THE DIRECT SUMMAND CONJECTURE

OBERSEMINAR ARITHMETISCHE GEOMETRIE IM WINTERSEMESTER 2018

The direct summand conjecture ("DSC") is charmingly straight forward to state:

Conjecture. *Let R be a regular noetherian ring and $f\colon R \longrightarrow S$ a finite R-algebra. Then f splits as a morphism of R-modules.*

Interest for this conjecture stems from the fact that the mixed characteristic case would imply many other conjectures (cf. [Hoc83]).

A few cases are comparatively easy to crack: DSC is almost obvious if R contains the rationals (basically, one can construct a trace map, which is surjective if $\mathbb{Q} \subseteq R$), and still possible if R contains a field of positive characteristic (cf. [Hoc73, theorem 2]). If $\dim R \leq 2$, the Auslander-Buchsbaum formula can be used to show DSC.

But apart form these results, progress was slow. Heitmann achieved a breakthrough in 2002, when he proved the conjecture for $\dim R = 3$ (cf. [Hei02]). Heitmann's proof however is based on a very delicate construction and shed little light on higher-dimensional R.

In 2016 André published a preprint proving the direct summand conjecture in full generality using a perfectoid Abhyankar lemma (cf. [And18a; And18b]). Shortly afterwards, Bhatt significantly simplified the proof (cf. [Bha18]), using only a key construction of André's, but circumventing the full perfectoid Abhyankar lemma. He also proved a derived variant:

Theorem. *Let A be a regular noetherian ring and $X \longrightarrow \operatorname{Spec} A$ a proper surjective map. Then $A \longrightarrow \mathbf{R}\Gamma(X, \mathcal{O}_X)$ splits in the derived category of A-modules.*

Both proofs heavily rely on the application of perfectoid spaces and the strategy is often summarized as follows: Using perfectoid geometry, construct a (very large) ring extension $A|R$, which is almost faithfully flat and where A is perfectoid. Afterwards, show that the splitting problem has an almost solution after base change to A and finally descend the almost solution to an actual solution.

In this seminar, we will first recall the necessary facts of *almost ring theory* and then develop the theory of Huber's adic and Scholze's perfectoid spaces. We will prove a few of the foundational results of perfectoid spaces of [Sch12] before finally giving Bhatt's proof of the DSC.

1. Time and Place

We meet on *Thursdays, 11ct* in *SR 4 INF 205.* The first meeting will be on October 18. Please contact me if you would like to give one of the first talks.

2. Contact

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3. The Talks

Our main source will be the lecture notes of Bhatt on the subject, cf. [Bha17a; Bha17b]. We want to point out however that the original articles are also very accessible and that a recent *Hot Topics* workshop on the very same subject produced excellent notes and interesting videos (cf. [BINS]).

Talk 1: **Perfectoid fields**. Introduce the notions of perfectoid fields and tilting. State the almost purity theorem in dimension zero. Prove as much and give as many examples as you can – but try to fit it into one session. Refer to [Bha17a, chapters 2, 3] for relevant material.

Talk 2: **Almost DSC**. This talk should over the course of two sessions motivate how p-adic Hodge theory can be used in the context of the DSC. Its main source is [Bha14], but the review of almost ring theory should be extended to cover most of [Bha17a, chapter 4].

Talk 3: **Non-Archimedean Banach Algebras**. There are multiple equivalent definitions of perfectoid algebras, one of which uses ultrametric Banach algebras. This talk, covering [Bha17a, chapter 5], is supposed to give an overview of the subject and explain how the category of uniform Banach algebra has a more algebraic description. It shouldn't take more than one session.

Talk 4: **Perfectoid Algebras**. After our discussion of perfectoid fields, we need to introduce perfectoid algebras. They also have a tilting correspondence, which is the most important result of this talk, which should cover [Bha17a, chapter 6]. Two sessions should suffice to prove these results.

Talk 5: **Adic Spaces**. The "correct" topological framework for perfectoid spaces are Huber's *adic spaces*. Introduce adic spaces as done in [Wei17, section 1]. The structure sheaf should be discussed in further detail, for this refer to [Bha17a, section 7.5]. Discussion of this topic should be thorough, but two sessions shall suffice.

Talk 6: **Tilting of Rational Subsets**. We need to examine the tilting process when applied to rational subsets. This is the content of [Sch12, theorem 6.3.(i)-(ii)] or [Bha17a, sections 9.1-2]. Especially lemma 6.5 in [Sch12] consists of a rather lengthy calculation, which should be omitted. One session should suffice for this talk.

Talk 7: **Tate Acyclicity**. Prove Tate acyclicity for perfectoid affinoid K-algebras as done in [Sch12, theorem 6.3.(iii)-(iv)] or [Bha17a, section 9.3]. It implies that perfectoid algebras give rise to sheafy Huber pairs and can thus be glued together to form perfectoid spaces. The talk should culminate in the tilting correspondence for perfectoid spaces. One session should suffice for this talk.

Talk 8: **Almost Purity**. A key ingredient to Bhatt's proof of the DSC is a perfectoid version of Faltings's almost purity theorem. This is done in [Bha17a, chapter 10] and originally in [Sch12, theorem 7.9.(iii)]. One session should suffice for this talk.

Talk 9: **DSC**. Finally we can prove the direct summand conjecture. Following André, one constructs a suitable ring extension and shows the DSC after base change to this ring with a "Riemann Hebbarkeitssatz," which one then descends afterwards. In [Bha17a, chapter 11], only the non-derived variant is proved there, so consult [Bha18] as well. Two sessions will be more than enough time for this talk.

References

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