakly invariant* if $\mathcal{K}_R(M \otimes_R I) < \mathcal{K}_R(R/I)$ and $\mathcal{K}_R(I \otimes_R M) < \mathcal{K}_R(R/I)$ for all finitely generated right or left modules M with $\mathcal{K}_R(M) < \mathcal{K}_R(R/I)$, respectively. Finally a ring R is called *homogeneous* if $\mathcal{K}_R(R) = \mathcal{K}_R(L)$

for all nonzero left or right ideals L of R. Now proposition 6.8.21 of [38] implies that $\mathcal{T}$ is an Ore set once we have shown the following

Lemma 6.11. *The ideal* $\mathfrak{m}_G(H)$

(i) *is a prime weakly invariant ideal,*
(ii) *has the Artin-Rees property, and*
(iii) *the ring* $\Omega := \Omega(\Gamma) := \kappa[\![\Gamma]\!] = \Lambda(G)/\mathfrak{m}_G(H)$ *is homogeneous (note that* $\mathcal{K}_{\Lambda(G)}(M)$ *and* $\mathcal{K}_\Omega(M)$ *coincide for all* Ω*-modules* M*).*

Proof. The crucial point is that Ω being a DVR is pure of Krull dimension $\mathcal{K}_\Omega(\Omega)$ equal to 1. In particular, Ω is homogeneous. To prove (i) let M be a nontrivial (say left) $\Lambda(G)$-module with

$$\mathcal{K}_{\Lambda(G)}(M) < \mathcal{K}_{\Lambda(G)}(\Omega) = \mathcal{K}_\Omega(\Omega) = 1.$$

Thus M has finite length and is finite because κ is the unique simple $\Lambda(G)$-module. Therefore it suffices to show that $\mathfrak{m}_G(H) \otimes_{\Lambda(G)} M$ is finite for any finite $\Lambda(G)$-module. This is equivalent to $\mathrm{Tor}_1^{\Lambda(G)}(\Omega, M)$ being finite for all finite modules M. That in turn follows immediately from the well known fact that for the (topologically) finitely generated pro-p group H the first homology group $\mathrm{H}_1(H, M) \cong \mathrm{Tor}_1^{\Lambda(G)}(\Lambda(\Gamma), M)$ is finite for finite M. Finally, the Artin-Rees property will follow analogously as in Corollary 6.6 from the next Proposition. □

Consider $\Lambda(G)$ and $\Lambda(G')$ as filtered rings endowed with the $\mathfrak{m}_G(H)$- and $\mathfrak{m}_{G'}(H)$-adic filtration, respectively. Then $\mathrm{gr}\Lambda(G)$ is a $\mathrm{gr}\Lambda(G')$-module in a natural way. More precisely we have the following

Proposition 6.12. *The associated graded ring* $\mathrm{gr}\Lambda(G)$ *is a free* $\mathrm{gr}\Lambda(G')$*-module of finite rank* $q = [G : G'] = [\Gamma : \Gamma']$*. In particular,* $\mathrm{gr}\Lambda(G)$ *is Noetherian.*

Proof. Pick a set of representatives $e_1 = 1, e_2, \ldots, e_q$ in G for $G/G' \cong \Gamma/\Gamma'$. Then we have an isomorphism of $\Lambda(G')$-modules

$$\Lambda(G) \cong \bigoplus_{i=1}^{q} \Lambda(G')e_i,$$

which induces an isomorphism of $\mathrm{gr}\Lambda(G')$-modules

$$\begin{aligned}
\mathrm{gr}\Lambda(G) &= \bigoplus_{n\geq 0} \mathfrak{m}_G(H)^n/\mathfrak{m}_G(H)^{n+1} \\
&\cong \bigoplus_{n\geq 0} \Big(\mathfrak{m}_H^n/\mathfrak{m}_H^{n+1} \otimes_{\Lambda(H)} \Lambda(G)\Big) \\
&\cong \bigoplus_{n\geq 0} \Big(\mathfrak{m}_H^n/\mathfrak{m}_H^{n+1} \otimes_{\Lambda(H)} \bigoplus_{i=1}^{q} \Lambda(G')e_i\Big) \\
&\cong \bigoplus_{i=1}^{q} \bigoplus_{n\geq 0} \Big(\mathfrak{m}_H^n/\mathfrak{m}_H^{n+1} \otimes_{\Lambda(H)} \Lambda(G')\Big)e_i \\
&\cong \bigoplus_{i=1}^{q} \mathrm{gr}\Lambda(G')\overline{e_i}
\end{aligned}$$

where $\overline{e_i}$ denotes the principal symbol of e_i. Since $\mathrm{gr}\Lambda(G')$ is Noetherian by Theorem 6.5 the ring $\mathrm{gr}\Lambda(G)$ is Noetherian, too. □

7. Twisting

The twisting of the complex L-function by an Artin character corresponds on the algebraic side to tensoring the associated modules (e.g. the dual of the Selmer group of an elliptic curve) by the accordant representation. Basic properties of the latter formalism are studied in the first subsection. In the second subsection we apply it in order to define the evaluation of an characteristic class or element at certain p-adic representations. In the third subsection we discuss several definitions of (equivariant) Euler-characteristics.

7.1. **Twisting of Λ-modules.** Let T be a free $\mathcal{O}$-module of finite rank r with continuous G-action given by

$$\rho : G \to \mathrm{Aut}_{\mathcal{O}}(T) = Gl_r(\mathcal{O}).$$

Definition 7.1. For a finitely generated $\Lambda = \Lambda(G)$-module M we define the finitely generated Λ-module

$$M(\rho) := M \otimes_{\mathcal{O}} T = \mathrm{Hom}_{cont.,\mathcal{O}}(M, A)^\vee$$

with diagonal G-action and where $A = T^\vee$ denotes the Pontryagin dual of T,

Note that the functor $-(\rho)$ is exact.

Lemma 7.2. *For any choice of an $\mathcal{O}$-basis of T, i.e. of an $\mathcal{O}$-module isomorphism $\phi : T \cong \mathcal{O}^r$, there is a canonical isomorphism*

$$\Lambda(\rho) \to \Lambda \otimes_{\mathcal{O}} \mathcal{O}^r \cong \Lambda^r,$$

induced by mapping $g \otimes t$, $g \,\epsilon\, G$, $t \,\epsilon\, T$, to $g\phi(\rho(g^{-1})t)$.

Proof. Fix an isomorphism of $\mathcal{O}$-modules $\phi : T \cong \mathcal{O}^r$ and, for pairs (U, m) consisting of an $m \,\epsilon\, \mathbb{N}$ and an open normal subgroup $U \trianglelefteq G$ such that U acts trivially on T/p^m, consider the well-known isomorphism of Λ-modules

$$\mathcal{O}[G/U] \otimes_{\mathcal{O}} T/p^m T \cong \mathcal{O}[G/U] \otimes_{\mathcal{O}} \mathcal{O}^r/p^m,$$

which sends $gU \otimes (t + p^m T)$ to $gU \otimes (\phi(g^{-1}t) + p^m\mathcal{O}^r)$. It is easily seen that these isomorphisms form an compatible system, i.e.

$$\begin{aligned}
\Lambda \otimes_{\mathcal{O}} A^\vee &= \Lambda \widehat{\otimes}_{\mathcal{O}} A^\vee \\
&= \varprojlim_r (U, m)\mathcal{O}[G/U] \otimes_{\mathcal{O}} (T)/p^m \\
&= \varprojlim_r (U, m)\mathcal{O}[G/U] \otimes_{\mathcal{O}} \mathcal{O}^r/p^m \\
&= \varprojlim_r U, m\mathcal{O}/p^m[G/U]^r \\
&= \Lambda^r.
\end{aligned}$$

□

From this lemma, it follows that if P is a projective Λ-module, then so is $P[A]$.

Proposition 7.3. *For every $i \geq 0$ we have canonical isomorphisms*

$$\mathrm{E}^i(M(\rho)) \cong \mathrm{E}^i(M)(\rho^d),$$

where ρ^d is the contragredient representation, i.e. $\rho^d(g) = \rho(g^{-1})^t$ is the transpose matrix of $\rho(g^{-1})$.

Proof. By homological algebra (and using a free presentation of M) it suffices to prove the case $i = 0$ for free modules. Finally, we only have to check the commutativity of the following diagram which is associated to an arbitrary homomorphism $\phi : \Lambda \to \Lambda$

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda(\Lambda(\rho), \Lambda) & \xrightarrow{\phi(\rho)^*} & \mathrm{Hom}_\Lambda(\Lambda(\rho), \Lambda) \\
\downarrow & & \downarrow \\
\Lambda^r & & \Lambda^r \\
\downarrow & & \downarrow \\
\mathrm{Hom}_\Lambda(\Lambda, \Lambda)(\rho) & \xrightarrow{\phi^*(\rho^d)} & \mathrm{Hom}_\Lambda(\Lambda, \Lambda)(\rho).
\end{array}$$

First note that via the identification $\Lambda^r \overset{\psi_\rho}{=\!=} \Lambda(\rho)$ the matrix representing $\phi(\rho)$ is $A := \sum a_g g\rho(g^{-1})$, where we assume for simplicity that $\phi(1) =: a = \sum a_g g \,\epsilon\, \mathcal{O}[G]$.

We denote by ι both, the involution $\Lambda \to \Lambda$, $g \mapsto g^{-1}$ (also extended to matrices with coefficients in Λ) and the isomorphism of left Λ-modules $\Lambda \to \mathrm{Hom}_\Lambda(\Lambda, \Lambda), g \mapsto (1 \mapsto g^{-1})$. Then its easy to see that the following two diagrams commute

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda(\Lambda(\rho), \Lambda) & \xrightarrow{\phi(\rho)^*} & \mathrm{Hom}_\Lambda(\Lambda(\rho), \Lambda) \\
\downarrow{\scriptstyle (\psi_\rho)^*} & & \downarrow{\scriptstyle (\psi_\rho)^*} \\
\mathrm{Hom}_\Lambda(\Lambda^r, \Lambda) & & \mathrm{Hom}_\Lambda(\Lambda^r, \Lambda) \\
\downarrow{\scriptstyle i^r} & & \downarrow{\scriptstyle i^r} \\
\Lambda^r & \xrightarrow{\iota(A^t)} & \Lambda^r,
\end{array}$$

$$\begin{array}{ccc}
\Lambda^r & \xrightarrow{B} & \Lambda^r \\
\downarrow{\scriptstyle \psi_{\rho^d}} & & \downarrow{\scriptstyle \psi_{\rho^d}} \\
\Lambda(\rho) & \xrightarrow{\iota(a)(\rho^d)} & \Lambda(\rho) \\
\downarrow{\scriptstyle i(\rho^d)} & & \downarrow{\scriptstyle i(\rho^d)} \\
\mathrm{Hom}_\Lambda(\Lambda, \Lambda)(\rho) & \xrightarrow{\phi^*(\rho^d)} & \mathrm{Hom}_\Lambda(\Lambda, \Lambda)(\rho),
\end{array}$$

where $B = \sum a_g g^{-1} \rho^d(g)$, because $\iota(a) = \sum a_g g^{-1}$. We are done if we can verify $B = \iota(A^t)$. But

$$\begin{aligned}
\iota(A^t) &= \sum a_g g^{-1} \rho(g^{-1})^t \\
&= \sum a_g g^{-1} \rho^d(g) = B.
\end{aligned}$$

□

Now let $\rho : G \to GL(V)$ be a continuous linear representation on a finite dimensional Vector space V over a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We choose a G-invariant $\mathcal{O}$-lattice $T \subseteq V$. We denote by $\Lambda(G) = \mathcal{O}[\![G]\!]$ the Iwasawa-algebra of G with coefficients in $\mathcal{O}$.

The following lemma is immediately verified.

Lemma 7.4. *For any finitely generated Λ-module M there are canonical isomorphism of $\mathcal{O}$-modules*

$$T \otimes_{\mathcal{O}[\![G]\!]} M \;\cong\; (T \otimes_{\mathcal{O}} M) \otimes_{\mathcal{O}[\![G]\!]} \mathcal{O}, \tag{7.10}$$

where $t \otimes m$ is mapped to $t \otimes m \otimes 1$ while the inverse map is induced by mapping $t \otimes m \otimes o$ to $o(t \otimes m) = (ot) \otimes m = t \otimes (om)$. They induce isomorphisms

$$\mathrm{Tor}_i^{\mathcal{O}[\![G]\!]}(T, M) \;\cong\; \mathrm{Tor}_i^{\mathcal{O}[\![G]\!]}(T \otimes_{\mathcal{O}} M, \mathcal{O}) \tag{7.11}$$

for all $i \geq 0$.

Now let us again assume to be in the relative situation of section 5 with G being pro-p and $G/H \cong \mathbb{Z}_p$. The exact functor $T \otimes_{\mathcal{O}} -$ induces the following homomorphism of K-groups

$$\rho_* : K_0(\Lambda(G)\text{-mod}^H) \to K_0(\Lambda(G)\text{-mod}^H),$$

indeed, if M is finitely generated over $\Lambda(H)$, so is $T \otimes_{\mathcal{O}} M$. In fact, ρ_* is independent of the choice of the lattice T.

Lemma 7.5. *Assume that the Weierstrass preparation theorem holds or that $\kappa[\![G]\!]$ is an integral domain. Let $M \in \Lambda(G)\text{-mod}^H$ and T' a further G-invariant lattice of V. Then we have*

$$[T \otimes_{\mathcal{O}} M] = [T' \otimes_{\mathcal{O}} M]$$

in $K_0(\Lambda(G)\text{-mod}^H)$.

Proof. By a standard argument we may assume that $T' \subseteq T$ and that $M = \Lambda/\Lambda f$ with an element f of finite reduced order because classes of such modules generate the K-group. It follows easily from our assumption that M is a torsionfree $\mathcal{O}$-module. Denoting by E the finite G-module T/T' we obtain thus the following exact sequence

$$0 \longrightarrow T' \otimes_{\mathcal{O}} M \longrightarrow T \otimes_{\mathcal{O}} M \longrightarrow E \otimes_{\mathcal{O}} M \longrightarrow 0.$$

Using a Jordan Hölder series of E it is sufficient to show that $[\kappa \otimes_{\mathcal{O}} M]$ vanishes. But this follows from the following exact sequence in $\Lambda(G)\text{-mod}^H$

$$0 \longrightarrow M \xrightarrow{\pi} M \longrightarrow \kappa \otimes_{\mathcal{O}} M \longrightarrow 0,$$

where π denotes an uniformizer of $\mathcal{O}$. □

Using the isomorphism $K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}}) \cong K_0(\Lambda(G)\text{-mod}^H)$ established in Proposition 6.4, we obtain also a homomorphism

$$\rho_* : K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}}) \to K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}}),$$

which can be described as follows:

Consider a triple (P_1, P_2, λ) representing a class of $K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}})$ with P_i projective (thus free) $\Lambda(G)$-modules and $\lambda : P_1 \otimes_{\Lambda(G)} \Lambda(G)_{\mathcal{T}} \to P_2 \otimes_{\Lambda(G)} \Lambda(G)_{\mathcal{T}}$ an isomorphism of $\Lambda(G)_{\mathcal{T}}$-modules. Since $P_i \otimes_{\Lambda(G)} \Lambda(G)_{\mathcal{T}},\ i = 1, 2$, are free $\Lambda(G)_{\mathcal{T}}$-modules of rank m, say, and since λ can be described by a invertible matrix with coefficients in $\Lambda(G)_{\mathcal{T}}$ it is easily seen by finding a common denominator of the matrix elements that there exist matrices $A_i \in M_m(\Lambda(G)) \cap GL_m(\Lambda(G)_{\mathcal{T}})$ such that $A_1(A_2)^{-1}$ represents λ (for a certain choice of bases). Now we twist the $\Lambda(G)$-homomorphisms λ_i given by the A_i with T and denote the composite $T \otimes_{\mathcal{O}} \lambda_1 \circ (T \otimes_{\mathcal{O}} \lambda_2)^{-1}$ by $T \otimes_{\mathcal{O}} \lambda$. Now the triple (P_1, P_2, λ) is sent to $(T \otimes_{\mathcal{O}} P_1, T \otimes_{\mathcal{O}} P_2, T \otimes_{\mathcal{O}} \lambda)$.

We also would like to see how twisting by ρ operates on $K_1(\Lambda(G)_{\mathcal{T}}) \cong (\Lambda(G)_{\mathcal{T}}^\times)^{ab}$. To this end, for a finite dimensional continuous $\mathcal{O}$-representation $\rho : G \to \mathrm{Aut}_O(T)$ we define the twist operator

$$\mathrm{tw}_\rho : \Lambda(G) \to \mathrm{End}_{\mathcal{O}}(T) \otimes_{\mathcal{O}} \Lambda(G)$$

as follows. By continuity we may assume that $\lambda \in \Lambda(G)$ is of the form $\sum a_g g$ where almost all $a_g \in \mathcal{O}$ are zero. Then we set $\mathrm{tw}_\rho(\lambda) := \sum a_g \rho(g^{-1}) \otimes g$.

The restriction of tw_ρ to $\mathcal{T}$ and the choice of an $\mathcal{O}$-basis of T induces the multiplicative map

$$\mathrm{tw}_\rho : \mathcal{T} \to M_m(\Lambda) \cap GL_m(\Lambda_{\mathcal{T}}),$$

if $\mathcal{T}$ is an Ore-set of $\mathcal{O}[\![G]\!]$ and where $m = \mathrm{rk}_{\mathcal{O}} T$. Thus tw_ρ extends to a ring homomorphism

$$\mathrm{tw}_\rho : \Lambda_{\mathcal{T}} \to M_m(\Lambda_{\mathcal{T}}).$$

Restricting it to the units we obtain a group homomorphism

$$\mathrm{tw}_\rho : \Lambda_{\mathcal{T}}^\times \to GL_m(\Lambda_{\mathcal{T}}).$$

If we compose this map with the "determinant"

$$\det : GL_m(\Lambda_{\mathcal{T}}) \to K_1(\Lambda_{\mathcal{T}}) \to (\Lambda_{\mathcal{T}}^\times)^{ab}$$

we have

$$\det \circ \mathrm{tw}_\rho : \Lambda_{\mathcal{T}}^\times \to (\Lambda_{\mathcal{T}}^\times)^{ab},$$

which explicitly describes the action on $K_1(\Lambda_{\mathcal{T}})$. From a functorial point of view this action is nothing else that the $K_1(-)$-functor applied to the ring homomorphism tw_ρ and using Morita equivalence:

$$K_1(\Lambda_{\mathcal{T}}) \to K_1(M_m(\Lambda_{\mathcal{T}})) \cong K_1(\Lambda_{\mathcal{T}}).$$

In fact, we can easily extend componentwise the above twisting operator to

$$\mathrm{tw}_\rho : M_n(\Lambda(G)) \to \mathrm{End}_{\mathcal{O}}(T) \otimes_{\mathcal{O}} M_n(\Lambda(G)).$$

Note that the latter ring can be identified - for a chosen $\mathcal{O}$-basis of T - with $M_{nm}(\mathcal{O}[\![G]\!])$. The augmentation map $\pi_H : \mathcal{O}[\![G]\!] \to \mathcal{O}[\![\Gamma]\!]$ induces the map

$$\pi_H : M_n(\mathcal{O}[\![G]\!]) \to M_n(\mathcal{O}[\![\Gamma]\!])$$

which we denote by the same symbol by abuse of notation.

Then we have the more general

Lemma 7.6. *Let g be in $M_n(\Lambda(G))$ for some n and assume that* $\mathrm{coker}(g)$ *is finitely generated as $\Lambda(H)$-module. Then $\mathrm{tw}_\rho(g)$ is in $GL_{nm}(\Lambda_{\mathcal{T}})$ if $\mathcal{T}$ is an Ore-set of Λ. In any case*

$$\pi_H(\mathrm{tw}_\rho(g)) \in GL_{nm}(Q(\Gamma)).$$

Proof. Applying $T \otimes_{\mathcal{O}} -$ to the short exact sequence

$$0 \longrightarrow \Lambda(G)^n \xrightarrow{g} \Lambda(G)^n \longrightarrow \mathrm{coker}(g) \longrightarrow 0.$$

gives the short exact sequence

$$0 \longrightarrow \mathcal{O}[\![G]\!]^{nm} \xrightarrow{\mathrm{tw}_\rho(g)} \mathcal{O}[\![G]\!]^{nm} \longrightarrow T \otimes_{\mathcal{O}} \mathrm{coker}(g) \longrightarrow 0.$$

after choosing a $\mathcal{O}$-basis of T and using the isomorphism

$$T \otimes_{\mathcal{O}} \Lambda(G) \cong \mathcal{O}[\![\Gamma]\!]^m$$

which is induced by $t \otimes g \mapsto \rho(g^{-1})t \otimes g$ with $g \in G$ and $t \in T$ (cf. 7.2). It is well-known that under our assumptions $T \otimes_{\mathcal{O}} \mathrm{coker}(g)$ is again finitely generated as $\Lambda(H)$-module which implies the first statement. Taking H-coinvariants we obtain the short exact sequence

$$0 \longrightarrow \mathcal{O}[\![\Gamma]\!]^{nm} \xrightarrow{\pi_H(\mathrm{tw}_\rho(g))} \mathcal{O}[\![\Gamma]\!]^{nm} \longrightarrow (T \otimes_{\mathcal{O}} \mathrm{coker}(g))_H \longrightarrow 0.$$

which is injective since the kernel is $\mathcal{O}[\![\Gamma]\!]$-torsionfree of rank zero. Since $(T \otimes_{\mathcal{O}} \mathrm{coker}(g))_H$ is a finitely generated $\mathcal{O}$-module the last claim follows after tensoring with $Q_{\mathcal{O}}(\Gamma)$. □

Lemma 7.7. *Let M be in Λ-mod^H and T as above. Then, for every choice of characteristic elements $F_{T\otimes_{\mathcal{O}} M}$ and F_M, we have the equality*

$$F_{T\otimes_{\mathcal{O}} M} = \det_{\Lambda_{\mathcal{T}}} \circ \mathrm{tw}_\rho(F_M)$$

in $(\Lambda_{\mathcal{T}}^\times)^{ab}/\mathrm{im}(\Lambda^\times)$.

Proof. By multiplicativity and additivity of F_M and $[M]$, respectively, we may assume that $F_M \in \mathcal{T}$. Then $[M] = [\mathrm{coker}(F_M)]$ and applying ρ_* gives immediately $[T \otimes_{\mathcal{O}} M] = [\mathrm{coker}(\mathrm{tw}_\rho(F_M)]$ using that for every $g \in \mathcal{T}$

$$T \otimes_{\mathcal{O}} \mathrm{coker}(g) \cong \mathrm{coker}(\mathrm{tw}_\rho(g))$$

via the isomorphism 7.2. Now the result follows from the definition of the determinant. □

7.2. **Evaluating at representations.** We would like to evaluate the characteristic elements $F_M \in \Lambda(G)_{\mathcal{T}}$ at "0" - in other words at the trivial representation -, i.e. we would like to extend the augmentation map $\Lambda(G) \to \mathcal{O}$ to $\Lambda(G)_{\mathcal{T}} \to K$, in order to relate the value to the G-Euler characteristic of M. But since we have to be careful with those denominators which map to zero we define it in two steps, firstly for elements of $Q(\Gamma)$ and secondly of $\Lambda(G)_{\mathcal{T}}$:

First let $F = hg^{-1}$ be an element of $Q(\Gamma)$ with h and g in $\Lambda(\Gamma)$ prime to each other. If the image $g(0)$ of g under the augmentation $\Lambda(\Gamma) \to \mathcal{O}$ is not zero, we say that F can be evaluated at zero and set $F(0) := h(0)g(0)^{-1}$. With other words,

$F(0)$ is defined, if F belongs to the localisation $\Lambda(\Gamma)_{I(\Gamma)}$ at the augmentation ideal $I(\Gamma)$, and then equals the image of F under the extended augmentation map $\Lambda(\Gamma)_{I(\Gamma)} \to K$.

Now let $F = hg^{-1}$ be an element of $\Lambda(G)_{\mathcal{T}}$ with h in $\Lambda(G)$ and g in $\mathcal{T} \subseteq \Lambda(G)$. We say that $F(0)$ is defined if $\pi_H(F)(0)$ is defined and then we set $F(0) := \pi_H(F)(0)$, where

$$\pi_H : \Lambda(G)_{\mathcal{T}} \to Q(\Gamma)$$

is the extended augmentation map (with respect to H).

We want to evaluate $F \epsilon \Lambda_{\mathcal{T}}$ at certain representations using the twist operator. Thus let $\rho : G \to GL(V)$ be a continuous linear representation on a finite dimensional vector space V of dimension m over a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We choose a G-invariant $\mathcal{O}$-lattice $T \subseteq V$ and fix an $\mathcal{O}$-basis.

Consider the following composition of ring homomorphisms

$$\Lambda_{\mathcal{T}} \xrightarrow{\mathrm{tw}_\rho} M_m(\Lambda_{\mathcal{T}}) \xrightarrow{\pi_H} M_m(Q(\Gamma)).$$

It induces a group homomorphism

$$\Lambda_{\mathcal{T}}^\times \xrightarrow{\mathrm{tw}_\rho} GL_m(\Lambda_{\mathcal{T}}) \xrightarrow{\pi_H} GL_m(Q(\Gamma)) \xrightarrow{\det} Q(\Gamma)^\times,$$

which factorizes over the abelianization $(\Lambda_{\mathcal{T}}^\times)^{ab}$ of $\Lambda_{\mathcal{T}}^\times$. Moreover, the composite $\det_{Q(\Gamma)} \circ \pi_H \circ \mathrm{tw}_\rho$ factorizes mod $\Lambda(\Gamma)^\times$ over the quotient $(\Lambda_{\mathcal{T}}^\times)^{ab}/\mathrm{im}(\Lambda^\times)$. We say that $F(\rho)$ is defined if $(\det_{Q(\Gamma)} \circ \pi_H \circ \mathrm{tw}_\rho(F))(0) \epsilon Q(\Gamma)$ is so, and then we take the latter as value of F at ρ. Since the change to another lattice and $\mathcal{O}$-basis corresponds to conjugation by some matrix $A \epsilon GL_2(\mathbb{Q}_p)$ and due to taking determinant, the value of $F(\rho)$ in $K^\times$ is independent of the choice of T and a basis.

From the functoriality of determinants and by Lemma 7.7 we obtain immediately the following

Lemma 7.8. (i) *For every F in $\Lambda_{\mathcal{T}}^\times$ it holds that*

$$\pi_H \circ \det_{\Lambda_{\mathcal{T}}} \circ \mathrm{tw}_\rho(F) = \det_{Q(\Gamma)} \circ \pi_H \circ \mathrm{tw}_\rho(F).$$

(ii) *For every $M \epsilon \Lambda\text{-mod}^H$ and T as above we have, for every choice of characteristic elements $F_{T\otimes_{\mathcal{O}}M}$ and F_M,*

$$F_M(\rho) = F_{T\otimes_{\mathcal{O}}M}(0)$$

in $K^\times/O^\times$.

There is a second but less general way to describe the evaluation at representations. The continuous homomorphism $\rho : G \to GL_{\mathcal{O}}(T)$ of groups induces a homomorphism

$$\rho : \Lambda(G) \to \mathrm{End}_{\mathcal{O}}(T)$$

of $\mathcal{O}$-algebras and we can compose this map with

$$\det_K : \mathrm{End}_{\mathcal{O}}(T) \to \mathcal{O} \subseteq K \subseteq \overline{\mathbb{Q}_p}$$

in order to evaluate elements $f \in \Lambda(G)$ at the representation ρ :

$$f(\rho) := \det_K(\rho^d(f)) \in \mathcal{O}_{\overline{\mathbb{Q}_p}},$$

where $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ denotes the ring of integers of $\overline{\mathbb{Q}_p}$ and where we have chosen the contragredient representation ρ^d of ρ, which belongs to the representation on $\mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})$ induced by ρ, i.e. in terms of matrices $\rho^d(g)$ is the transpose matrix of $\rho(g^{-1})$ (after choosing a basis of T). The reason for this is that we want compatibility with the twisting operator, see Lemma 7.9. Note that this value is independent of the choice of the lattice T. Also one easily verifies that

$$f(\rho^d) = f^{\iota}(\rho)$$

where $-^{\iota} : \Lambda(G) \to \Lambda(G)$ denotes the involution which maps g to g^{-1}.

We still have to justify that this new definition is compatible with our earlier definition of $f(\rho)$.

Lemma 7.9. *Let $F = fg^{-1}$ be in $\Lambda_{\mathcal{T}}$ with $f \in \Lambda(G), g \in \mathcal{T}$. If $g(\rho) \neq 0$, then we have*

$$F(\rho) = \frac{f(\rho)}{g(\rho)}.$$

In particular, the quotient is independent of the choice of fraction $F = fg^{-1}$ with $g(\rho) \neq 0$ (if such fraction exists at all).

Proof. If $g(\rho) \neq 0$, then we have

$$\begin{aligned}
\pi_\Gamma(\det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(F)))) &= \pi_\Gamma(\det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(f)) \cdot \pi_H(\mathrm{tw}_\rho(g))^{-1})) \\
&= \pi_\Gamma(\det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(f)))) \cdot \pi_\Gamma(\det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(g))))^{-1} \\
&= \det_K(\pi_G(\mathrm{tw}_\rho(f))) \cdot \det_K(\pi_G(\mathrm{tw}_\rho(g)))^{-1} \\
&= \frac{f(\rho)}{g(\rho)}.
\end{aligned}$$

Note that the last equality relies on our choice to define $f(\rho)$ using the contragredient representation ρ^d and thus we have $\pi_G(\mathrm{tw}_\rho(f)) = \rho^d(f)^t \in \mathrm{End}_{\mathcal{O}}(T)$ where t indicates the dual or transpose endomorphism. □

7.3. **Equivariant Euler-characteristics.** Let $\mathcal{O}$ be a complete discrete valuation ring which is a finitely generated $\mathbb{Z}_p$ -module, and let K be its field of quotients. Let G a compact (not necessarily pro-p) p-adic Lie group and $U \subseteq G$

a normal open subgroup. Then we set $\Delta := G/U$. It is well known as a consequence of Maschke's theorem and general Wedderburn theory that the group algebra with coefficients in K decomposes as a product of matrix algebras

$$K[\Delta] \cong \prod_{l=1}^{k} M_{n_l}(D_l^o),$$

where D_l^o denotes the opposite of the division fields of endomorphisms $\mathrm{End}_{K[\Delta]}(V_l)$ corresponding to a system of representatives of irreducible K-representations V_l, $1 \leq l \leq k$, of Δ. The integers n_l is the length of $\mathrm{End}_{D_l}(V_l)$ and equals the multiplicity with which the representation V_l occurs in the regular representation of $K[\Delta]$. The centers K_l of $D_l = \mathrm{End}_{K[\Delta]}(V_l)$ (and $M_{n_l}(D_l^o) = \mathrm{End}_{D_l}(V_l)$) are finite field extensions of K, whose ring of integers we denote by $\mathcal{O}_l$.

Henceforth we suppose that $D_l = K_l$ for all $1 \leq l \leq k$ which holds e.g. if K is a splitting field of Δ but we don't assume this stronger condition. Then $\prod_{l=1}^{k} M_{n_l}(\mathcal{O}_l)$ is a maximal order in $K[\delta]$. By choosing Δ-invariant $\mathcal{O}_l$-lattices T_l of V_l and a basis of T_l the above isomorphism induces an embedding of $\mathcal{O}$-algebras

$$\Omega : \mathcal{O}[\Delta] \subseteq \prod_{l=1}^{k} M_{n_l}(\mathcal{O}_l), \tag{7.12}$$

whose cokernel is finite and annihilated by the order $|\Delta_p|$ of a p-Sylow group δ_p of Δ. I am grateful to D. Vogel for pointing out to me the following

Example 7.10. If Δ is a p-group, by a theorem of Fong one obtains an embedding of $\mathbb{Z}_p$-algebras with finite cokernel

$$\mathbb{Z}_p[\Delta] \subseteq \prod_{l=1}^{k} M_{n_l}(\mathbb{Z}_p[\zeta_{p^{m_l}}]),$$

i.e. $K_l = \mathbb{Q}_p(\zeta_{p^{m_l}})$ where $\zeta_{p^{m_l}}$ denotes a primitive p^{m_l}th root of unity for some m_l.

In order to define equivariant Euler-characteristics we will first discuss which different possibilities of (relative) K-groups exist in which they could live. To this end consider the following diagram

$$\begin{array}{ccccc}
K_0(\mathcal{F}(\mathcal{O}[\Delta])) & \xleftarrow{\psi} & K_0(\mathcal{O}[\Delta],\mathbb{Q}_p) & \xrightarrow{\theta} & K_0(\prod_{l=1}^k M_{n_l}(\mathcal{O}_l),\mathbb{Q}_p) \\
 & & & & \| \\
 & & & & \prod_{l=1}^k K_0(M_{n_l}(\mathcal{O}_l),\mathbb{Q}_p) \\
{\scriptstyle forget}\downarrow & & {\scriptstyle forget}\downarrow & & {\scriptstyle Morita}\| \\
 & & & & \prod_{l=1}^k K_0(\mathcal{O}_l,\mathbb{Q}_p) \cong \prod_{l=1}^k \mathbb{Z} \\
 & & & & {\scriptstyle forget}\| {\scriptstyle \prod[\kappa_l:\kappa]} \\
K_0(\mathcal{F}(\mathcal{O})) & = & K_0(\mathcal{O},\mathbb{Q}_p)\cong\mathbb{Z} & \xleftarrow{\Sigma'} & \prod_{l=1}^k K_0(\mathcal{O},\mathbb{Q}_p)\cong \prod_{l=1}^k\mathbb{Z}
\end{array}$$

Here we write $\mathcal{F}(\mathcal{O}[\Delta])$ and $\mathcal{F}(\mathcal{O})$ for the categories of finite $\mathcal{O}[\Delta]$- and $\mathcal{O}$-modules, respectively. For a $\mathbb{Z}_p$-algebra A which is free and finitely generated as $\mathbb{Z}_p$ -module we denote by $K_0(A,\mathbb{Q}_p)$ the relative K-group associated to the ring homomorphism $A \to A\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ in the sense of Swan [54]. But we mention that $K_0(A,\mathbb{Q}_p)$ can be identified with the Grothendieck group of the category of finite A-modules of finite projective dimension. Therefore the exact functor

$$\{\text{finite } A\text{-modules of finite projective dimension}\} \to \{\text{finite } A\text{-modules}\}$$

induces a homomorphism ψ, but note that $K_0(\mathcal{O}[\Delta],\mathbb{Q}_p)$ and $K_0(\mathcal{F}(\mathcal{O}[\Delta])$ are not isomorphic in general.

Since $\mathcal{O}$ is regular we have the first isomorphism in the bottom line which is induced by the analogue of ψ. The maps labelled "forget" are induced by the forgetful functor $\mathcal{O}[\Delta]$-modules $\to$ $\mathcal{O}$-modules and $\mathcal{O}_l$-modules $\to$ $\mathcal{O}$-modules, respectively. Thus the first quadrant obviously commutes. In the second quadrant the map θ is induced by the base change functor $\prod_{l=1}^k M_{n_l}(\mathcal{O}_l)\otimes_{\mathcal{O}[\Delta]} -$, it is surjective because of the canonical isomorphism $K_1(K[\Delta])\cong K_1(\prod_{l=1}^k M_{n_l}(K_l))$ (use the (long) exact sequence of relative K-theory and functoriality of base change). The map labelled "Morita" comes from Morita equivalence, more precisely it is induced by the functor $T_l\otimes_{M_{n_l}(\mathcal{O}_l)}-$, where T_l is considered as $\mathcal{O}_l - M_{n_l}(\mathcal{O}_l)$- bimodule in the obvious way. Note that we have natural isomorphisms of $\mathcal{O}_l - M_{n_l}(\mathcal{O}_l)$-bimodules

$$M_{n_l}(\mathcal{O}_l)\cong\bigoplus_{i=1}^{n_l} e_{n_l}^i M_{n_l}(\mathcal{O}_l)\cong T_l^{n_l},$$

where $e_{n_l}^i$, $1\le i\le n_l$, denotes the idempotent

$$e_{n_l}^i \in M_{n_l}(\mathcal{O}_l)$$

whose unique non-zero entry is 1 at the ith diagonal place. Thus the functor $T_l\otimes_{M_{n_l}(\mathcal{O}_l)}-$ is equivalent to applying $e_{n_l}^i$ for some i, say 1.

Identifying $K_0(\mathcal{O}_l, \mathbb{Q}_p)$ with $\mathbb{Z}$ by associating to a finite $\mathcal{O}_l$-module M its $\mathrm{length}_{\mathcal{O}_l}(M)$ and analogously for $K_0(\mathcal{O}, \mathbb{Q}_p)$ one immediately verifies that the forgetful functor induces the map

$$[\kappa_l : \kappa] : \mathbb{Z} \to \mathbb{Z}, a \mapsto [\kappa_l : \kappa]a,$$

where κ_l and κ denote the residue class fields of $\mathcal{O}_l$ and $\mathcal{O}$, respectively.

Finally, by definition the map $\sum' : \prod_{l=1}^k \mathbb{Z} \to \mathbb{Z}$ maps $(a_l)_l$ to $\sum_{l=1}^k n_l \cdot a_l$ and we claim

Lemma 7.11. *The above diagram is commutative.*

Proof. Note that the group $K_0(\mathcal{O}[\Delta], \mathbb{Q}_p)$ can be generated by triples $(\mathcal{O}[\Delta], \mathcal{O}[\Delta], f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ where $f : \mathcal{O}[\Delta] \to \mathcal{O}[\Delta]$ is a $\mathcal{O}[\Delta]$-module homomorphism with finite kernel and cokernel. The image of such a class in $K_0(\mathcal{O}, \mathbb{Q}_p) \cong \mathbb{Z}$ is $\mathrm{length}_{\mathcal{O}}(\mathrm{coker}(f))$.

Using the embedding 7.12 one obtains the following commutative diagram of $\mathcal{O}$-modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}[\Delta] & \longrightarrow & \prod_{l=1}^k M_{n_l}(\mathcal{O}_l) & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}[\Delta] & \longrightarrow & \prod_{l=1}^k M_{n_l}(\mathcal{O}_l) & \longrightarrow & C & \longrightarrow & 0,
\end{array}$$

where $F = \prod_{l=1}^k M_{n_l}(\mathcal{O}_l) \otimes_{\mathcal{O}[\Delta]} f$. Since C is finite and observing that $\ker(f) = \ker(F) = 0$ one concludes by the snake lemma that

$$\mathrm{length}_{\mathcal{O}}(\mathrm{coker}(f)) = \mathrm{length}_{\mathcal{O}}(\mathrm{coker}(F)).$$

On the other hand $F = \sum_{l=1}^k T_l^{n_l} \otimes_{\mathcal{O}[\Delta]} f$ and thus

$$\mathrm{length}_{\mathcal{O}}(\mathrm{coker}(F)) = \sum_{l=1}^k n_l \cdot \mathrm{length}_{\mathcal{O}}(\mathrm{coker}(F_l)),$$

where $F_l = T_l \otimes_{\mathcal{O}[\Delta]} f$. This implies the lemma. □

Now we are in a position to discuss different Euler-characteristics: Let M be a finitely generated $\mathcal{O}[\![G]\!]$-module with finite projective dimension (if G does not have an element of order p this holds for every $\mathcal{O}[\![G]\!]$-module) and assume that all homology groups $\mathrm{H}_i(U, M)$ are finite. With other words, choosing a projective resolution $P^\bullet \to M$ we obtain a bounded complex of finitely generated projective $\mathcal{O}[\Delta]$-modules

$$P_U^\bullet = \mathcal{O}[\Delta] \otimes^{\mathbb{L}}_{\mathcal{O}[\![G]\!]} M$$

with finite homology groups. Using as before the category of virtual objects we obtain an element $(P_U^\bullet, \lambda_{P_U^\bullet}) \in V(\mathcal{O}[\Delta], K[\Delta])$ whose class in $K_0(\mathcal{O}[\Delta]), \mathbb{Q}_p)$ we denote by

$$\chi_u(U, M)$$

because it should be considered as the "universal" U-Euler characteristic in this situation. On the other hand we define

$$\chi_f(U, M) := \sum_i (-1)^i \, [\mathrm{H}_i(P_U^\bullet)] \in K_0(\mathcal{F}(\mathcal{O}[\Delta])) \tag{7.13}$$

$$= \sum_i (-1)^i \, [\mathrm{Tor}_i^{\mathcal{O}[\![G]\!]}(\mathcal{O}[\Delta], M)] \tag{7.14}$$

$$= \sum_i (-1)^i \, [\mathrm{H}_i(U, M)]. \tag{7.15}$$

If it happens that all $\mathrm{H}_i(P_U^\bullet)$ are in $D^p(\mathcal{O}[\Delta])$, then it follows from (4.4) that

$$\psi(\chi_u(U, M)) = \chi_f(U, M),$$

but we don't know whether this relation fails/holds /makes sense in general. In any case their images under the forgetful functor coincide by the regularity of $\mathcal{O}$ and (4.4) and they equal the "absolute" U-Euler characteristic

$$\chi_a(U, M) := \sum_i (-1)^i \, \mathrm{length}_{\mathcal{O}}(\mathrm{H}_i(U, M)) \in \mathbb{Z} \cong K_0(\mathcal{O}, \mathbb{Q}_p). \tag{7.16}$$

Considering the right hand side of the above diagram we are lead to consider G-Euler characteristics of twisted modules. For any finitely generated free $\mathcal{O}$-module with continuous G-action we define

$$\chi_a(G, T, M) := \sum_i (-1)^i \, \mathrm{length}_{\mathcal{O}}(\mathrm{Tor}_i^{\mathcal{O}[\![G]\!]}(T, M)) \in \mathbb{Z} \cong K_0(\mathcal{O}, \mathbb{Q}_p), \tag{7.17}$$

if all Tor-groups are finite, compare with [26, §3] where it is shown that this *twisted Euler characteristic* only depends on $V = T \otimes_{\mathcal{O}} K$ if G is a pro-p-group, a fact which will turn out automatically in the situation we will consider in section 8.1. For the trivial representation we also write

$$\chi_a(G, M) := \chi_a(G, \mathcal{O}, M),$$

which is conform with our definition of $\chi_a(U, M)$ for $U = G$.

From Lemma 7.11 and its proof we obtain

Proposition 7.12. *The Euler characteristic $\chi_u(U, M)$ is defined if and only if $\chi_a(G, T_l, M)$ is defined for all $1 \le l \le k$. If these equivalent statements hold, then the image of $\chi_u(U, M)$ under the above map $K_0(\mathcal{O}[\Delta], \mathbb{Q}_p) \to \prod_{l=1}^k K_0(\mathcal{O}, \mathbb{Q}_p)$*

equals $(\chi_a(G,T_l,M))_l$. *Consequently we have*

$$\chi_a(U,M) = \sum_{l=1}^{k} n_l \cdot \chi_a(G,T_l,M) = \chi_a(G,M) + \sum_{l\neq 1} n_l \cdot \chi_a(G,T_l,M),$$

where we assume without loss of generality $T_1 = \mathcal{O}$.

Let us consider special cases.

Case I: Δ is a p-group.

Then $\mathcal{O}[\Delta]$ is a local ring with unique simple module κ (with trivial action) [39, ch. V]. Thus $K_0(\mathcal{F}(\mathcal{O}[\Delta]) = K_0(\mathcal{F}(\mathcal{O})) = \mathbb{Z}$ and therefore $\chi_f(U,M)$ and $\chi_a(U,M)$ coincide. By example 7.10 we may take $\mathcal{O} = \mathbb{Z}_p$ here and then $[\kappa_l : \kappa] = 1$ for all l.

Case II: Δ is of order prime to p.

Now the embedding Ω is an isomorphism [12, prop. 27.1] and thus $\chi_u(U,M)$ is completely determined by the tuple $(\chi_a(G,T_l,M))_l$. Moreover, $\mathcal{O}[\Delta]$ is regular now and thus $\psi : K_0(\mathcal{O}[\Delta],\mathbb{Q}_p) \cong K_0(\mathcal{F}(\mathcal{O}[\Delta]))$ is an isomorphism.

8. Descent of K-theory

Let G be a compact p-adic Lie group satisfying the following conditions: (i) G is pro-p, (ii) G has no element of order p, and (iii) G has a closed normal subgroup H such that $\Gamma := G/H \cong \mathbb{Z}_p$. In particular, G is isomorphic to the semi-direct product $H \rtimes \Gamma$. By $\Lambda(G)$-mod and $\Lambda(G)$-mod^H we denote the category of finitely generated $\Lambda(G)$-modules and its full subcategory consisting of modules which are finitely generated as $\Lambda(H)$-module,respectively. The latter is a full subcategory of the category of finitely generated torsion $\Lambda(G)$-modules.

In [9, §4] Coates-Schneider-Sujatha define the alternating characteristic ideal of $M \in \Lambda(G)$-mod^H as follows:

$$\mathrm{Ak}_G(H,M) := \prod_{i\geq 0} \mathrm{char}_\Gamma(\mathrm{H}_i(H,M))^{(-1)^i}.$$

This can be considered as a relative Euler characteristic and following J. Coates we call it the *Akashi series* of M. This is a fractional ideal of the quotient field $Q(\Gamma)$ of $\Lambda(\Gamma)$ and can alternatively be interpreted as an element of $Q(\Gamma)^\times/\Lambda(\Gamma)^\times$. Then the above invariant induces the following map of K-groups

$$\mathrm{Ak}_G(H,-) : K_0(\Lambda(G)\text{-mod}^H) \to K_0(\Lambda(\Gamma), Q(\Gamma)) \cong Q(\Gamma)^\times/\Lambda(\Gamma)^\times,$$

where $K_0(\Lambda(\Gamma), Q(\Gamma))$ denotes the relative K-group of the ring-homomorphism $\Lambda(\Gamma) \to Q(\Gamma)$ in the sense of Swan [54], see section 4.

Now assume that the multiplicative set $\mathcal{T}$ defined in subsection 6 is an Ore-set and recall from section 4 that we had associated with every $M \in \Lambda(G)\text{-mod}^H = \Lambda(G)\text{-mod}_{\mathcal{T}-\mathrm{tor}}$ an characteristic class

$$\mathrm{char}_G(M) := \mathrm{char}_{\Lambda(G)}(M)$$

in the group $K_0(\Lambda, \Lambda_{\mathcal{T}})$ which fits into the following short exact sequence

$$\begin{array}{ccccccc} K_1(\Lambda) & \longrightarrow & K_1(\Lambda_{\mathcal{T}}) & \longrightarrow & K_0(\Lambda, \Lambda_{\mathcal{T}}) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ \Lambda^\times/[\Lambda^\times, \Lambda^\times] & \longrightarrow & \Lambda_{\mathcal{T}}^\times/[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times] & \longrightarrow & \Lambda_{\mathcal{T}}^\times/[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times]\Lambda^\times & \longrightarrow & 1. \end{array}$$

Thus we considered $\mathrm{char}_G(M)$ also as an element $F_M[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times]\Lambda^\times \in \Lambda_{\mathcal{T}}^\times/[\Lambda_{\mathcal{T}}^\times, \Lambda_{\mathcal{T}}^\times]\Lambda^\times$ and we called (any such choice) $F_M \in \Lambda_{\mathcal{T}}^\times$ a *characteristic* element of M.

Now consider the canonical ring homomorphism $\pi_H : \Lambda(G) \to \Lambda(\Gamma)$ which is induced by the group homomorphism $G \to \Gamma$. It induces a commutative diagram of K-groups with exact rows

$$\begin{array}{ccccccc} K_1(\Lambda) & \longrightarrow & K_1(\Lambda_{\mathcal{T}}) & \longrightarrow & K_0(\Lambda, \Lambda_{\mathcal{T}}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle (\pi_H)_*} & & \downarrow{\scriptstyle (\pi_H)_*} & & \downarrow{\scriptstyle (\pi_H)_*} & & \\ K_1(\Lambda(\Gamma)) & \longrightarrow & K_1(Q(\Gamma)) & \longrightarrow & K_0(\Lambda(\Gamma), Q(\Gamma)) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ \Lambda(\Gamma)^\times & \longrightarrow & Q(\Gamma)^\times & \longrightarrow & Q(\Gamma)^\times/\Lambda(\Gamma)^\times & \longrightarrow & 1, \end{array}$$

where the middle map $(\pi_H)_*$ is induced by $\Lambda_{\mathcal{T}}^\times \to Q(\Gamma)^\times$. The following result is now almost self-proving.

Proposition 8.1. *Let $M \in \Lambda(G)\text{-mod}^H$. Then the following holds:*

$$\mathrm{Ak}_G(H, M) \equiv \pi_H(\mathrm{char}_G(M)) \equiv \pi_H(F_M) \bmod \Lambda(\Gamma)^\times,$$

i.e. there is a commutative diagram

$$\begin{array}{ccccc} K_0(\Lambda(G)\text{-mod}^H) & = & K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}}) & = & \Lambda(G)_{\mathcal{T}}^\times/[\Lambda(G)_{\mathcal{T}}^\times, \Lambda(G)_{\mathcal{T}}^\times]\Lambda(G)^\times. \\ & {\scriptstyle \mathrm{Ak}_G(H,-)}\searrow & \downarrow{\scriptstyle (\pi_H)_*} & \swarrow{\scriptstyle (\pi_H)_*} & \\ & & K_0(\Lambda(\Gamma), Q(\Gamma)) & & \end{array}$$

Proof. The second $\equiv$ follows from the functoriality of our identifications. To prove the first choose a projective resolution $P^\bullet \to M$ by a perfect complex $P^\bullet$. As explained in the paragraph before Proposition 4.1 $\mathrm{char}_G(M)$ is given by the class of $([P^\bullet], \lambda_{P^\bullet}) \in V(\Lambda, \Lambda_{\mathcal{T}})$ where

$$\lambda_{P^\bullet} : \Lambda_{\mathcal{T}} \otimes_\Lambda [P^\bullet] = [\Lambda_{\mathcal{T}} \otimes_\Lambda P^\bullet] \cong \mathbf{1}_{V(\Lambda_{\mathcal{T}})}$$

is the canonical isomorphism of virtual objects in $V(\Lambda_{\mathcal{T}})$ associated to the quasi-isomorphism $\Lambda_{\mathcal{T}} \otimes_{\Lambda} P^{\bullet} \to 0$. Now the base chance maps $\pi_H : \Lambda \to \Lambda(\Gamma)$ and $\pi_H : \Lambda_{\mathcal{T}} \to Q(\Gamma)$ induce a morphism of Picard categories

$$(\pi_H)_* : V(\Lambda, \Lambda_{\mathcal{T}}) \to V(\Lambda(\Gamma), Q(\Gamma))$$

under which $([P^{\bullet}], \lambda_{P^{\bullet}})$ is mapped to $([\Lambda(\Gamma) \otimes_{\Lambda} P^{\bullet}], Q(\Gamma) \otimes_{\Lambda_{\mathcal{T}}} \lambda_{P^{\bullet}})$ where we use the compatibility of (non-commutative) determinants $[-]$ with arbitrary change of rings:

$$Q(\Gamma) \otimes_{\Lambda_{\mathcal{T}}} \Lambda_{\mathcal{T}} \otimes_{\Lambda} [P^{\bullet}] = Q(\Gamma) \otimes_{\Lambda(\Gamma)} [\Lambda(\Gamma) \otimes_{\Lambda} P^{\bullet}],$$
$$Q(\Gamma) \otimes_{\Lambda_{\mathcal{T}}} \mathbf{1}_{V(\Lambda_{\mathcal{T}})} = \mathbf{1}_{V(Q(\Gamma))}$$

and thus

$$Q(\Gamma) \otimes_{\Lambda_{\mathcal{T}}} \lambda_{P^{\bullet}} : Q(\Gamma) \otimes_{\Lambda(\Gamma)} [\Lambda(\Gamma) \otimes_{\Lambda} P^{\bullet}] \xrightarrow{\cong} \mathbf{1}_{V(Q(\Gamma))}.$$

Since for the commutative rings $\Lambda(\Gamma)$ and $Q(\Gamma)$ we can replace $[-]$ by the functor $\det(-)$, the above shows that $\pi_H(\mathrm{char}_G(M))$ in $K_0(\Lambda(\Gamma), Q(\Gamma))$ is represented by the pair $(\det_{\Lambda(\Gamma)}(\Lambda(\Gamma) \otimes_{\Lambda} P^{\bullet}), \lambda_{\Lambda(\Gamma)\otimes_{\Lambda} P^{\bullet}})$, where

$$\lambda_{\Lambda(\Gamma)\otimes_{\Lambda(G)} P^{\bullet}} = Q(\Gamma) \otimes_{\Lambda_{\mathcal{T}}} \lambda_{P^{\bullet}} : \det_{Q(\Gamma)}(Q(\Gamma) \otimes_{\Lambda} P^{\bullet}) \cong \det_{Q(\Gamma)}(0) = (Q(\Gamma), 0).$$

Since $\Lambda(\Gamma)$ is regular we obtain by [32] or [4, (9)] a canonical isomorphism

$$\begin{aligned} \det_{\Lambda(\Gamma)}(\Lambda(\Gamma) \otimes_{\Lambda(G)} P^{\bullet}) &\cong \bigotimes_{i \,\epsilon\, \mathbb{Z}} \det_{\Lambda(\Gamma)}(H^i(\Lambda(\Gamma) \otimes_{\Lambda(G)} P^{\bullet})^{(-1)^{i-1}} \\ &= \bigotimes_{i \,\epsilon\, \mathbb{Z}} \det_{\Lambda(\Gamma)}(H_i(H, M))^{(-1)^{i-1}}. \end{aligned}$$

But Kato [29, prop. 6.1] observed that $\mathrm{char}_{\Gamma}(N)$" = "$(\det_{\Lambda(\Gamma)}(N))^{-1}$ for every torsion $\Lambda(\Gamma)$-module N, in the sense that under the canonical isomorphism

$$\det_{Q(\Gamma)}(Q(\Gamma) \otimes_{\Lambda(\Gamma)} N) \cong \det_{Q(\Gamma)}(0) = (Q(\Gamma), 0)$$

$\det_{\Lambda(\Gamma)}(N) \subseteq \det_{Q(\Gamma)}(Q(\Gamma) \otimes_{\Lambda(\Gamma)} N)$ is mapped to the fractional ideal generated by $\mathrm{char}_{\Gamma}(N)^{-1}$. This implies that $\det_{\Lambda(\Gamma)}(\Lambda(\Gamma) \otimes_{\Lambda(G)} P^{\bullet})$ is mapped under $\lambda_{\Lambda(\Gamma)\otimes_{\Lambda(G)} P^{\bullet}}$ to $\Lambda(\Gamma)\mathrm{Ak}_G(H, M) \subseteq Q(\Gamma)$ and the proposition follows.

In order to understand better the different constructions we now sketch a second proof which avoids the use of virtual objects (which have nice functorial and universal properties as we have seen above but which are quite difficult to imagine). To this end we represent M by the class $[P^+, P^-, \phi]$ where $\phi : P^+_{\mathcal{T}} \to P^-_{\mathcal{T}}$ is an isomorphism obtained by choosing successively sections, see section 4. Since M is torsion P^+ and P^- are free of the same rank, r say, and thus the class of $\phi \,\epsilon\, \mathrm{Aut}_{\Lambda_{\mathcal{T}}}(\Lambda^r_{\mathcal{T}})$ in $K_1(\Lambda_{\mathcal{T}})$ is a characteristic element of M. Base change with respect to π_H gives a class $[P^+_H, P^-_H, \phi_H]$ where

$$\phi_H = Q(\Gamma) \otimes_{\Lambda_{\mathcal{T}}} \phi : Q(\Gamma) \otimes_{\Lambda(\Gamma)} P^+_H \cong Q(\Gamma) \otimes_{\Lambda(\Gamma)} P^-_H$$

is the induced isomorphism of free $Q(\Gamma)$-modules. Of course, $\phi_H = (\pi_H)_*(\phi)$ is a characteristic element of the complex $P_H^\bullet$. Now using Proposition 4.1 one can proceed as above using determinants or - if one even wants to avoid them - it is not difficult but tedious to show directly that $[P_H^+, P_H^-, \phi_H]$ can be expressed as alternating sum of classes associated to the homology groups of $P_H^\bullet$ (cf. [13]), which are $\Lambda(\Gamma)$-torsion modules and thus represented by classes $[\Lambda(\Gamma), \Lambda(\Gamma), \mathrm{char}_\Gamma(\mathrm{H}^i(H, M)]$.

But the most elegant way is the following: Use the fact that $\Lambda(G)_{\mathcal{T}}^\times$ is generated by the elements in $\mathcal{T}$ and check the commutativity just for $t \in \mathcal{T}$. Obviously, t is mapped to the module $\Lambda/\Lambda t$ whose Akashi series is just $\pi_H(t)$. □

Remark 8.2. (The commutative case) Let G be isomorphic to ${\mathbb{Z}_p}^d$ for some integer $d \geq 2$ and fix for the moment a subgroup H such that $G/H \cong \mathbb{Z}_p$. Recall from [2, §4.5] that in this situation we have canonical isomorphisms

$$K_0(\Lambda(G)\text{-mod}_{\mathrm{tor}}/\mathcal{PN}) \cong K_0(\Lambda(G)\text{-mod}_{\mathrm{tor}}) \cong \mathrm{Div}(\Lambda(G)),$$

where $\mathrm{Div}(\Lambda(G)) \cong \bigoplus_{P \in \mathcal{P}} \mathbb{Z}$ denotes the group of divisors of $\Lambda(G)$ and $\mathcal{P}$ denotes a system of representatives of classes of irreducible elements of $\Lambda(G)$. In particular, using this identification an element $f = (unit) \prod_{\mathcal{P}} P^{v_P(f)} \in Q(G)^\times \cong K_1(Q(G))$ is mapped to its divisor $\mathrm{div}(f) = \sum_{\mathcal{P}} v_P(f) P$ under the connecting map in the localization sequence of K-theory. Now one sees immediately that the injection $\Lambda_{\mathcal{T}}^\times \to Q(G)^\times$ induces a commutative diagram

$$\begin{array}{ccc} \Lambda_{\mathcal{T}}^\times/\Lambda(G)^\times & = & K_0(\Lambda(G)\text{-mod}^H) \\ \downarrow & & \downarrow \\ Q(G)^\times/\Lambda(G)^\times & = & K_0(\Lambda(G)\text{-mod}_{\mathrm{tor}}). \end{array}$$

In particular, we see that our characteristic class or element for modules which are finitely generated over $\Lambda(H)$ can be identified with the usual one of the full torsion category. In the next remark we will see that this fails in the non-commutative situation because the big commutator subgroups of the units destroy the injectivity. The image of $K_0(\Lambda(G)\text{-mod}^H)$ in $\mathrm{Div}(\Lambda(G))$ is precisely

$$\mathrm{Div}(\Lambda(G))_H := \bigoplus_{P \in \mathcal{P}_{\mathrm{red}}} \mathbb{Z},$$

where $\mathcal{P}_{\mathrm{red}}$ denotes the subset of $\mathcal{P}$ consisting of all elements having finite reduced order (with respect to H). In geometric terms $\mathrm{Div}(\Lambda(G))_H$ consists precisely of those cycles, which have codimension 1 and which have good intersection with the closed subscheme defined by $\mathfrak{m}(H)$.

If now H ranges over all subgroups of G having a quotient isomorphic to $\mathbb{Z}_p$ one concludes using [19, lem. 2] that

$$\varinjlim_{H} K_0(\Lambda(G)\text{-mod}^H) \cong \mathrm{Div}(\Lambda(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong \bigoplus_{p \neq P \,\epsilon\, \mathcal{P}} \mathbb{Z},$$

i.e. up to $\mathbb{Z}_p$-torsion modules one obtains all $\Lambda(G)$-torsion modules this way.

Remark 8.3. In Remark 4.2 we have discussed why we have to replace the full torsion category $\Lambda(G)\text{-mod}_{\mathrm{tor}}$ by a smaller subcategory like $\Lambda(G)\text{-mod}^H \cong \Lambda\text{-mod}_{\mathcal{T}-\mathrm{tor}}$. Note that the example mentioned there also shows that the canonical map

$$K_0(\Lambda(G)\text{-mod}^H) \to K_0(\Lambda(G)\text{-mod})$$

is not injective in general. In contrast to our expectation from the commutative theory (e.g. Gersten's conjecture) it is also remarkable that the canonical map

$$K_0(\Lambda\text{-mod}_{\Lambda(H)-\mathrm{tor}}) \to K_0(\Lambda(G)\text{-mod}^H)$$

is *not* trivial, i.e. also pseudo-null $\Lambda(G)$-modules (in $\Lambda(G)\text{-mod}^H$) can give rise to non-trivial classes.

Remark 8.4. Let $F_M = gh^{-1}$ be a characteristic element for $M \,\epsilon\, \Lambda(G)\text{-mod}^H$ with $g, h \,\epsilon\, \Lambda_{\mathcal{T}}$. Then we have

$$\mathrm{rk}_{\Lambda(H)} M = \mathrm{ord}^{\mathrm{red}}(F_M) := \mathrm{ord}^{\mathrm{red}}(g) - \mathrm{ord}^{\mathrm{red}}(h).$$

In particular, $\mathrm{ord}^{\mathrm{red}}(F_M) \geq 0$ is independent of the chosen fraction. We think of $\mathrm{rk}_{\Lambda(H)} M$ as a generalized λ-invariant. The claim follows from the fact that the $\Lambda(H)$-rank is additive on short exact sequences and thus induces a group homomorphism $\mathrm{rk}_{\Lambda(H)} : K_0(\Lambda(G)\text{-mod}^H) \to \mathbb{Z}$. Now $[M] = [\Lambda/\Lambda g] - [\Lambda/\Lambda h]$ and from the Weierstrass preparation theorem we conclude that $\mathrm{rk}_{\Lambda(H)} \Lambda/\Lambda g = \mathrm{ord}^{\mathrm{red}}(g)$.

Proposition 8.5. *Assume that M in $\Lambda(G)\text{-mod}^H$ satisfies a relation $[M] = [\mathrm{coker}(f)] - [\mathrm{coker}(g)]$ for some $f, g \,\epsilon\, M_n(\Lambda(G))$ such that $\mathrm{coker}(f), \mathrm{coker}(g)$ belong to $\Lambda(G)$-mod^H. Then for all representations $\rho : G \to \mathrm{Aut}_{\mathcal{O}}(T)$ we have*

$$\mathrm{Ak}_G(H, T \otimes_{\mathcal{O}} M) = \det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(f)) \cdot \pi_H(\mathrm{tw}_\rho(g))^{-1})$$

in $K_0(\mathcal{O}[\![\Gamma]\!], Q(\Gamma))$. In particular, it holds for any choice of F_M that

$$\mathrm{Ak}_G(H, T \otimes_{\mathcal{O}} M) = \det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(F_M))).$$

Proof. Applying ρ_* to the relation $[M] = [\mathrm{coker}(f)] - [\mathrm{coker}(g)]$ gives the relation

$$[T \otimes_{\mathcal{O}} M] = [\mathrm{coker}(\mathrm{tw}_\rho(f))] - [\mathrm{coker}(\mathrm{tw}_\rho(g))]$$

which in turn induces after evaluating the functor $\mathrm{Ak}_G(H,-)$ (with $\mathcal{O}$-coefficients)

$$\begin{aligned}\mathrm{Ak}_G(H, T\otimes_{\mathcal{O}} M) &= \mathrm{Ak}_G(H, \mathrm{coker}(\mathrm{tw}_\rho(f)))\cdot \mathrm{Ak}_G(H, \mathrm{coker}(\mathrm{tw}_\rho(f)))^{-1}\\ &= \det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(f)))\cdot \det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(g)))^{-1}\\ &= \det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(f))\cdot \pi_H(\mathrm{tw}_\rho(g))^{-1}),\end{aligned}$$

where the second equality is easily verified, compare with the proof of Lemma 7.6. □

In subsection 7.3 we have defined an additive Euler characteristic, but for our arithmetic applications the following multiplicative version is more suitable

$$\chi(G,T,M) := (\#\kappa)^{\chi_a(G,T,M)} \text{ (if } \chi_a(G,T,M) \text{ is defined).}$$

Similarly we write $\chi(U,M)$ for the multiplicative version of $\chi_a(U,M)$.

Proposition 8.6. *Assume that M in $\Lambda(G)$-mod^H satisfies a relation $[M] = [\mathrm{coker}(f)] - [\mathrm{coker}(g)]$ for some $f,g \,\epsilon\, M_n(\Lambda(G))$ such that $\mathrm{coker}(f), \mathrm{coker}(g)$ are finitely generated over $\Lambda(H)$. Then, if $\chi(G,T,M)$ is finite, $fg^{-1}(\rho)$ is defined (and non-zero) and we have*

$$\chi(G,T,M) = |fg^{-1}(\rho)|_p^{-[K:\mathbb{Q}_p]},$$

where the norm is normalized by $|p|_p = \frac{1}{p}$. In particular, we have for any choice of F_M

$$\chi(G,T,M) = |F_M(\rho)|_p^{-[K:\mathbb{Q}_p]}.$$

This proposition in junction with Lemma 7.5 shows that $\chi(G,T,M)$ actually depends only on the p-adic representation $V = T\otimes_{\mathcal{O}} K$ and thus we will sometimes write $\chi(G,V,M)$ for it.

Proof. Under the assumptions $\mathrm{Ak}_G(H, T\otimes_{\mathcal{O}} M)(0)$ is defined in the sense of [9, §4] and $\chi(G, T\otimes_{\mathcal{O}} M) = |\mathrm{Ak}_G(H, T\otimes_{\mathcal{O}} M)(0)|_p^{-[K:\mathbb{Q}_p]}$ by [9, lem. 4.2]. Actually, the lemma is only stated for $\mathbb{Z}_p$-coefficients but it obviously generalizes to the case of coefficients in $\mathcal{O}$ using the well known fact that

$$\#(\mathcal{O}/\mathcal{O}f) = |f|_p^{[K:\mathbb{Q}_p]}$$

for every $f \,\epsilon\, \mathcal{O}\setminus\{0\}$. The idea of that lemma is the following: evaluating an element of $K_0(\Lambda(\Gamma, Q(\Gamma))$ at zero means calculating the Γ-Euler characteristic of that element (if defined), i.e. in this case the Γ-Euler characteristic of $\prod_{i\geq 0} \mathrm{char}_\Gamma(\mathrm{H}_i(H,M))^{(-1)^i}$, which is related to the G-Euler characteristic of M via the (almost degenerating) spectral sequence

$$\mathrm{H}_p(\Gamma, \mathrm{H}_q(H,M)) \Rightarrow \mathrm{H}_{p+q}(G,M).$$

Inspection of the latter gives their lemma.

By Prop. 8.5 we have $\det_{Q(\Gamma)}(\pi_H(\mathrm{tw}_\rho(f))\cdot\pi_H(\mathrm{tw}_\rho(g))^{-1})(0) = \mathrm{Ak}_G(H, T\otimes_{\mathcal{O}} M)(0)$ and thus $fg^{-1}(\rho)$ is defined and the claim follows from the previous Lemma. □

For $G \cong \mathbb{Z}_p$ it is well known that also a converse statement holds: If $f_M(0) \neq 0$, then $\chi(G, M)$ is defined. But in higher dimensions one easily constructs counterexamples in which lemma 4.2 of [9] and thus this sort of statement fails to be true.

Corollary 8.7. *Let U be an open normal subgroup of G and $M \in \Lambda(G)\text{-mod}^H$. Then, for any choice of F_M, we have*

$$\chi(U, M) = \prod_{\rho} |F_M(\rho)|_p^{-[K:\mathbb{Q}_p]n_\rho}$$

where ρ runs through a system of representatives of irreducible K-representations of $\Delta := G/U$ and where K is a splitting field of Δ.

This result should be compared with well-known formulas in the $\mathbb{Z}_p$-situation, see [39, Ch. V § 3 Ex. 3].

At the end of this section we want to determine characteristic elements in several examples and use them to calculate the Euler characteristics.

Example 8.8. (i) Concerning example 5.13 D. Vogel [59] shows that there is an exact sequence of Λ-modules

$$0 \longrightarrow \Lambda/\Lambda N \xrightarrow{d} \Lambda/\Lambda f \longrightarrow \Lambda/L \longrightarrow 0,$$

where $N = Y + 1 - u$ and the map d is given by $\lambda + \Lambda N \mapsto \lambda k + \Lambda f$ for $\lambda \in \Lambda$. Thus setting

$$F_{\Lambda/L} = fN^{-1} = \{Y^2 + (2-\beta)Y + (u - \beta + 1)\}\{Y + 1 - u\}^{-1}$$

where

$$\beta = \frac{u\xi + \sigma^2(\xi)}{\sigma(\xi)}$$

is a characteristic element for Λ/L and $\mathrm{Ak}_G(H, \Lambda/L)$ is generated by

$$\pi_H(F_{\Lambda/L}) = \frac{Y^2 + (1-u)Y}{Y + 1 - u} = Y.$$

In particular, $\chi(G, \Lambda/L)$ is not defined.

(ii) As in the first example let $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$, where now the action is defined by an arbitrary non-trivial continuous character $\rho : \mathbb{Z}_p \to \mathbb{Z}_p^\times$. In particular, we have $\gamma h \gamma^{-1} = h^\delta$ for fixed topological generators h, γ and with non-trivial $\delta = \rho(\gamma) \in \mathbb{Z}_p^\times$. Again we identify $\Lambda = \Lambda(G)$ with $\mathbb{Z}_p[[X, Y; \sigma, \delta]]$, where $X = h - 1$ and $Y = \gamma - 1$. In this case it is a little exercise to deduce that the pro-p Fox-Lyndon resolution ([39, 5.6.6], see also [33, Satz 7.7]) $P^\bullet \to \mathbb{Z}_p$ has the following form

$$0 \longrightarrow \Lambda \xrightarrow{d_2} \Lambda^2 \xrightarrow{d_1} \Lambda \longrightarrow \mathbb{Z}_p \longrightarrow 0,$$

where d_2 is given by right multiplication with the matrix $\begin{pmatrix} M & N \end{pmatrix}$ where $N = -\sigma(X)$ and $M = Y + \frac{X-\sigma(X)}{X}$ (or equivalently, if δ is a positive integer, $N = 1 - \gamma h \gamma^{-1}$ and $M = \gamma - \sum_{i=0}^{\delta-1} h^i$). The map d_1 corresponds to the matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$. Tensoring with $\Lambda_{\mathcal{T}}$ gives a short split exact sequence

$$P_{\mathcal{T}}^{\bullet}: \qquad 0 \longrightarrow \Lambda_{\mathcal{T}} \underset{t}{\overset{d_2}{\rightleftarrows}} \Lambda_{\mathcal{T}}^2 \underset{s}{\overset{d_1}{\rightleftarrows}} \Lambda_{\mathcal{T}} \longrightarrow 0.$$

A possible choice for s is for instance given by the matrix $(0\ Y^{-1})$ and then $d \circ t = \mathrm{id} - s \circ d_1$ is given by

$$\begin{pmatrix} 1 & -XY^{-1} \\ 0 & 0 \end{pmatrix}.$$

We conclude that thus t corresponds to $\begin{pmatrix} M^{-1} \\ 0 \end{pmatrix}$. Hence, the map

$$P_{\mathcal{T}}^- \cong \Lambda_{\mathcal{T}}^2 \xrightarrow{d_1 \oplus t} \Lambda_{\mathcal{T}}^2 \cong P_{\mathcal{T}}^+$$

is given by the matrix

$$\begin{pmatrix} M^{-1} & X \\ 0 & Y \end{pmatrix}$$

and using Proposition 4.1 we see that

$$F_{\mathbb{Z}_p} = M^{-1}Y = \left(Y + \frac{X - \sigma(X)}{X}\right)^{-1} Y$$

is a characteristic element for $\mathbb{Z}_p$. For any continuous character $\psi : \mathbb{Z}_p \to \mathbb{Z}_p^{\times}$ we obtain characteristic elements

$$\begin{aligned} F_{\mathbb{Z}_p(\psi)} &= \mathrm{tw}_{\psi}(F_{\mathbb{Z}_p}) \\ &= \left(\mathrm{tw}_{\psi}(Y) + \frac{X - \sigma(X)}{X}\right)^{-1} \mathrm{tw}_{\psi}(Y) \end{aligned}$$

where

$$\mathrm{tw}_{\psi}(Y) = \psi(\gamma)^{-1}(Y+1) - 1$$

(viewing ψ via $G \twoheadrightarrow \mathbb{Z}_p$ as a character of G). Thus $\mathrm{Ak}_G(H, \mathbb{Z}_p(\psi))$ is generated by

$$\pi_H(F_{\mathbb{Z}_p(\psi)}) = \frac{\psi(\gamma)^{-1}(Y+1) - 1}{\psi(\gamma)^{-1}(Y+1) - \rho(\gamma) + 1}$$

and hence we have

$$\chi(G, \mathbb{Z}_p(\psi)) = |\frac{1 - \psi(\gamma)}{1 - \psi(\gamma)(\rho(\gamma) - 1)}|_p = |\psi(\gamma) - 1|_p,$$

which is different from 1 for non-trivial ψ.

(iii) Finally we complete the discussion of Example 3.7: Using sequence (3.2) to calculate the H-homology Coates-Schneider-Sujatha show that the ideal $\mathrm{Ak}_G(H, M)$ is generated by

$$f_M(T) = \frac{T - \omega(0) + 1}{T - u(0) + 1},$$

where for any $z \in \Lambda(H)$ we write $z(0)$ for the image of z under the augmentation map in $\mathbb{Z}_p$. Since ϕ is injective we have

$$\omega(0) = 1 + p^r \neq \phi(h_2) + p^r = u(0)$$

and thus f_M is not a unit in $\Lambda(\Gamma)$. From the short exact sequence (3.2) one sees immediately that

$$F_M = (c - \omega)(c - u)^{-1} \in \Lambda_{\mathcal{T}}$$

is a characteristic element of M. This gives a second calculation of f_M by 8.1. Finally, by 8.6 the G-Euler characteristic is given by

$$\chi(G, M) = |\frac{\phi(h_2) - 1 + p^r}{p^r}|_p,$$

which is generically non-trivial.

9. Characteristic elements of Selmer groups

In this section we are going to apply the techniques developed so far to study properties of the Selmer group of an elliptic curve over a p-adic Lie extension k_∞. Needless to say that we could also take arbitrary abelian varieties or motives instead, but all the phenomena we want to discuss occur already for elliptic curves, for which moreover lots of examples have been discussed recently (e.g. [8], [9], [22],[25], [26], [41]). We shall consider the characteristic element associated with the Pontryagin dual of the Selmer group over k_∞ (provided this module is finitely generated over $\Lambda(H)$, where H denotes the Galois group $G(k_\infty/k_{cyc})$) and discuss its relation with the characteristic polynomial of E over the cyclotomic $\mathbb{Z}_p$-extension k_{cyc}. To this end we first have to recall some facts from the latter theory.

We fix an odd prime p. Let k be a number field, S a finite set of places of k containing the set S_p of places lying above p and the set S_∞ of infinite places. By k_S we denote the maximal outside S unramified extension of k and, for any intermediate extension $k_S|L|k$, we write $G_S(L) := G(k_S/L)$ for the Galois group of k_S over L. Suppose that $G_S(k)$ acts continuously and linearly on a vector space V over $\mathbb{Q}_p$ of dimension d. Let T be a Galois invariant $\mathbb{Z}_p$-lattice in V. Then $A = V/T$ is a discrete $G_S(k)$-module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^d$ as an $\mathbb{Z}_p$-module. We set $V^* = \mathrm{Hom}(V, \mathbb{Q}_p(1))$, $T^* = \mathrm{Hom}(T, \mathbb{Z}_p(1)))$ and $A^* = \mathrm{Hom}(T, \mu_{p^\infty})$. Then it is easy to see that $A^* \cong V^*/T^*$.

Let k_{cyc} denote the cyclotomic $\mathbb{Z}_p$-extension of k and set $\Gamma := G(k_{cyc}/k)$. For an arbitrary number field L and a place ν of L we denote by L_ν the completion of L at ν. If L is an infinite extension of $\mathbb{Q}$ we write L_ν for the limit of the completions of the finite subextensions of L with respect to the induced valuations. Note that the decomposition groups $\Gamma_\nu := G(k_{cyc,\nu}/k_\nu)$ have finite index in Γ.

9.1. **Local Euler factors.** In the context of the Selmer group of A over k_{cyc} the following local cohomology groups show up: $\mathrm{H}^1(k_{cyc,v}, A)^\vee$ and its global version $\mathrm{Ind}_\Gamma^{\Gamma_\nu}\mathrm{H}^1(k_{cyc,v}, A)^\vee$. They are finitely generated $\Lambda(\Gamma_\nu)$- and $\Lambda(\Gamma)$-modules, respectively. *For the rest of this section we shall assume that ν is* not *lying above p.* Then we will see that the above modules are torsion and the aim of this subsection is to determine their characteristic ideals. I am very grateful to Yoshitaka Hachimori for discussions on this problem and for pointing out to me that Greenberg and Vatsal [21] have given a description of the characteristic power series in a similar context. We follow closely their approach:

The structure of $\mathrm{H}^1(k_{cyc,v}, A)^\vee$ is studied in [20, Prop 2] and also in [42], [58]. In particular, it is known that their μ-invariant is zero and that there is a canonical Galois equivariant isomorphism

$$\mathrm{H}^1(k_{cyc,\nu}, A)^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V^*(k_{cyc,\nu}) \subseteq (V^*)^{I_\nu},$$

where I_ν denotes the inertia subgroup of G_{k_ν}.

By $\mathrm{Frob}_\nu \in G(k_\nu^{nr}/k)$ we denote the (arithmetic) Frobenius automorphism where k_ν^{nr} denotes the maximal unramified extension of k_ν, which contains $k_{cyc,\nu}$ because $\nu \nmid p$. Furthermore, we write γ_ν for the image of Frob_ν in Γ_ν.

Let $\alpha_1, \ldots, \alpha_{e_\nu}$ denote the eigenvalues of Frob_ν (counting with multiplicities) acting on on the maximal quotient V_{I_ν} of V on which I_ν acts trivially; i.e. $e_\nu = \dim_{\mathbb{Q}_p} V_{I_\nu}$. Then the eigenvalues of Frob_ν acting on

$$(V^*)^{I_\nu} \cong \mathrm{Hom}_{\mathbb{Q}_p}(V_{I_\nu}, \mathbb{Q}_p(1))$$

are $q_\nu\alpha_1^{-1}, \ldots, q_\nu\alpha_{e_\nu}^{-1}$, where q_ν denotes the order of the residue class field of ν. We shall apply the next lemma to $W = (V^*)^{I_\nu}$ and $F = \mathrm{Frob}_\nu$.

Lemma 9.1. *Let $< F >= \widehat{\mathbb{Z}} \to GL(W)$ a continuous representation of the free profinite group with topological generator F on a finite dimensional $\mathbb{Q}_p$-vectorspace. Let $F = F_pF_{p'}$ the unique decomposition of F corresponding to $\widehat{\mathbb{Z}} \cong \mathbb{Z}_p \times \widehat{\mathbb{Z}}_{(p')}$, where $\widehat{\mathbb{Z}}_{(p')} = \prod_{l \neq p} \mathbb{Z}_l$. Then the eigenvalues of F_p (counting with multiplicities) acting on $W^{<F_{p'}>}$ are precisely those eigenvalues of F (counting with multiplicities) acting on W which are principal units (in some extension of $\mathbb{Q}_p$).*

For the proof just note that the image of $< F_{p'} >$ in $GL(W)$ is a finite group of order prime to p such that W decomposes into the eigenspaces of $F_{p'}$. Of course, $W^{<F_{p'}>}$ is nothing else than the eigenspace with eigenvalue 1, i.e. the eigenvalues of F and F_p coincide on this subspace while on the other eigenspaces the eigenvalues of F have a non-trivial prime to p part.

Let

$$P_\nu(T) = \det(1 - \mathrm{Frob}_\nu T|V_{I_\nu}) \,\epsilon\, \mathbb{Z}_p[T] = \det(1 - \mathrm{Frob}_\nu^{-1} T|(V^*)^{I_\nu}) = \prod_{i=1}^{e_\nu}(1-\alpha_i T)$$

and put

$$\mathcal{P}_\nu = \mathcal{P}_\nu(A/k) = P_\nu(q_\nu^{-1}\gamma_\nu) \,\epsilon\, \Lambda(\Gamma_\nu) \subseteq \Lambda(\Gamma).$$

If one identifies $\Lambda(\Gamma_\nu)$ and $\Lambda(\Gamma)$ with the power series rings $\mathbb{Z}_p[[T_\nu]]$, $T_\nu = \gamma_\nu - 1$, and $\mathbb{Z}_p[[T]]$, $T = \gamma - 1$, for a fixed generator γ of Γ, then $\mathcal{P}_\nu$ corresponds to

$$P_\nu(q_\nu^{-1}(T_\nu + 1)) = P_\nu(q_\nu^{-1}(T+1)^{f_\nu}),$$

where $f_\nu \,\epsilon\, \mathbb{Z}_p$ is uniquely determined by the condition

$$q_\nu \omega(q_\nu^{-1}) = \chi_{cyc}(\gamma)^{f_\nu}.$$

Here $\omega : \mathbb{Z}_p{}^\times \to \mu_{p-1} \subseteq \mathbb{Z}_p{}^\times$ and $\chi_{cyc} : \Gamma \to 1+p\mathbb{Z}_p \subseteq \mathbb{Z}_p{}^\times$ denote the Teichmüller and cyclotomic character, respectively.

For a finitely generated $\Lambda(G)$-module M let $\mathrm{char}_G(M)$ denote its characteristic ideal in $\Lambda(G)$, where we assume $G \cong \mathbb{Z}_p$. From the above considerations one obtains immediately the following

Proposition 9.2. (*cf.* [21, prop. 2.4])

(i) $\mathrm{char}_{\Gamma_\nu}(\mathrm{H}^1(k_{cyc,v}, A)^\vee) = \Lambda(\Gamma_\nu)\mathcal{P}_\nu$,
(ii) $\mathrm{char}_{\Gamma}(\mathrm{Ind}_\Gamma^{\Gamma_\nu}\mathrm{H}^1(k_{cyc,v}, A)^\vee) = \Lambda(\Gamma)\mathcal{P}_\nu$,
(iii) *The μ-invariants of $\mathrm{H}^1(k_{cyc,\nu}, A)^\vee$ and $\mathrm{Ind}_\Gamma^{\Gamma_\nu}\mathrm{H}^1(k_{cyc,\nu}, A)^\vee$ are zero.*
(iv) *The λ-invariant $\lambda(\mathrm{Ind}_\Gamma^{\Gamma_\nu}\mathrm{H}^1(k_{cyc,v}, A)^\vee)$ is equal to $s_\nu d_\nu$, where $s_\nu = (\Gamma : \Gamma_\nu)$ equals $[(k_{cyc} \cap k(\mu_t)) : k]$ with t the largest power of p dividing $\widetilde{(q_\nu^{p-1} - 1)}$ and $d_\nu = \lambda(\mathrm{H}^1(k_{cyc,v}, A)^\vee)$ is the multiplicity of $1 - \widetilde{q_\nu^{-1}}T$ in $\widetilde{P_\nu(T)} \,\epsilon\, \mathbb{F}_p[T]$. Here $\widetilde{\cdot}$ means reduction modulo p.*

9.2. **The characteristic element of an elliptic curve over k_∞.** Now we come to our arithmetic main results. Assume that E is an elliptic curve over k with good ordinary reduction at all places S_p.

Throughout the whole paper we assume that E has good reduction at all places in S_p.

As usual the p-Selmer group of E is defined as

$$\begin{aligned}\mathrm{Sel}_{p^\infty}(E/L) &:= \ker\Big(H^1(L, E_{p^\infty}) \to \bigoplus_w H^1(L_w, E(\overline{L_w}))_{p^\infty}\Big)\\ &\cong \ker\Big(H^1(G_S(L), E_{p^\infty}) \to \bigoplus_{w \,\epsilon\, S(L)} H^1(L_w, E(\overline{L_w}))_{p^\infty}\Big).\end{aligned}$$

Here, L is a finite extension of k and, in the first line, w runs through all places of L while, in the second line, $S(L)$ denotes the set of all places of L lying above some place of S. As usual, L_w denotes the completion of L at the place w and for any field K we fix an algebraic closure $\bar{K}$. For infinite extensions K of k, $\mathrm{Sel}_{p^\infty}(E/K)$ is defined to be the direct limit of $\mathrm{Sel}_{p^\infty}(E/L)$ over all finite intermediate extensions L.

Suppose now that $k_\infty|k$ is an torsionfree pro-p p-adic Lie extension inside k_S and containing k_{cyc}. Then its Galois group is isomorphic to the semidirect product $G := G(k_\infty/k) \cong H \rtimes \Gamma$ where H denotes the Galois group of $k_\infty|k_{cyc}$ and $\Gamma = G(k_{cyc}/k)$ as before.

The Selmer group $\mathrm{Sel}_{p^\infty}(E/k_\infty)$ bears a natural structure as an discrete (left) G-module. For some purposes it is more convenient to deal with (left) compact G-modules, thus we take the Pontryagin duals $-^\vee$ and set

$$X(k_\infty) := (\mathrm{Sel}_{p^\infty}(E/k_\infty))^\vee.$$

9.3. **The false Tate curve case.** We first consider the case where G is 2-dimensional, i.e. isomorphic to the semidirect product $\mathbb{Z}_p \rtimes \mathbb{Z}_p$. In this case Theorem 6.2 tells us that $\mathcal{T}$ is an Ore set.

For simplicity we shall assume that k contains the p^{th} roots of unity. Let $\mathfrak{M}_0(k_\infty/k)$ be a set of all primes of k which are not lying above p and are ramified for k_∞/k_{cyc}. We put

$$\text{(9.18)}\quad \mathfrak{M}_1(k_\infty/k, E) := \{v \,\epsilon\, \mathfrak{M}_0(k_\infty/k) |\ E/k \text{ has split multiplicative reduction at } v\},$$

$$\text{(9.19)}\quad \mathfrak{M}_2(k_\infty/k, E) := \{v \,\epsilon\, \mathfrak{M}_0(k_\infty/k) |\ E/k \text{ has good reduction at } v \text{ and } E(k_v)_{p^\infty} \neq 0.\}$$

and $\mathfrak{M} = \mathfrak{M}(k_\infty/k, E) := \mathfrak{M}_1(k_\infty/k, E) \cup \mathfrak{M}_2(k_\infty/k, E)$.

In a joint work with Y. Hachimori we have proven the following

Theorem 9.3. ([22, thm. 3.7]) *Assume that $X(k_\infty)$ is non-zero and finitely generated as a $\Lambda(H)$-module. Then, it is a faithful torsion $\Lambda(G)$-module which is not pseudo-null and whose image in the quotient category (up to pseudo-isomorphism) is completely faithful and cyclic.*

This is in striking disharmony with the commutative situation in which the characteristic element of a torsion module annihilates this module (at least up to pseudo-isomorphism). In particular, the characteristic element associated with $X(k_\infty)$ in this paper cannot have this property. Anyway, other properties it does have we will see now.

The following result, which relies heavily on the vanishing of higher H-homology groups of $X(k_\infty)$, generalizes partly the Euler characteristic formula [22, thm. 4.11]. More precisely the mentioned formula will be reobtained in the Corollary below by "evaluating the characteristic power series $\mathrm{Ak}_G(H, X(k_\infty))$ at 0" and applying the p-adic valuation.

Theorem 9.4. *Assume that $X(k_{cyc})$ is a torsion $\Lambda(\Gamma)$-module with vanishing μ-invariant. Then $X(k_\infty)$ is finitely generated over $\Lambda(H)$ and it holds modulo $\Lambda(\Gamma)^\times$ that*

$$\pi_H(\mathrm{char}_G(X(k_\infty))) \equiv \mathrm{Ak}_G(H, X(k_\infty)) \equiv \mathrm{char}_\Gamma(X(k_{cyc})) \cdot \prod_{\nu \,\in\, \mathfrak{M}} \mathcal{P}_\nu(E(p)/k),$$

where the local factors are those defined in section 9.1.

Before we evaluate at "0" we have to introduce some more notation. We define the p-Birch-Swinnerton-Dyer constant as

$$\rho_p(E/k) := \frac{\sharp Ш(E/k)_{p^\infty}}{(\sharp E(k)_{p^\infty})^2 \prod_v |c_v|_p} \times \prod_{v|p} (\sharp \tilde{E}_v(\kappa_v)_{p^\infty})^2.$$

Here, $Ш(E/k)$ is the Tate-Shafarevich group of E over k, κ_v is the residue field of k at v and $\tilde{E}_v$ is the reduction of E over κ_v. We denote by c_v the local Tamagawa factor at v, $[E(k_v) : E_0(k_v)]$, where $E_0(k_v)$ is the subgroup of $E(k_v)$ consisting from all of the points which maps to smooth points by reduction modulo v. $|*|_p$ denotes the p-adic valuation normalized such that $|p|_p = \frac{1}{p}$. For any prime v of k, let $L_v(E, s)$ be the local L-factor of E at v. Let $P_0(k_\infty/k)$ be the set of all primes of k which are not lying above p and ramified for k_∞/K_{cyc}. As mentioned above using Proposition 8.6) we (re)obtain

Corollary 9.5. *In the situation of the theorem and assuming that the G-Euler characteristic $\chi(G, X(k_\infty))$ of $X(k_\infty)$ is finite let $F_{X(k_\infty)} \in \Lambda_{\mathcal{T}}$ be a characteristic element of $X(k_\infty)$. Then $F_{X(k_\infty)}(0)$ is defined and non-zero, and it holds that*

$$\chi(G, X(k_\infty)) = |F_{X(k_\infty)}(0)|_p^{-1} = \rho_p(E/k) \times \prod_{v \,\in\, \mathfrak{M}} |L_v(E, 1)|_p.$$

Before we give the proof of the theorem we introduce the modified Selmer group

$$\mathrm{Sel}'_{p^\infty}(E/K_{cyc}) := \mathrm{Ker}(H^1(k_S/k_{cyc}, E_{p^\infty}) \to \bigoplus_{S \setminus \mathfrak{M}} J_\nu(k_{cyc})),$$

where $J_\nu(k_{cyc}))$ is the Pontryagin dual of $\mathrm{Ind}_\Gamma^{\Gamma_\nu}\mathrm{H}^1(k_{cyc,\nu},E(p))^\vee$, see section 9.1. Then we have the following exact sequence

$$0 \to \mathrm{Sel}_{p^\infty}(E/k_{cyc}) \to \mathrm{Sel}'_{p^\infty}(E/k_{cyc}) \to \bigoplus_{\mathfrak{M}} J_\nu(k_{cyc}) \to 0. \tag{9.20}$$

Proof. First note that $\mathrm{H}^i(H, \mathrm{Sel}_{p^\infty}(E/k_\infty)) = 0$ for $i \geq 1$ by the proof of [22, thm. 4.11]. Thus $\mathrm{Ak}_G(H, X(k_\infty)) = \mathrm{char}_\Gamma(X(k_\infty)_H)$. But since according to the proof of (loc. cit.) the dual of the restriction map

$$\mathrm{res} : \mathrm{Sel}'_{p^\infty}(E/k_{cyc}) \to \mathrm{Sel}_{p^\infty}(E/k_\infty)^H$$

is a pseudo-isomorphism, the statement follows from the short exact sequence 9.20, the determination of the local factors in Proposition 9.2 and by proposition 8.1. □

Remark 9.6. If the rank of the Mordell-Weil group is positive we expect the vanishing of $F_{X(k_\infty)}(0)$ by the Birch and Swinnerton-Dyer conjecture. In this case one has to modify the Euler characteristic as is well known in the cyclotomic situation ([44] or [46],[47]) and as was proposed in [9] in the GL_2-case. For simplicity we assume that M is a $\Lambda(G)$-module with $\mathrm{H}_i(H, M) = 0$ for all $i \geq 1$. Let

$$\phi_M : \mathrm{H}_1(G, M) \to \mathrm{H}_0(G, M)$$

be the following composition of maps

$$\mathrm{H}_1(G, M) \xrightarrow[\cong]{inf} \mathrm{H}_1(\Gamma, \mathrm{H}_0(H, M) = (M_H)^\Gamma \xrightarrow{\psi_M} (M_H)_\Gamma = \mathrm{H}_0(G, M),$$

where $\psi_M : (M_H)^\Gamma \to (M_H)_\Gamma$ is induced by the identity on M_H. We say that M has *finite truncated G-Euler characteristic,* if both $\mathrm{coker}(\phi_M)$ and $\ker(\phi_M)$ are finite, and we define the truncated G-Euler characteristic of M by

$$\chi_t(G, M) = \#\mathrm{coker}(\phi_M)/\#\ker(\phi_M).$$

Setting formally $H = 1$, e.g. $G = \Gamma$, in the above we reobtain the definition of the generalized Γ-Euler characteristic $\chi_t(\Gamma, N)$ of a $\Lambda(\Gamma)$-module N. Recall from [46, lem. 3] that if $\chi_t(\Gamma, N)$ is defined ("semi-simplicity at zero"), then it equals $|c(N)|_p^{-1}$ where $c(N) := c(f_N) := [f_N(t) \cdot t^{-m(N)}]_{t=0} \,\epsilon\, \mathcal{O}$ denotes the leading coefficient of the characteristic power series f_N of N (if the multiplicity of zero of $f_N(t)$ at 0 is $m(N)$). Similarly, we introduce the *leading coefficient* $c(F)$ for any $F \,\epsilon\, \Lambda_{\mathcal{T}}$ by

$$c(F) := c(\pi_H(F)).$$

Now under the assumptions of Theorem 9.4 we have the following: If $\chi_t(G, X(k_\infty))$ is defined then

$$\chi_t(G, X(k_\infty) = |c(F_{X(k_\infty)})|_p = \chi_t(\Gamma, X(k_{cyc})) \times \prod_{v \,\epsilon\, \mathfrak{M}} |L_v(E, 1)|_p.$$

This follows immediately from [22, thm. 4.10] and its proof.

Proposition 9.7. *In the situation of the theorem let* $F_{X(k_\infty)} \in \Lambda_{\mathcal{T}}$ *be a characteristic element of* $X(k_\infty)$. *If* $\rho_G \to \mathrm{Aut}(V)$ *is a finite dimensional representation of* G *over a finite extension* K *of* $\mathbb{Q}_p$ *such that the* G*-Euler characteristic* $\chi(G, V, X(k_\infty))$ *of* $X(k_\infty)$ *is finite, then* $F_{X(k_\infty)}(\rho)$ *is defined and non-zero, and it holds that*

$$\chi(G, V, X(k_\infty)) = |F_{X(k_\infty)}(\rho)|_p^{-[K:\mathbb{Q}_p]}.$$

9.4. **The GL_2-case.** Now suppose that $k_\infty = k(E(p))$ and hence $G = H \rtimes \Gamma$, where H is an open subgroup of $SL_2(\mathbb{Z}_p)$ and thus of dimension 3. Assume that H is uniform. Then by Theorem 6.10 the multiplicative set $\mathcal{T} = \Lambda \setminus \mathfrak{m}(H)$ is an Ore set and we can apply the theory of characteristic elements.

Theorem 9.8. *Assume that* $X(k_{cyc})$ *is a torsion* $\Lambda(\Gamma)$*-module with vanishing* μ*-invariant. Then* $X(k_\infty)$ *is finitely generated over* $\Lambda(H)$ *and it holds that*

$$\pi_H(\mathrm{char}_G(X(k_\infty))) \equiv \mathrm{Ak}_G(H, X(k_\infty)) \equiv \mathrm{char}_\Gamma(X(k_{cyc})) \cdot \prod_{\nu \,\in\, \mathfrak{M}} \mathcal{P}_\nu(E(p)/k),$$

where $\mathfrak{M}$ *is the set of places of* k *with non-integral* j*-invariant and the local factors are those defined in section 9.1.*

This result generalizes partly [9, thm. 3.1]. Its proof is analogous to that of theorem 9.4, using now [9, rem. 2.6, lem. 3.3, lem. 3.6]. We leave it to the reader to derive an analogue of Corollary 9.5 from this theorem. Also an analogue of Remark 9.6 holds in this situation, see [9, prop. 2.9, thm. 3.1].
We expect similar results for other p-adic Lie extensions of dimension at least 2 and containing the cyclotomic $\mathbb{Z}_p$-extension and for other p-adic Galois representations.

10. Towards a main conjecture

Let L denote either the complex numbers $\mathbb{C}$ or a fixed algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. By an *Artin representation* $\rho : G \to \mathrm{Aut}_L(V)$ *over* L we mean a finite dimensional representation of G over L which factorizes through a finite quotient, say Δ, of G. We fix embeddings of a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ both into $\mathbb{C}$ and $\overline{\mathbb{Q}_p}$. Since an Artin representation over $\mathbb{C}$ is already defined over a finite extension $K \subseteq \bar{\mathbb{Q}}$ of $\mathbb{Q}$ we can also interpret it as a finite-dimensional $\overline{\mathbb{Q}_p}$-representation, and viceversa. Using character theory one sees immediately that the equivalence classes of (absolutely irreducible) Artin representations over $\mathbb{C}$ and $\overline{\mathbb{Q}_p}$ are naturally equivalent. Let E be an elliptic curve over k. By $L(E, \rho, 1)$ we denote the Hasse-Weil L-series, twisted by an Artin representation ρ.

Suppose as before that $k_\infty|k$ is an torsionfree pro-p p-adic Lie extension inside k_S and containing k_{cyc}. Then its Galois group is isomorphic to the semidirect product $G := G(k_\infty/k) \cong H \rtimes \Gamma$ where H denotes the Galois group of $k_\infty|k_{cyc}$ and $\Gamma = G(k_{cyc}/k)$ as before.

As we have seen in subsection 6, any $M \in \Lambda(G)\text{-mod}^H$ gives rise to an element of $K_0(\Lambda(G), \Lambda(G)_{\mathcal{T}})$ and any element of the latter group be represented by some $g \in (\Lambda(G)_{\mathcal{T}})^\times$. Assume that $\mathcal{T}$ is an Ore-set (e.g. if H is uniform pro-p and $G = H \times \Gamma$). We are quite optimistic and make the following

Conjecture 10.1. *Assume that $\mathcal{T}$ is an Ore-set. Let E be an elliptic curve over k with good ordinary reduction at S_p and assume that $X(k_{cyc})$ is a $\Lambda(\Gamma)$-torsion module with vanishing μ-invariant. Then*

(i) (existence of distribution) *there exist $F \in \Lambda(G)^\times_{\mathcal{T}}$ such that for all irreducible Artin-representations ρ of G such that $L(E, \rho, 1)$ is defined, i.e. $L(E, \rho, s)$ has no pole at $s = 1$, $F(\rho)$ is defined and*

$$F(\rho) = C(\rho) \cdot \prod_{\mathfrak{M}} \mathrm{Euler}_\nu(E, \rho, 1) \cdot \frac{L(E, \rho, 1)}{\Omega_E(\rho)},$$

where $C(\rho)$ should be a generalization of Gauß-sums times a contribution from the Euler factors of $L(E, \rho, 1)$ at primes above p, $\mathrm{Euler}_\nu(E, \rho, 1)$ the local Euler factors of $L(E, \rho, 1)$ at primes in $\mathfrak{M}$ and $\Omega_E(\rho)$ a (Deligne)-Period.

(ii) (main conjecture) *F is a characteristic element of $X(k_\infty)$.*

Remark 10.2. If a distribution F with the interpolation property in (i) exists, then F^ι interpolates $C(\rho^d) \cdot \prod_{\mathfrak{M}} \mathrm{Euler}_\nu(E, \rho^d, 1) \cdot \frac{L(E, \rho^d, 1)}{\Omega_E(\rho^d)}$.

Of course, this is nothing but a proposal for the shape of a main conjecture and it will only be complete once we have a good guess for the occurring (epsilon-)factors, periods (depending on ρ). For the cyclotomic $\mathbb{Z}_p$-extension and motives over $\mathbb{Q}$ this has been discussed by J. Coates [6]. We hope that his formalism can be adapted to our situation and we will come back to this item in a subsequent paper. Also we are aware that at moment our formalism applies only under restrictions on the base field, in particular we cannot take $\mathbb{Q}$ as base field in all our examples we are interested in. In order to illustrate a possible setting consider the following example, already discussed in [8] and [10, example 8.7].

Example 10.3. Let E be the elliptic curve $X_1(11)$, given by $E\ :\ y^2 + y = x^3 - x^2$, of conductor 11. Take $p = 5$, put $k = \mathbb{Q}(\mu_5)$, $k_\infty = k(E_{5^\infty})$ and $G = G(k_\infty/k)$, $H = G(k_\infty/\mathbb{Q}(\mu_{5^\infty}))$ as well as $\Gamma = G(\mathbb{Q}(\mu_{5^\infty})/k)$. It is shown in [16] that G is isomorphic to the congruence subgroup

$$\Gamma(5) := \{g \in GL_2(\mathbb{Z}_5) | g \text{ is congruent the identity modulo } 5\}.$$

We have $G \cong H \times \Gamma$ where we have identified the center of G with the quotient Γ and we point out that H is uniform. Thus, in this example $\mathcal{T}$ is known to be an Ore-set. Since $G(\mathbb{Q}(E_{5^\infty})/\mathbb{Q})/H$ has the simple form $\Gamma \times \Delta$ where Δ is a finite group of order prime to 5 we expect that one can even localize the Iwasawa algebra $\Lambda(G(\mathbb{Q}(E_{5^\infty})/\mathbb{Q}))$ of the non pro-5 group $G(\mathbb{Q}(E_{5^\infty})/\mathbb{Q})$ suitably. Then it would be possible to formulate a main conjecture over the base field $\mathbb{Q}$ as one would actually like.

A slightly stronger reformulation of the above conjecture reads as follows.

Conjecture 10.4. *Let E be an elliptic curve over k with good ordinary reduction at S_p and assume that $X(k_{cyc})$ is a $\Lambda(\Gamma)$-torsion module with vanishing μ-invariant. Then*

(i) (existence of distribution) *there exist $f, g \in \mathcal{T}$ such that for all irreducible Artin-representations ρ of G such that $L(E, \rho, 1)$ is defined, i.e. $L(E, \rho, s)$ has no pole at $s = 1$, one has $g(\rho) \neq 0$ and*

$$f(\rho)g(\rho)^{-1} = C(\rho) \cdot \prod_{\mathfrak{M}} \mathrm{Euler}_\nu(E, \rho, 1) \cdot \frac{L(E, \rho, 1)}{\Omega_E(\rho)}.$$

(ii) (main conjecture) *we have the identity*

$$[X(k_\infty)] = [\Lambda/\Lambda f] - [\Lambda/\Lambda g]$$

in $K_0(\Lambda(G)\text{-}mod^H)$.

In view of Proposition 8.5 and Theorem 9.4 there should be a third equivalent version of the above conjectures in the following style

Conjecture 10.5. *Let E be an elliptic curve over k with good ordinary reduction at S_p and assume that $X(k_{cyc})$ is a $\Lambda(\Gamma)$-torsion module with vanishing μ-invariant. Then*

(i) (existence of distribution) *there exist $F \in \Lambda(G)^\times_{\mathcal{T}}$ such that for all irreducible Artin-representations ρ of G one has*

$$\det_{Q_{\mathcal{O}}(\Gamma)}(\pi_H(\mathrm{tw}_\rho(F)) = \mathcal{L}(E \otimes \rho) \cdot \prod_{\nu \in \mathfrak{M}} \mathcal{P}_\nu(E \otimes \rho/k),$$

where $\mathcal{L}(E \otimes \rho)$ denotes the (conjectural) cyclotomic p-adic L-function associated with $E \otimes \rho$ and $\mathcal{P}_\nu(E \otimes \rho/k)$ are the corresponding Euler factors.

(ii) (main conjecture) *F is a characteristic element of $X(k_\infty)$.*

A. Huber and G. Kings propose in [27] an (non-commutative) Iwasawa main conjecture for motives from the point of view of the equivariant Tamagawa number (TNC) conjecture. Indeed, the validity of their main conjecture is equivalent to the validity of the equivariant TNC at each level of the tower of number fields. We are convinced that our Conjecture is coherent with their main conjecture (in cases where both conjectures are "defined", also Huber-Kings consider the TNC specialized to the motive associated with E only "away from the critical point" and thus one has to consider an analogous version of their conjecture at this point). After a first version of this article was finished, T. Fukaya and K.

Kato [17] formulated a slightly different main conjecture as Huber-Kings, though in a similar spirit. Their approach could be considered as an intermediate version between that of Huber-Kings and ours because on the one hand they derive their main conjecture also from Tamagawa number conjectures, using conjectural ϵ-elements with non-commutative coefficient rings, and on the other hand they also construct a localized K_1. But instead of localizing the ring Λ they localize in some sense a certain category of bounded complexes of Λ-modules, a construction which works in full generality but which is less explicit. The comparison of both approaches with ours could also lead to the determination of the precise constants $C(\rho)$ etc. in the above formulation.

Appendix A. Filtered rings

Let R be a filtered ring with filtration $F_\bullet R := \{F_n R | n \in \mathbb{Z}\}$, which we shall always assume is indexed by $\mathbb{Z}$ and increasing, mostly also exhaustive and separated. We write $\mathrm{gr} R = \bigoplus_{n \in \mathbb{Z}} F_n R / F_{n-1} R$ for its associated graded ring. A similar notation and convention is used for filtered left R-modules. For each $m \in \mathbb{Z}$ we denote by $R(m)$ and $M(m)$ the ring R or module, respectively, with shifted filtration $F_n R(m) := F_{n+m} R$ and $F_n M(m) := F_{n+m} M$, respectively. The filtration of inductive and projective limits of filtered R-modules is defined by $F_n \varinjlim_i M_i := \varinjlim_i F_n M_i$ and $F_n \varprojlim_i M_i := \varprojlim_i F_n M_i$, respectively. A filtered module of the form $\bigoplus_I R(m_i)$ with integers m_i is called *filtered free.*

Assume that $F_\bullet R$ is exhaustive, separated, complete and trivial in positive degrees ($F_n R = R$ for all $n \geq 0$). If $m_i, i \in I$, is a sequence of positive integers which tends to infinity (with respect to the filter of complements of finite subsets of I), then $\prod_I R(m_i)$ is the completion of $\bigoplus_I R(m_i)$ and called *complete free.*

For any ideal I we define the I-adic filtration by $F_n R := I^{-n}$, where we use the convention $I^{-n} = R$ for $n \geq 0$.

Lemma A.1. *Let $\phi : R \to S$ be a morphism of complete separated filtered rings whose filtration is trivial in positive degrees and let m_i, $i \in I$, be any sequence tending to infinity in the above sense.*

(i) *If $\mathrm{gr} R$ is right Noetherian and $\mathrm{gr} S$ finitely generated as $\mathrm{gr} R$-module, then there is a canonical isomorphism of filtered S-modules*

$$S \otimes_R \prod_I R(m_i) \cong \prod_I S(m_i).$$

Moreover, there is a canonical isomorphism of graded $\mathrm{gr} S$-modules

$$\mathrm{gr}(S \otimes_R \prod_I R(m_i)) \cong \bigoplus_I (\mathrm{gr} S)(m_i) \cong \mathrm{gr} S \otimes_{\mathrm{gr} R} \mathrm{gr}\big(\prod_I R(m_i)\big).$$

(ii) *If* gr$S$ *is a torsionfree* gr$R$*-module, then there is a canonical isomorphism of filtered* S*-modules*

$$S\widehat{\otimes}_R \prod_I R(m_i) \cong \prod_I S(m_i),$$

where $\widehat{\otimes}$ *denotes the completed tensor product. Moreover, there is a canonical isomorphism of graded* gr$S$*-modules*

$$\mathrm{gr}(S\widehat{\otimes}_R \prod_I R(m_i)) \cong \bigoplus_I (\mathrm{gr}S)(m_i) \cong \mathrm{gr}S \otimes_{\mathrm{gr}R} \mathrm{gr}\big(\prod_I R(m_i)\big).$$

Proof. We first show the isomorphisms of filtered rings. In both cases they are induced by mapping $s \otimes (r_i)$ to $\big(s\phi(r_i)\big)$, which gives a morphism of filtered rings. Thus it is sufficient to see that this map is an isomorphism of filtered R-modules. In the case that $S = \bigoplus_J R(n_j)$ is filtered free of finite rank $|J|$ as R-module, the statement is easily checked using the well-known fact that shifting commutes with direct products and that for any filtered R-module M we have a canonical isomorphism $R(k) \otimes_R M \cong M(k)$ filtered modules. In the general case we proceed as follows: By assumption gr$S$ has a finite presentation by graded free gr$R$-modules of finite rank

$$G_2 \longrightarrow G_1 \longrightarrow \mathrm{gr}S \longrightarrow 1,$$

which can be lifted to a finite presentation by filtered free R-modules of finite rank

$$F_2 \longrightarrow F_1 \longrightarrow S \longrightarrow 1,$$

see [55, lem. 8.2, proof of prop. 8.1]. Note that the filtration of F_1 induces the filtration of S, i.e. the epimorphism is strict. Shifting and taking products on the one hand and tensoring this exact sequence with $\prod_I R(m_i)$ on the other hand leads to a commutative diagram of filtered R-modules

$$\begin{array}{ccccccc}
\prod_I F_2(m_i) & \longrightarrow & \prod_I F_1(m_i) & \longrightarrow & \prod_I S(m_i) & \longrightarrow & 1, \\
\cong\uparrow & & \cong\uparrow & & \uparrow & & \\
F_2 \otimes_R \prod_I R(m_i) & \longrightarrow & F_1 \otimes_R \prod_I R(m_i) & \longrightarrow & S \otimes_R \prod_I R(m_i) & \longrightarrow & 1,
\end{array}$$

with exact rows and isomorphisms in the first two columns by the previous case above. Thus, by the five lemma the right vertical map is an isomorphism of R-modules. Moreover, since the epimorphism in the top row is again strict we see immediately that it is also an isomorphism of *filtered* R-modules (recall that the category of filtered modules is not abelian in general, thus we cannot apply the five lemma directly). To prove (ii) we observe that

$$\begin{aligned} S\widehat{\otimes}\prod_I R(m_i) &\cong S\widehat{\otimes}\bigoplus_I R(m_i) \\ &\cong \big(S\otimes\bigoplus_I R(m_i)\big)^\wedge \\ &\cong \prod_I S(m_i) \end{aligned}$$

by [35, I 3.2.6.3] and the definition of the completed tensor product.

In the graded situation we have the following isomorphisms, using that $\prod_I R(m_i)$ is the completion of $\bigoplus_I R(m_i)$ and similarly for R replaced by S,

$$\begin{aligned} \mathrm{gr}(S\otimes_R\prod_I R(m_i)) &\cong \mathrm{gr}\big(\prod_I S(m_i)\big) \\ &\cong \mathrm{gr}\big(\bigoplus_I S(m_i)\big) \\ &\cong \bigoplus_I(\mathrm{gr}S)(m_i) \\ &\cong \mathrm{gr}S\otimes_{\mathrm{gr}R}\big(\bigoplus_I(\mathrm{gr}R)(m_i)\big) \\ &\cong \mathrm{gr}S\otimes_{\mathrm{gr}R}\mathrm{gr}\big(\prod_I R(m_i)\big). \end{aligned}$$

The case of the completed tensor product is analogous. □

Appendix B. Induction

For a closed subgroup $H\subseteq G$ we denote by $\mathrm{Ind}_G^H(M)=M\widehat{\otimes}_{\Lambda(H)}\Lambda(G)$ the compact induction of a $\Lambda(H)$-module to a $\Lambda(G)$-module. Note that if M is a finitely generated $\Lambda(H)$-module, then $\mathrm{Ind}_G^H(M)$ equals just the usual tensor product $M\otimes_{\Lambda(H)}\Lambda(G)$. The next lemma I learned from Yoshihiro Ochi.

Lemma B.1. *$\Lambda(G)$ is a flat $\Lambda(H)$-module.*

Proof. To see that $\Lambda(G)$ is a flat (left) $\Lambda(H)$-module, we just need to check that for any (right) ideal $\mathfrak{a}$ of $\Lambda(H)$, the natural map $\mathfrak{a}\otimes_{\Lambda(H)}\Lambda(G)\to\Lambda(G)$ is injective. Write $G_n=G/U_n$, where U_n runs through a (linearly ordered) basis of neighbourhood of the neutral element consisting of open normal subgroups, and H_n for $H/(H\cap U_n)$. Then $\Lambda(G)=\varprojlim_n\mathcal{O}[G_n]$ and $\Lambda(H)=\varprojlim_n\mathcal{O}[H_n]$. Let f_n be the natural surjective map $\Lambda(H)\to\mathcal{O}[H_n]$ and $\mathfrak{a}$ any right ideal of $\Lambda(H)$.

Then $\mathfrak{a} = \varprojlim_n \mathfrak{a}_n$, where $\mathfrak{a}_n$ is the right ideal of $\mathcal{O}[H_n]$ generated by $f_n(\mathfrak{a})$. Now the following sequence is exact since $\mathcal{O}[G_n]$ is a projective $\mathcal{O}[H_n]$-module:

$$0 \to \mathfrak{a}_n \otimes_{\mathcal{O}[H_n]} \mathcal{O}[G_n] \to \mathcal{O}[G_n].$$

Taking projective limits we obtain the exact sequence

$$0 \to \varprojlim_n (\mathfrak{a}_n \otimes_{\mathcal{O}[H_n]} \mathcal{O}[G_n]) \to \varprojlim_n \mathcal{O}[G_n] = \Lambda(G).$$

But $\varprojlim_n (\mathfrak{a}_n \otimes_{\mathcal{O}[H_n]} \mathcal{O}[G_n]) \cong \mathfrak{a} \otimes_{\mathcal{O}[[G_v]]} \mathcal{O}[[G]]$ by [3, lem. A.4, lem. 2.1(ii)]. □

This lemma implies immediately the following

Proposition B.2. *Let H be a closed subgroup of G.*

(i) *For any $M \in \Lambda(H)$-mod and any $i \geq 0$ we have an isomorphism of Λ-modules*
$$\mathrm{E}^i_{\Lambda(G)}(\mathrm{Ind}^H_G M) \cong \mathrm{Ind}^H_G \mathrm{E}^i_{\Lambda(H)}(M).$$

(ii) *If, in addition, H is an open subgroup, then there is an isomorphism of $\Lambda(H)$-modules*
$$\mathrm{E}^i_{\Lambda(G)}(M) \cong \mathrm{E}^i_{\Lambda(H)}(M).$$

Proof. The first statement is proved in [41, lemma 5.5] while the second one can be found in [28, lemma 2.3]. □

Proposition B.3. ([56, prop. 4.9]) *Let G be a compact p-adic analytic group without p-torsion, $H \subseteq G$ a closed subgroup and M a finitely generated $\Lambda(H)$-module. If $d_{\Lambda(G)}$ (resp. $d_{\Lambda(H)}$) denotes the projective and $\delta_{\Lambda(G)}$ (resp. $\delta_{\Lambda(H)}$) the δ- dimension of $\Lambda(G)$ (resp. $\Lambda(H)$), then the following holds:*

(i) $j_{\Lambda(G)}(\mathrm{Ind}^H_G M) = j_{\Lambda(H)}(M)$,
(ii) $\delta_{\Lambda(G)}(\mathrm{Ind}^H_G M) = \delta_{\Lambda(H)}(M) + d_{\Lambda(G)} - d_{\Lambda(H)}$,
(iii) $\mathrm{pd}_{\Lambda(G)}(\mathrm{Ind}^H_G M) = \mathrm{pd}_{\Lambda(H)}(M)$.

Lemma B.4. ([58, lem. 2.9]) *Let G be a profinite group, $H \subseteq G$ a closed subgroup and $U \trianglelefteq G$ an open normal subgroup. Then for any compact $\mathbb{Z}_p[\![H]\!]$-module M the following is true:*

(i) $(\mathrm{Ind}^H_G(M))_U \cong \mathrm{Ind}^{HU/U}_{G/U}(M_{U\cap H})$ *and*
(ii) $\mathrm{H}_i(U, (\mathrm{Ind}^H_G(M))) \cong \mathrm{Ind}^{HU/U}_{G/U} \mathrm{H}_i(U \cap H, M)$ *for all $i \geq 0$.*

Proof. The dual statement of (i) is proved in [34] while (ii) follows from (i) by homological algebra. □

Finally we recall the behaviour of the annihilator ideal under induction. So let H be a closed subgroup of a p-adic Lie group G.

Proposition B.5. ([57, prop. 4.8]) *Let M be a finitely generated $\Lambda(H)$-module. Then the global annihilator ideal of* $\mathrm{Ind}_G^H M$ *is contained in ideal generated by the global annihilator of M :*

$$Ann_{\Lambda(G)}(\mathrm{Ind}_G^H M) \subseteq \Lambda(G) Ann_{\Lambda(H)}(M).$$

In particular, if M is a faithful $\Lambda(H)$-module, then $\mathrm{Ind}_G^H M$ *is a faithful $\Lambda(G)$-module.*

This result should be compared to a theorem of Harris [23] which claims that the module $\mathrm{Ind}_G^H \mathbb{Z}_p$ is bounded whenever $2 \dim H > \dim G$; unfortunately its proof contains a gap discovered by Konstantin Ardakov. Using the above proposition we can produce a series of faithful $\Lambda(G)$-modules where G is either an appropriate pro-p-subgroup of $SL_2(\mathbb{Z}_p)$ or $GL_2(\mathbb{Z}_p)$ (and thus $SL_n(\mathbb{Z}_p)$ or $GL_n(\mathbb{Z}_p)$). Indeed, the closed subgroup H of $GL_2(\mathbb{Z}_p)$ generated by the matrices $t = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $c = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, where e.g. $a = 1 + p$ and $b = 1$, i.e. more or less “the” (pro-p-) Borel subgroup of $SL_2(\mathbb{Z}_p)$, is isomorphic to the semidirect product $H \cong C \rtimes T$ of $C = c^{\mathbb{Z}_p} = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and $T = t^{\mathbb{Z}_p}$. Hence all the induced modules $\mathrm{Ind}_G^H(\Lambda(H)/\Lambda(H)F)$ where F denotes a distinguished polynomial in $\Lambda(H) \cong \mathbb{Z}_p[[X, Y; \sigma, \delta]]$ are faithful. We should mention that the faithfulness of $\mathrm{Ind}_G^H(\Lambda(H)/\Lambda(H)(t-1)) \cong \mathrm{Ind}_G^T \mathbb{Z}_p$ was proved by Greenberg (private communication) using “p-adic harmonic analysis.”

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