Extensions of semistable vector bundles on the Fargues-Fontaine-curve

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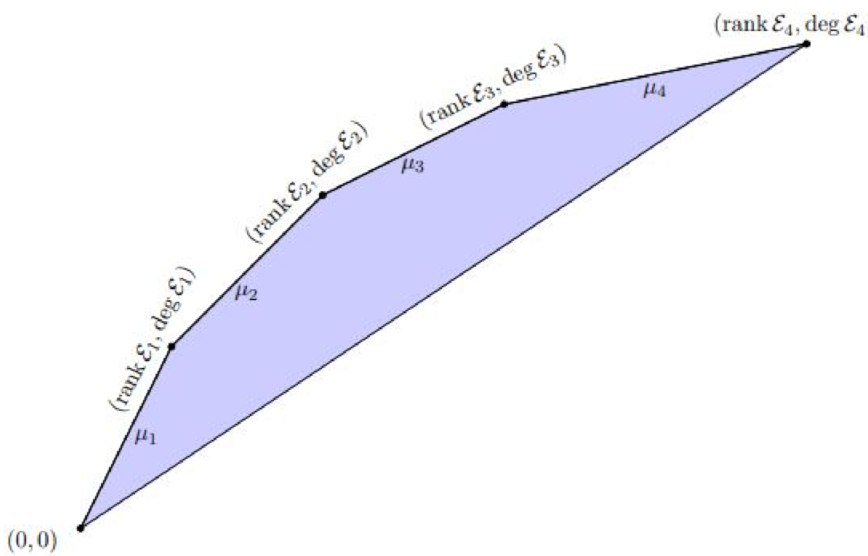
Let E be a p-adic field with residue field $\mathbb{F}_q$ and let $F/\mathbb{F}_q$ be an algebraically closed complete nonarchmidean field. For any such pair Fargues and Fontaine defined a scheme $X_{E,F}$, the Fargues-Fontaine curve. Concepts from p-adic Hodge theory can be reinterpreted in terms of vector bundles over $X_{E,F}$ and this allows for conceptual proofs of e.g. 'weakly admissible = admissible'. Furthermore closed points of $X_{E,F}$ correspond to Frobenius equivalence classes of 'untilts' of F. Any vector bundle $\mathcal{E}$ over $X_{E,F}$ admits a canonical Harder-Narasimhan-filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E},$$

where each successive quotient is a semistable vector bundle and Fargues and Fontaine show that $\mathcal{E}$ is (up to isomorphism) determined by its filtration. It is possible to associate a polygon to any such filtration by taking the upper convex hull of the points $(\mathrm{rank}(\mathcal{E}_i), \deg(\mathcal{E}_i))$ in the Euclidian plane[1].



Example of a HN-polygon for a 4-step filtration (cf. [BFH$^+$17, Example 2.2.9])

[1]The degree $\deg(\mathcal{F})$ is defined as the degree of the determinant line bundle of $\mathcal{F}$ which is the degree of the divisor of a non-zero meromorphic section of $\det\mathcal{F}$.

One can further show, that any bundle splits into a direct sum of semistable bundles. In view of these results it is natural to ask : Which bundles $\mathcal{E}$ can arise as an extension of two semistable bundles

$$0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{F}_2 \to 0?$$

One can show that the HN-polygon of $\mathcal{E}$ has to lie below the HN-polygon of $\mathcal{F}_1 \oplus \mathcal{F}_2$. Together with the convexivity one obtains 'obvious' nescessary conditions on the HN-polygon of $\mathcal{E}$. It is shown in [BFH$^+$17], that these obvious conditions are also sufficient. While this problem can be stated in a purely scheme-theoretic way, the proof uses the theory of diamonds in a crucial way and finally reduces to a combinatorial argument with HN-polygons. The goal of this seminar is to understand these results and 'dive' into non-trivial applications of the theory of diamonds. The classification results of vector bundles on the Fargues-Fontaine curve will be presented as a black box.

Talks

Talk 1 (Rustam): **Outline: General strategy/ overview**
Talk 2: **Preliminaries I: Perfectoid spaces**
Explain the notions of a perfectoid space (def. 3.19 in [Sch17]), pro-étale morphisms (def. 7.8) and the big pro-étale site (def. 8.1). Assume that the reader is familiar with the notion of an adic space. Also state corollary 8.6 from [Sch17].
Talk 3: **Preliminaries II: Diamonds**
Explain the notion of a diamond and its underlying topological space. Define open subdiamonds and prove the first part of proposition 11.15 in [Sch17]. Define the classes of diamonds and morphisms of diamonds from definition 3.1.7 in [BFH$^+$17] and if time permits sketch proposition 3.1.8. (Proofs can be found in [Han16]).
Talk 4 (Judith): **Black Box: The Fargues-Fontaine curve and vector bundles**
Talk 5: **Diamonds I: Dimension theory for diamonds**
Present the material of Section 3.2 in [BFH$^+$17] p. 14-17.
Talk 6: **Diamonds II: Spaces of bundle maps as diamonds**
Present the first half of section 3.3 including 3.3.7. The main result is 3.3.6. (p. 17-20).
Talk 7: **Diamonds III: Spaces of bundle maps continued**
Continue with the rest of 3.3. The main result is that $Surj(\mathcal{E}, \mathcal{F})^{\mathcal{K}}$ is a locally spatial partially proper diamond and the dimension formula.
Talk 8: **Polygons I: Proof of 4.1.1 in [BFH$^+$17]**
Present the proof of 4.1.1. (p.24-30)
Talk 9 (Oliver): **Polygons II: Proof of 5.1.1 in [BFH$^+$17]**
Present the (informal) proof of 5.1.1 and prove the extension theorem 1.1.2. If time permits, explain the formal version of the proof and sketch how to generalize 1.1.2 to 1.1.4.

Time and Date

We meet on Thursdays 11-13h in Seminarraum 4 (Mathematikon).
The first session will be on 25th october.

References

[BFH+17] Christopher Birkbeck, Tony Feng, David Hansen, Serin Hong, Qirui Li, Anthony Wang, and Lynnelle Ye. Extensions of vector bundles on the fargues-fontaine curve. `https://arxiv.org/pdf/1705.00710v3.pdf`, 2017.

[Han16] David Hansen. Notes on diamonds. `https://pdfs.semanticscholar.org/0e0e/309b56554ae49b2addd854765536a038454e.pdf`, 2016.

[Sch17] Peter Scholze. Etale cohomology of diamonds. `https://arxiv.org/pdf/1709.07343.pdf`, 2017.