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Diplomarbeit

Tamagawa Number Conjecture for Semi-Abelian Varieties

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Contents

Preface

This thesis aims to deduce the assumed form of the leading coefficient L^* in the L-function's Taylor expansion of the motive $h^1(G)(1)$ – G being a semi-abelian variety – using the Tamagawa Number Conjecture in the form stated by Fontaine and Perrin-Riou. As a semi-abelian variety is associated with a short exact sequence of algebraic groups

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

with a torus T and an abelian variety A, this generalises both the analytic class number formula and the conjecture of Birch and Swinnerton-Dyer.

In the first chapter, we will initially gather some basic material. In order to later state the Tamagawa Number Conjecture, determinants and some of their properties will become relevant, so naturally they will be mentioned here. We will also review continuous Galois cohomology which is fundamental to the conjecture. In particular, we will need to discuss the distinguished subspaces H^1_f as they are the local building blocks for the motivic construct. Seeing that semi-abelian varieties unite affine and projective algebraic groups, a proper understanding of the according basic concepts is crucial. Afterwards, ℓ-adic completions will be given a short review in order to be prepared for subtle issues which may arise. At last, as the Tamagawa Number Conjecture for a torus generalises the analytic class number formula, the analogues of the objects involved in the class number formula need to be defined, particularly the class number and regulator of a torus.

The second chapter then briefly reviews the notion of mixed motives with the topic being profound and so interrelated that any attempt to illuminate a situation more closely would only marginally enhance the understanding. Therefore, the reader may consult books such as [JKS94] which illustrate various views on the landscape of motives.

The third chapter introduces 1-motives which are believed to be a well-behaving subcategory of mixed motives, their main advantage being their relatively simple and concrete definition as opposed to the yet unknown definition of mixed motives. The chapter is primarily an expansion of chapter §8.1/2 in [Fon92] and defines the different realisation functors. Moreover, the motivic cohomology of a 1-motive will be calculated explicitly.

In the fourth chapter we will state the Tamagawa Number Conjecture using the formulation of Fontaine and Perrin-Riou. As a well-thought-out introduction to the topic would require a paper like [FPR94], we restrict ourselves here to describe Fontaine and Perrin-Riou's constructions which will ultimately lead us to their formulation of the Tamagawa Number Conjecture.

The fifth and last chapter finally derives the desired formula for the leading coefficient of the L-function up to sign (and up to a power of two for technical reasons). Firstly, we will be replacing the motive $h^1(G)(1)$ by a 1-motive, consequently leading us to a setting which is much more accessible and which enables us to compute everything we may need. A large portion of the chapter is devoted to computing the different H^1_f groups. We will then be able to use techniques as employed by Venjakob [Ven07] in order to derive formulas from the Tamagawa Number Conjecture for the leading coefficient. In a first step, this will be done up to a rational multiple, and in a second step up to the aforementioned sign and power of two.

For the deduction of the leading coefficient's formula it is vital that throughout the whole thesis we assume the following conjecture:

Conjecture 0.0.1 (Tate-Shafarevich conjecture)**.** *For an abelian variety A over a number field K, the Tate-Shafarevich group $Ш^1(K, A)$ is finite.*

Notation: Let p and ℓ always be primes, potentially the same if not stated otherwise. K is a number field and $K_\mathfrak{p}$ is its completion with respect to a prime ideal $\mathfrak{p}$ over p, i.e. $K_\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$.

Define $S(K)$ as all places ℓ of K. Furthermore, let $S_f(K)$, $S_\infty(K)$, $S_\ell(K)$ be the subset of all finite places, resp. all infinite places, resp. all primes over ℓ. We will drop the field K from notation and leave it as implicit where it is obvious from the context. By $\prod_\mathfrak{p}$ resp. $\prod_\nu$, we denote the product over all finite primes resp. all archimedean valuations.

We use the notation $A^{\wedge\ell}$ to denote the ℓ-adic completion $\varprojlim_n A/\ell^n A$ of an abelian group A. We write A_{tor} for the torsion subgroup of A and $A_{tor.fr.}$ for the torsion free quotient A/A_{tor}.

All occurring groups are assumed to be abelian.

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1 Preliminaries

We start by collecting basic knowledge which serves as a foundation for later chapters.

1.1 Determinants

The construction of the determinant functor for a principal ideal domain R which we are using is based on Fontaine and Perrin-Riou [FPR94, §0.4 − 0.5]. For explicit constructions see Knudsen and Mumford [KM76]. Note that more general notions exist but will not be needed in this context; an overview may be found in Venjakob [Ven07, §0.1].

Definition 1.1.1 ([FPR94, §0.5]). Let A be a free R-module of rank d. Then define $\mathbf{d}_R A$ as $\bigwedge_R^d A$. For an R-module A of finite type, i.e. there is a free resolution

$$0 \longrightarrow N \longrightarrow M \longrightarrow A \longrightarrow 0$$

where N and M are finite rank modules, define $\mathbf{d}_R A$ as $\mathbf{d}_R(N)^{-1} \wedge \mathbf{d}_R M$ where $\mathbf{d}_R(N)^{-1}$ is $\mathbf{d}_R N^*$, the determinant of the dual of N.

Remark 1.1.2 ([Ven07, §0.1]). This determinant has many properties, a non-exhaustive list is given here:

(i) There is an associative and commutative product structure $\otimes$ with a unit object $\mathbf{1}_R = \mathbf{d}_R(0)$.

(ii) $\mathrm{Aut}(\mathbf{1}_R) = R^\times$.

(iii) It is compatible with direct sums:

$$\mathbf{d}_R(A \oplus B) = \mathbf{d}_R A \otimes \mathbf{d}_R B.$$

(iv) It is compatible with base change:

$$(\mathbf{d}_R A)_{R'} := R' \otimes_R \mathbf{d}_R A = \mathbf{d}_{R'}(R' \otimes_R A)$$

(v) For a k-vector space with an automorphism f, the determinant $\mathbf{d}_k f$ coincides with $\det f \in k^\times$.

(vi) Let $C^\bullet$ be a bounded complex of projective R-modules, then define

$$\mathbf{d}_R C^\bullet := \bigotimes_i \mathbf{d}_R(C^i)^{(-1)^i}.$$

Furthermore, there is a canonical isomorphism

$$\mathbf{d}_R C^\bullet = \bigotimes_i \mathbf{d}_R H^i(C^\bullet)^{(-1)^i}.$$

The functor factors over the image of $C^\bullet$ in the derived category of perfect complexes.

(vii) For a short exact sequence of finite type R-modules

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

the determinant functor induces a canonical isomorphism

$$s:\ \mathbf{d}_R A' \otimes \mathbf{d}_R A'' \cong \mathbf{d}_R A.$$

This even extends to short exact sequences of perfect complexes.

(viii) Let $R = \mathbb{Z}_p$ and A be a finite module. Then the definition yields

$$\mathbf{d}_{\mathbb{Z}_p} A = \ \mathbf{d}_{\mathbb{Z}_p}(N)^{-1} \otimes \ \mathbf{d}_{\mathbb{Z}_p} M = \mathbf{1}_{\mathbb{Z}_p}.$$

On the other hand, the map $N \to M$ becomes an isomorphism after tensoring with $\mathbb{Q}_p$, i.e.

$$\mathbf{1}_{\mathbb{Q}_p} \cong (\mathbf{d}_{\mathbb{Z}_p}(N)^{-1})_{\mathbb{Q}_p} \otimes (\mathbf{d}_{\mathbb{Z}_p} M)_{\mathbb{Q}_p} = (\mathbf{d}_{\mathbb{Z}_p} A)_{\mathbb{Q}_p},$$

hence

$$\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{\sim} (\mathbf{d}_{\mathbb{Z}_p} A)_{\mathbb{Q}_p} = \mathbf{1}_{\mathbb{Q}_p}.$$

The above corresponds to an element $\mathbb{Q}_p^\times$ which is actually nothing other than the inverse of the cardinality of A modulo $\mathbb{Z}_p^\times$, see [Ven07, Rem. 1.3] and [BV11, App. C]. We refer to this process as the *trivialisation of the identity.*

Proposition 1.1.3 ([KM76, Def. 1])**.** *The determinant of a quasi-isomorphism of (bounded) complexes is an isomorphism. Furthermore, for a diagram of complexes*

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d, "\lambda_A", "\wr"'] & B^\bullet \arrow[r] \arrow[d, "\lambda_B", "\wr"'] & C^\bullet \arrow[r] \arrow[d, "\lambda_C", "\wr"'] & 0 \\
0 \arrow[r] & A^{\bullet\prime} \arrow[r] & B^{\bullet\prime} \arrow[r] & C^{\bullet\prime} \arrow[r] & 0
\end{tikzcd}$$

with quasi-isomorphisms λ_A, λ_B and λ_C, we get a commutative diagram

$$\begin{tikzcd}
\mathbf{d}_R A^\bullet \otimes \mathbf{d}_R C^\bullet \arrow[r, "s", "\sim"'] \arrow[d, "\mathbf{d}_R\lambda_A \otimes \mathbf{d}_R\lambda_C"', "\wr"] & \mathbf{d}_R B^\bullet \arrow[d, "\mathbf{d}_R\lambda_B", "\wr"'] \\
\mathbf{d}_R A^{\bullet\prime} \otimes \mathbf{d}_R C^{\bullet\prime} \arrow[r, "s'", "\sim"'] & \mathbf{d}_R B^{\bullet\prime}
\end{tikzcd}$$

where s and s' are induced by the first and second exact row of the original diagram respectively.

Remark 1.1.4. Consider an acyclic complex $A^\bullet$, for example an exact sequence. Then there is a canonical quasi-isomorphism $A^\bullet \xrightarrow{\sim} 0^\bullet$ and hence $\mathbf{d}_R A^\bullet \xrightarrow{\sim} \mathbf{1}_R$.

Differing with the convention used in [Ven07], we will regard an exact sequence $0 \to A_1 \to \ldots$ as an (acyclic) complex where A_1 is placed in degree zero.

Remark 1.1.5. Let A, B, C be free R-modules with chosen bases and a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Denote the sequence by $S^\bullet$. Then there is an isomorphism

$$\pi : \quad \mathbf{1}_R \xrightarrow[\sim]{can} \mathbf{d}_R A \otimes \mathbf{d}_R B^{-1} \otimes \mathbf{d}_R C = \mathbf{d}_R S^\bullet \xrightarrow[\sim]{t} \mathbf{1}_R$$

where *can* is induced by the choice of bases and t is induced by acyclicity of S. As π is an automorphism of $\mathbf{1}_R$, we can also regard it as an element of $R^\times$. Moreover, there is a more explicit description of this element:

Let φ be a section of β. Consider the isomorphism of complexes

$$\begin{tikzcd}
S^{\bullet\prime} \arrow[r, equal] \arrow[d, "\wr"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & A \oplus C \arrow[r] \arrow[d, "\alpha \oplus \varphi"] & C \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
S^\bullet \arrow[r, equal] & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where the line $A = A$ is placed in degree zero. Because both complexes are acyclic, we have quasi-isomorphisms mapping to the complex $0^\bullet$. Hence, using proposition 1.1.3, we get

$$\begin{tikzcd}
\mathbf{d}_R S^{\bullet\prime} \arrow[r, "s", "\sim"'] \arrow[d, equal] & \mathbf{d}_R S^\bullet \arrow[d, "\wr", "t"'] \\
\mathbf{1}_R \arrow[r, equal] & \mathbf{1}_R
\end{tikzcd}$$

where s collapses to

$$\mathbf{d}_R(\alpha \oplus \varphi)^{-1} : \mathbf{d}_R(A \oplus C)^{-1} \xrightarrow{\sim} \mathbf{d}_R B^{-1}$$

and can be evaluated using the bases above. Let $M(\alpha)$ and $M(\varphi)$ be the transformation matrices of α resp. φ. This then leads us to

$$\pi = \det(\alpha \oplus \varphi) = \det \begin{pmatrix} M(\alpha) & M(\varphi) \end{pmatrix}$$

as the arrow s needs to be inverted and because $(\mathbf{d}_R f)^{-1} \cdot \mathbf{d}_R f = \mathrm{id}_{\mathbf{1}_R}$ holds.

1.2 Continuous Galois Cohomology

This section serves as a quick introduction to continuous Galois cohomology based on [NSW08] and [Rub00, App. B].

Definition 1.2.1 ([NSW08, §I.2])**.** For a (profinite) group G and a topological G-module A, i.e. an abelian topological group with a continuous action of G, we define $C^n(G,A) \coloneqq Map(G^n, A)$ as the *inhomogeneous n-cochains*, where $Map(G^n, A)$ denotes the *continuous* maps from G^n to A. There is the notion of coboundary operators

$$\partial^{n+1} : C^n(G,A) \to C^{n+1}(G,A),$$

cf. [NSW08, §I.2]. Then, we define *inhomogeneous n-cocycles* and *inhomogeneous n-coboundaries* as

$$Z^n(G,A) \coloneqq \ker(C^n(G,A) \overset{\partial^{n+1}}{\to} C^{n+1}(G,A)) \text{ resp.}$$
$$B^n(G,A) \coloneqq \operatorname{im}(C^{n-1}(G,A) \overset{\partial^{n}}{\to} C^{n}(G,A))$$

and the *n-th continuous Galois cohomology group* as

$$H^n(G,A) \coloneqq Z^n(G,A)/B^n(G,A).$$

Remark 1.2.2. Using the definition of ∂^i for $i = 0, 1$, we see that the zeroth cohomology group $H^0(G,A)$ coincides with A^G. Another important special case arises if A is a trivial G-module. Then we may easily see that $H^1(G,A) = \operatorname{Hom}(G,A)$, where we use the *continuous* homomorphisms.

Lemma 1.2.3 ([NSW08, Lem. II.7.2])**.** *Given a short exact sequence of topological G-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

such that the topology of A is induced by that of B and such that $B \to C$ has a continuous section (in the category of sets). Then this induces a long exact cohomology sequence

$$\cdots \longrightarrow H^{n-1}(G,C) \xrightarrow{\delta} H^n(G,A) \longrightarrow H^n(G,B) \longrightarrow H^n(G,C) \longrightarrow \cdots.$$

In the case of discrete G-modules, the condition on $B \to C$ can be dropped.

Remark 1.2.4. If C is a free $\mathbb{Z}_\ell$-module or a $\mathbb{Q}_\ell$-vector space, the section exists.

Remark 1.2.5 ([NSW08, §I.2]). There is an explicit description of the boundary morphism δ in the case $n = 1$. Let c be an element of $H^0(G,C)$. The goal is to find an element in $H^1(G,A) \subset Map(G,A)$. By assumption, $B \overset{\beta}{\to} C$ is surjective so we find an element b which maps to c. Let $\sigma \in G$, then $\beta(\sigma b - b) = \sigma c - c = 0$ because we started with an element of $H^0(G,C)$. Exactness yields $\sigma b - b \in A$, hence we may define $\delta(c)$ as $\sigma \mapsto (\sigma b - b)$. It can be shown that this map has all required properties and is independent of all choices.

Due to [NSW08, p. 138], we have the following proposition even in the case of modules with continuous group action.

Proposition 1.2.6 ([NSW08, Prop. I.6.1])**.** *Let A be a continuous G-module. Assume $n \geq 1$ and let H be an open subgroup of G such that $H^n(H, A) = 0$. Then*

$$(G : H) \cdot H^n(G, A) = 0.$$

In particular, if G is finite, then $H^n(G, A)$ is annihilated by the order of G. Furthermore, assuming A is finitely generated as $\mathbb{Z}$-module, $H^n(G, A)$ is finite.

For later reference, we want to explicitly state the connection between cocycles and extensions.

Proposition 1.2.7. *Assume A is a discrete $\mathbb{Z}[G]$-module. Then there is a canonical isomorphism $\operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, A) = H^1(G, A)$.*

Proof. The proof is inspired by the proof of [HM03, Thm. A.1].

Assume

$$0 \longrightarrow A \xrightarrow{u} E \xrightarrow{v} \mathbb{Z} \longrightarrow 0$$

is an extension in $\mathbb{Z}[G]$. Choose a lift $e \in E$ of $1 \in \mathbb{Z}$. Then this gives rise to a cocycle $G \to A$ defined by $\sigma \mapsto u^{-1}(\sigma(e) - e)$. The map is well-defined because

$$v(\sigma(e) - e) = \sigma v(e) - v(e) = \sigma 1 - 1 = 0$$

and different lifts result in equivalent elements in cohomology.

Starting with a cocycle $f : G \to A$, one can define an inverse map: we have an extension of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow A \xrightarrow{\iota} A \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

where the G-action on the direct sum of abelian groups is defined by

$$\sigma : (a, n) \mapsto (\sigma(a) + n \cdot f(\sigma), n).$$

These maps are mutually inverse, which can be seen as follows: in order to show that

$$\operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to H^1(G, A) \to \operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

is the identity, it is sufficient to find a $\mathbb{Z}[G]$-map $t : A \oplus \mathbb{Z} \to E$ such that

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus \mathbb{Z} \arrow[r] \arrow[d, dashed, "t"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "u"] & E \arrow[r, "v"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

is commutative. This is accomplished by $(a,n) \mapsto u(a) + n \cdot e$ where $e \in E$ is the lift of $1 \in \mathbb{Z}$ as above. Then one verifies

$$\begin{aligned} t(\sigma(a,n)) &= t(\sigma(a) + nf(\sigma), n) \\ &= u(\sigma(a) + n \cdot f(\sigma)) + n \cdot e \\ &= \sigma u(a) + n \cdot u(f(\sigma)) + n \cdot e \\ &= \sigma u(a) + n \cdot (\sigma(e) - e) + n \cdot e \\ &= \sigma(u(a) + n \cdot e) = \sigma t(a,n). \end{aligned}$$

Confirming

$$H^1(G,A) \to \mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to H^1(G,A)$$

as the identity works as follows: use as a lift of $1 \in \mathbb{Z}$ the element $(0,1)$. The resulting cocyle is then defined as

$$\sigma \mapsto u^{-1}(\sigma(e) - e) = u^{-1}(f(\sigma), 0) = f(\sigma).$$

□

Definition 1.2.8. Galois groups have the desired structure of profinite groups. Therefore, we define for a field k the object $H^i(k,-)$ as $H^i(\mathrm{Gal}(\overline{k}/k), -)$.

Many interesting properties can be proved for Galois cohomology, for example:

Theorem 1.2.9 (Hilbert's Satz 90, [NSW08, Thm. VI.2.1])**.** *Let K over k be a Galois extension with Galois group G. Then $H^1(G, K^\times) = 1$ if we use the discrete topology on $K^\times$.*

Lemma 1.2.10 ([NSW08, Thm. VIII.3.20b])**.** *Let A be a finitely generated free $\mathbb{Z}$-module with the discrete topology. Then $H^1(k,A)$ is finite.*

Proof. Let K over k be a finite field extension for which A becomes a trivial Galois module, i.e. $A \cong \mathbb{Z}^r$. Then

$$0 \longrightarrow H^1(\mathrm{Gal}(K/k), A) \xrightarrow{\sim} H^1(k,A) \longrightarrow H^1(K, \mathbb{Z}^r) = 0$$

because

$$H^1(G,\mathbb{Z}) = \varinjlim_U H^1(G/U, \mathbb{Z}) = \varinjlim_U \mathrm{Hom}(G/U, \mathbb{Z}) = 0$$

where U runs through the open normal subgroups of G, i.e. the quotient G/U is finite. The first group is finite due to proposition 1.2.6 and the result follows. □

Now we want to illuminate the situation of topological G-modules T, which are finitely generated $\mathbb{Z}_\ell$-modules and on which G acts $\mathbb{Z}_\ell$-linearly. Fix $V = T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, then V is a $\mathbb{Q}_\ell$-vector space. Our primary example for T will be Tate modules of abelian variety.

Proposition 1.2.11 ([NSW08, Cor. II.7.6])**.** *There is an isomorphism for $n = 0, 1$:*

$$H^n(G, T) = \varprojlim_n H^n(G, T/\ell^n T).$$

If G is finitely generated, this also holds for $n = 2$.

Proposition 1.2.12 ([NSW08, Prop. II.7.11])**.** *There is an isomorphism for all n,*

$$H^n(G, V) = H^n(G, T) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Proposition 1.2.13 ([Rub00, Prop. B.2.5(i)],[FO08, Prop. 0.107])**.** *For a closed, normal subgroup H of G, we have the first part of the five-term exact sequence given by*

$$0 \longrightarrow H^1(G/H, T^H) \longrightarrow H^1(G, T) \longrightarrow H^1(H, T)^{G/H}.$$

Remark 1.2.14. In the case of discrete modules we have the full five-term exact sequence at hand.

1.2.1 Distinguished Subspaces

This section renders the definition of $H^1_f \subseteq H^1$ for local and global cohomology. Let $K_\mathfrak{p}$ be a p-adic local field and K be a number field.

Definition 1.2.15. Define the dual of V as $V^* = \mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ and define the *Cartier* or *Kummer dual* as $V^D = V^*(1)$. The same holds for T with $\mathbb{Q}_\ell$ replaced by $\mathbb{Z}_\ell$.

Proposition 1.2.16 ([FPR94, §I.3.2.4])**.** *In general, there is the equality*

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Q}_p} H^i(K_\mathfrak{p}, V) = \begin{cases} -[K_\mathfrak{p} : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} V & \text{if } \ell = p \\ 0 & \text{else.} \end{cases}$$

We need to assume the existence of certain *period rings* B_{exp}, B_{crys} and B_{dR} associated with $K_\mathfrak{p}$. Their definition is sophisticated and as such exceeds the scope of this thesis. We suggest that readers refer to [FO08] for definitions and properties. Note that B_{exp} is denoted as $B_{crys}^{f=1}$ in [BK90] and as B_e in [FO08]. As a first indication towards the period ring's properties we state that they are filtered and we have the following inclusions

$$\overline{K_\mathfrak{p}} \subset B_{exp} \subset B_{crys} \subset B_{dR}.$$

Definition 1.2.17. For a p-adic representation of $\mathrm{Gal}_{K_\mathfrak{p}}$ we define

$$\mathbf{D}_\star(V) \coloneqq H^0(K_\mathfrak{p}, B_\star \otimes_{\mathbb{Q}_p} V)$$

for $\star = exp, crys, dR$. For $\star = crys, dR$, this is a functor in the category of filtered $K_\star$-vector spaces with field $K_\star = H^0(K_\mathfrak{p}, B_\star)$. One has $K_{dR} = K_\mathfrak{p}$ by [FO08, Prop. 5.23] and $K_{crys} = K_{\mathfrak{p},0}$, the maximal unramified subfield of $K_\mathfrak{p}$ by [FO08, Thm. 6.14]. We denote the filtration $\mathrm{Fil}^i \mathbf{D}_\star$ by $\mathbf{D}_\star^i$. Furthermore, we know $\dim_K \mathbf{D}_{dR}(V) \leq \dim_{\mathbb{Q}_p} V$ and we say the p-adic representation V is *de Rham* if equality holds.

Theorem 1.2.18 ([FO08, Thm. 5.28])**.** *The functor $\mathbf{D}_{dR}$ from de Rham representations to filtered $K_{\mathfrak{p}}$-vector spaces is an exact, faithful tensor functor.*

Proposition 1.2.19 ([BK90, p. 356])**.** *For a de Rham representation, we have*

$$\dim_{K_{\mathfrak{p}}} \mathbf{D}^0_{dR} V + \dim_{K_{\mathfrak{p}}} \mathbf{D}^0_{dR} V^D = \dim_{K_{\mathfrak{p}}} \mathbf{D}_{dR} V.$$

Definition 1.2.20 ([BK90, §3])**.** For $V \in \mathrm{Repr}_{K_{\mathfrak{p}}}(\mathbb{Q}_\ell)$, define the following subspaces of Galois cohomology. If $\ell \neq p$ (where p is the residue characteristic of $K_{\mathfrak{p}}$), define

$$\begin{aligned} H^1_e(K_{\mathfrak{p}}, V) &:= (0) \\ H^1_f(K_{\mathfrak{p}}, V) &:= \ker(H^1(K_{\mathfrak{p}}, V) \to H^1(K^{nr}_{\mathfrak{p}}, V)) \\ H^1_g(K_{\mathfrak{p}}, V) &:= H^1(K, V) \end{aligned}$$

where $K^{nr}_{\mathfrak{p}}$ is the maximal unramified extension of $K_{\mathfrak{p}}$. If $\ell = p$, define

$$\begin{aligned} H^1_e(K_{\mathfrak{p}}, V) &:= \ker(H^1(K_{\mathfrak{p}}, V) \to H^1(K_{\mathfrak{p}}, B_{exp} \otimes_{\mathbb{Q}_p} V)) \\ H^1_f(K_{\mathfrak{p}}, V) &:= \ker(H^1(K_{\mathfrak{p}}, V) \to H^1(K_{\mathfrak{p}}, B_{crys} \otimes_{\mathbb{Q}_p} V)) \\ H^1_g(K_{\mathfrak{p}}, V) &:= \ker(H^1(K_{\mathfrak{p}}, V) \to H^1(K_{\mathfrak{p}}, B_{dR} \otimes_{\mathbb{Q}_p} V)). \end{aligned}$$

Then there are inclusions

$$H^1_e(K_{\mathfrak{p}}, V) \subset H^1_f(K_{\mathfrak{p}}, V) \subset H^1_g(K_{\mathfrak{p}}, V) \subset H^1(K_{\mathfrak{p}}, V)$$

due to the same inclusion relations mentioned above for the period rings.

Remark 1.2.21. The definition of these subgroups is different from the definition in [FPR94]. Yet according to Remark I.3.3.3 in [FPR94], the definitions coincide if V is a de Rham representation. This we assume throughout this thesis.

The most striking result for a H^1_f group is

Lemma 1.2.22 ([HK03, Lem. A.1])**.** *The subgroup $H^1_f(K_{\mathfrak{p}}, \mathrm{V}_\ell \mathbb{G}_m) = H^1_f(K_{\mathfrak{p}}, \mathbb{Q}_p(1))$ of $H^1(K_{\mathfrak{p}}, \mathbb{Q}_p(1)) = (K^\times_{\mathfrak{p}})^{\wedge \ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is equal to $(\mathcal{O}^\times_{K_{\mathfrak{p}}})^{\wedge \ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Here, $\mathbb{G}_m$ is the multiplicative group which we will define later in 1.3.11.*

Remark 1.2.23. Almost all proofs in literature use the exponential map defined by Bloch-Kato. Still, it is not too hard to obtain this result just by explicit computation on cocycles using the logarithm $\log : R^\times \to B_{dR}$ defined in [FO08, §6.1.3] and the fact that the Galois group $\mathrm{Gal}_{K_{\mathfrak{p}}}$ operates on $t := \log \epsilon$ by $\sigma t = \chi(\sigma) \cdot t$, where ϵ is the generator of $\mathbb{Z}_p(1) \subset R^\times$ and χ is the cyclotomic character.

Lemma 1.2.24 ([Rub00, Lem. 1.3.2])**.** *Let $\ell \neq p$, then there is an exact sequence*

$$0 \longrightarrow H^0(K_{\mathfrak{p}}, V) \longrightarrow V^{\mathcal{I}_{K_{\mathfrak{p}}}} \xrightarrow{\mathrm{Fr}_{\mathfrak{p}} - 1} V^{\mathcal{I}_{K_{\mathfrak{p}}}} \longrightarrow H^1_f(K_{\mathfrak{p}}, V) \longrightarrow 0$$

where $\mathcal{I}_{K_\mathfrak{p}}$ is the inertia subgroup of $\mathrm{Gal}_{K_\mathfrak{p}}$ and $\mathrm{Fr}_\mathfrak{p}$ is the Frobenius. As a consequence, we get

$$\dim_{\mathbb{Q}_p} H^0(K_\mathfrak{p}, V) = \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, V).$$

Theorem 1.2.25 ([BK90, Prop. 3.8])**.** *If $\ell = p$, assume V to be de Rham. If $\ell \neq p$, no assumptions need to be made. Then there exists a perfect pairing*

$$H^1(K, V) \times H^1(K, V^D) \to H^2(K, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell.$$

In this pairing, the subspaces $H^1_?(K, V)$ and $H^1_¿(K, V^D)$ are orthogonal *for $(?, ¿) = (e, g), (f, f), (g, e)$.*

Proposition 1.2.26 ([BK90, Cor. 3.8.4])**.** *Assume $\ell = p$ and let V be a de Rham representation. Then there is a commutative diagram with exact rows*

$$\begin{tikzcd}
0 \arrow[r] & H^0(K_\mathfrak{p}, V) \arrow[r] \arrow[d, equal] & \mathbf{D}_{exp}(V) \oplus \mathbf{D}^0_{dR}(V) \arrow[r] \arrow[d, hook] & \mathbf{D}_{dR}(V) \arrow[r] \arrow[d, "{(0, id)}"] & H^1_e(K_\mathfrak{p}, V) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & H^0(K_\mathfrak{p}, V) \arrow[r] & \mathbf{D}_{crys}(V) \oplus \mathbf{D}^0_{dR}(V) \arrow[r] & \mathbf{D}_{crys}(V) \oplus \mathbf{D}_{dR}(V) \arrow[r] & H^1_f(K_\mathfrak{p}, V) \arrow[r] & 0.
\end{tikzcd}$$

Corollary 1.2.27. *With the same assumptions as in the proposition, it holds that*

$$\dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, V) = \dim_{\mathbb{Q}_p} \mathbf{D}_{dR} V - \dim_{\mathbb{Q}_p} \mathbf{D}^0_{dR} V + \dim_{\mathbb{Q}_p} H^0(K_\mathfrak{p}, V).$$

Note that the above definitions concern local fields. However, we also need the distinguished subgroups for global fields. We achieve this by imposing the standard Selmer group condition, using the aforementioned groups as local building blocks.

Definition 1.2.28 ([FPR94, Def. II.1.3.1])**.** For number fields K we define

$$H^1_\star(K, V) = \ker\left(H^1(K, V) \to \prod_\mathfrak{p} H^1(K_\mathfrak{p}, V)/H^1_\star(K_\mathfrak{p}, V)\right)$$

for $\star = e, f, g$.

By Poitou-Tate duality, we get the following

Proposition 1.2.29 ([Nek06, Prop. 12.5.9.5 iii)])**.** *For V there is*

$$\begin{aligned}
\dim H^1_f(K, V) = \dim H^1_f(K, V^D) & \\
&+ \dim H^0(K, V) - \dim H^0(K, V^D) \\
&+ \sum_{\mathfrak{p} \in S_\ell} \left(\dim \mathbf{D}_{dR} V_{|\mathrm{Gal}_{K_\mathfrak{p}}} - \dim \mathbf{D}^0_{dR} V_{|\mathrm{Gal}_{K_\mathfrak{p}}}\right) \\
&- \sum_{\nu \in S_\infty} \dim H^0(K_\nu, V).
\end{aligned}$$

Furthermore, there are also integral versions of $H^1_\star$:

Definition 1.2.30 ([FPR94, §I.4.1.2])**.** For p-adic and number fields k we define for any $\mathbb{Z}_\ell[\mathrm{Gal}_k]$-module T

$$H^1_\star(k, T) := \iota^{-1} H^1_\star(k, T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$$

for $\iota : H^1_\star(k, T) \to H^1_\star(k, T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ and $\star = e, f, g$.

1.3 Algebraic Groups

Most results of this section can be defined or proved the same way in a far more general setting. For convenience, we restrict ourselves to the case of working over a ground field of characteristic 0.

Definition 1.3.1. An *algebraic group* A over a field k is a separated k-group scheme of finite type. Denote the category of algebraic groups over k by $\mathrm{AlgGrp}(k)$.

Remark 1.3.2. Recall that we only consider commutative groups.

Remark 1.3.3. Algebraic groups automatically have various useful properties, e.g. they are reduced, cf. [Oor66].

Remark 1.3.4. We can regard an algebraic group A as a representable sheaf on the fppf site via $A_{fppf} = \mathrm{Hom}_{k_{fppf}}(-, A)$.[1]

Definition 1.3.5 ([Poo03, Def. 5.1.15])**.** A sequence of algebraic groups is called *exact* if it induces an exact sequence on the associated fppf sheaves.

We need to gather some results on the fppf topology.

Proposition 1.3.6. *Let $f : X \to Y$ be a morphism of schemes. Then this induces a functor $f^* : Sh(Y_{fppf}) \to Sh(X_{fppf})$ with the following properties:*

(i) Let A_{fppf} be the sheaf represented by A. Then f^*A_{fppf} is represented by $A \times_Y X$.

(ii) f^* is exact.

Proof. We essentially imitate the approach used in [Tam94, §II.1.4].

The map f induces a morphism of topologies $f_{fppf} : Y_{fppf} \to X_{fppf}$ via $Y' \mapsto Y' \times_Y X$. Then, employing general theory, we get a pullback functor $f^* = (f_{fppf})_s : Sh(Y_{fppf}) \to Sh(X_{fppf})$ for the category of abelian sheaves on Y_{fppf}. It follows that f^* is exact in complete analogy to the étale case using [Tam94, Prop. I.3.6.7].

In [Tam94, Eq. I.3.6.1], $(f_{fppf})_s$ is defined as $\#' \circ (f_{fppf})_p \circ i$ where i is the inclusion of sheaves into presheafs and $\#'$ is the sheafification functor. We apply this functor to A_{fppf} step by step, starting with i. The resulting presheaf is then still represented by A. Applying $(f_{fppf})_p$ yields a presheaf represented by $f_{fppf}(A) = A \times_Y X$, cf. [Tam94, Ex. I.2.3.3]. As the fppf topology is subcanonical, the sheafification is still represented by $A \times_Y X$. □

Corollary 1.3.7 (extension of scalars)**.** *A homomorphism of fields $k \to K$, i.e. a morphism of schemes $f : \operatorname{Spec} K \to \operatorname{Spec} k$, induces a functor $f^* : \mathrm{AlgGrp}(k) \to \mathrm{AlgGrp}(K)$. Let A be an algebraic group over k. Then the functor has the following properties:*

(i) $f^*A = A \otimes_k K$.

(ii) f^* is exact.

[1] We will use the big fppf topos, see [Poo03, Rem. 6.2.6].

(iii) For a scheme T over K the T-valued points are unchanged:

$$A(T) = (f^*A)(T)$$

where the left hand side T is a k-scheme via f.

Proof. The first statement follows from proposition 1.3.6 (i) because A_{fppf} is represented by A. So we need to check whether $A \otimes_k K$ is still an algebraic group. But this is obvious, cf. [GW10, Appendix C].

The second statement follows from proposition 1.3.6 (ii) because exactness is just defined via the embedding into the fppf topos.

The last result follows from

$$A(T) = \mathrm{Hom}_k(T, A) = \mathrm{Hom}_K(T, A \otimes_k K) = (f^*A)(T)$$

where the middle equality follows directly from the universal mapping property of the fibre product. □

Proposition 1.3.8 ([Mil86, Prop. 0.16])**.** *The functor which maps an algebraic group A over k to its $\overline{k}$-rational points $A(\overline{k})$ is exact.*

Remark 1.3.9. Let K be an extension of k. Then the group $A(K)$ inherits a Galois action from the absolute Galois group of K and this action is compatible in the sense that for a field extension $K \subset L$ over k and $G = \mathrm{Gal}(L/K)$

$$A(L)^G = \mathrm{Hom}_k(\mathrm{Spec}\, L, A)^G = \mathrm{Hom}_k(\mathrm{Spec}\, K, A) = A(K)$$

holds.

Definition 1.3.10. The previous remark enables us to apply Galois cohomology to the points of an algebraic group. For ease of notation we write $H^i(k, A)$ for $H^i(k, A(\overline{k}))$.

Definition 1.3.11. Define the *multiplicative group* as

$$\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Z}[T, T^{-1}].$$

Note that it can be given the structure of an algebraic group. Define the *multiplicative group over a basis S* via base change:

$$\mathbb{G}_{m,S} = \mathbb{G}_m \times_{\mathbb{Z}} S.$$

Definition 1.3.12. An algebraic group T is called a *torus* over k if $T \otimes_k \overline{k} \cong \mathbb{G}^r_{m,\overline{k}}$ where r is called the rank of T. Let K be a field extension of k and $T \otimes_k K \cong \mathbb{G}^r_{m,K}$, then we say that T is *split* over K.

Lemma 1.3.13 ([Ono61, Prop. 1.2.1])**.** *A torus already splits over a finite extension of k.*

Definition 1.3.14. For a torus T, we define for a field extension K of k the *character group* $X_K = X_K^*(T) = \mathrm{Hom}_K(T \times_k K, \mathbb{G}_{m,K})$. We drop the index K if $K = \overline{k}$. This group is a finitely generated free $\mathbb{Z}$-module.

Remark 1.3.15. It is apparent that the property of being a torus is stable under base change.

Definition 1.3.16. An algebraic group A is called an *abelian variety* over k if A is connected, geometrically integral and proper over k.

Lemma 1.3.17. *Being an abelian variety is stable under extension of scalars.*

Proof. Properness is stable under base change, cf. [GW10, Appendix C], and after extension of scalars it is still connected, see [GW10, Cor. 5.56 (2)]. □

Definition 1.3.18. A connected algebraic group G is called a *semi-abelian variety* over k if there is an exact sequence of algebraic groups

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

where T is a torus and A is an abelian variety.

Lemma 1.3.19. *Being a semi-abelian variety is stable under extension of scalars.*

Proof. This holds true for tori and for abelian varieties. Furthermore, extension of scalars is exact. □

1.3.1 Properties of Semi-Abelian Varieties

We are now gathering results for semi-abelian varieties. Let K be a number field and $K_\mathfrak{p}$ its $\mathfrak{p}$-adic completion at a finite place $\mathfrak{p}$.

Definition 1.3.20. For an abelian variety A over k, denote by A^D the connected component of the Picard scheme $\mathrm{Pic}_{A/k}$. It can be given the structure of an abelian variety and is consequently called the *dual abelian variety*, see [Kle05, Rem. 5.24].

One example for the significance of the dual abelian variety is the following

Theorem 1.3.21 (Local Tate-duality, [Mil86, Cor. I.3.4])**.** *Let A be an abelian variety over $K_\mathfrak{p}$. Then there is a canonical pairing*

$$H^r(K_\mathfrak{p}, A^D) \times H^{1-r}(K_\mathfrak{p}, A) \to \mathbb{Q}/\mathbb{Z}.$$

Theorem 1.3.22 (Mattuck-Tate, [Mat55, Thm. 7], [Tat67, p. 168-169])**.** *Let A be an abelian variety over $K_\mathfrak{p}$. Then there is an integer d such that*

$$A(K_\mathfrak{p}) \cong \mathcal{O}_{K_\mathfrak{p}}^d \oplus finite$$

where d is the dimension of A.

Proposition 1.3.23 ([Jos09, Prop. 3.3.3])**.** *Let G be a semi-abelian variety over $K_{\mathfrak{p}}$. Then there exists an exact sequence*

$$0 \longrightarrow \mathbb{Z}_p^a \longrightarrow G(K_{\mathfrak{p}}) \longrightarrow F \longrightarrow 0$$

where F is finitely generated as $\mathbb{Z}$-module.

Lemma 1.3.24 ([Mil86, Cor. 2.4])**.** *Let T be a torus over $K_{\mathfrak{p}}$. Then the group $H^1(K_{\mathfrak{p}}, T)$ is finite.*

Lemma 1.3.25. *For a semi-abelian variety over $K_{\mathfrak{p}}$ there is an exact sequence*

$$0 \longrightarrow T(K_{\mathfrak{p}}) \longrightarrow G(K_{\mathfrak{p}}) \longrightarrow A(K_{\mathfrak{p}}) \longrightarrow F_G(K_{\mathfrak{p}}) \longrightarrow 0$$

where $F_G(K_{\mathfrak{p}}) \subset H^1(K_{\mathfrak{p}}, T)$ is finite.

Proof. We have an exact sequence

$$0 \longrightarrow T(\overline{K_{\mathfrak{p}}}) \longrightarrow G(\overline{K_{\mathfrak{p}}}) \longrightarrow A(\overline{K_{\mathfrak{p}}}) \longrightarrow 0$$

as stated in proposition 1.3.8. The result follows by taking the long exact cohomology sequence 1.2.3. □

Remark 1.3.26. Now let G be a semi-abelian variety over a number field K. The isomorphism in corollary 1.3.7 (iii) $G(K_{\mathfrak{p}}) = G_{K_{\mathfrak{p}}}(K_{\mathfrak{p}})$ transforms the exact sequence of the previous lemma into an exact sequence

$$0 \longrightarrow T(\overline{K_{\mathfrak{p}}}) \longrightarrow G(\overline{K_{\mathfrak{p}}}) \longrightarrow A(\overline{K_{\mathfrak{p}}}) \longrightarrow F_G(K_{\mathfrak{p}}) \longrightarrow 0.$$

Therefore, $G(K_{\mathfrak{p}})$ has the above structure regardless of the base field over which the semi-abelian variety is defined. We will be using this fact throughout this thesis.

Similar results can be proved for global fields.

Definition 1.3.27. We say that a group is *almost free* if it is isomorphic to the direct sum of a free group and a finite group.

Remark 1.3.28. It is common knowledge that submodules of free R-modules are free if R is a principal ideal domain, e.g. $\mathbb{Z}$ or $\mathbb{Z}_p$.

We now prove specialised versions of Theorem 3.3.13 and Proposition 3.3.14 stated in [Jos09] regarding almost free groups and the structure of semi-abelian varieties.

Lemma 1.3.29. *Let A be a group and $B \subseteq A$ a subgroup. Then an exact sequence*

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow B' \xrightarrow{i'} B \xrightarrow{j'} B'' \longrightarrow 0$$

with $B' = i^{-1}(B) = B \cap A' \subseteq A'$ *and* $B'' = j(B) \subseteq A''$.

Proof. Injectivity and surjectivity are obvious. Let $x \in \ker j' = \ker j \cap B$. The original sequence is exact, hence $x \in \operatorname{im} i \cap B = A' \cap B = B' = \operatorname{im} i'$. □

Lemma 1.3.30. *A subgroup B of an almost free group A is almost free.*

Proof. Let F be a free group and T be a finite group such that $A \cong F \oplus T$. Then there is an exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0.$$

Due to the previous lemma, we get an exact sequence

$$0 \longrightarrow T' \longrightarrow B \longrightarrow F' \longrightarrow 0$$

where $T' \subseteq T$ is finite and $F' \subset F$ is free. Thus, there is a section and $B \cong T' \oplus F'$ is almost free. □

Lemma 1.3.31. *The quotient of an almost free group A by a finitely generated subgroup B is almost free.*

Proof. Let $A \cong F_A \oplus T_A$ with $F_A = \mathbb{Z}^S$ being a free group and T_A being a finite group. B decomposes into $B \cong F_B \oplus T_B$ with $F_B \subset F_A$ and $T_B \subset T_A$. Fix a basis b_i of F_B. By assumption, this basis is of finite length. Thus, the set $S' = \{s \in S \mid \exists i : b_{i,s} \neq 0\}$ is finite where $b_{i,s}$ denotes the s-th component of b_i. We then know $F_B \subset \mathbb{Z}^{S'} \subset F_A$ and the construction yields that

$$A/B \cong (\mathbb{Z}^{S \setminus S'} \oplus \mathbb{Z}^{S'} \oplus T_A)/(F_B \oplus T_B) \cong \mathbb{Z}^{S \setminus S'} \oplus (\mathbb{Z}^{S'} \oplus T_A)/(F_B \oplus T_B).$$

$\mathbb{Z}^{S'} \oplus T_A$ is finitely generated due to the finiteness of S'. The claim follows because quotients of finitely generated groups are finitely generated. □

Lemma 1.3.32. *Let A be an almost free group and let C be a finitely generated group. Then the exact sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is isomorphic to the direct sum of the short exact sequences

$$0 \longrightarrow F_{in} = F_{in} \longrightarrow 0 \longrightarrow 0$$

where F_{in} is free and

$$0 \longrightarrow \tilde{A} \longrightarrow \tilde{B} \longrightarrow C \longrightarrow 0$$

where $\tilde{A}$ and $\tilde{B}$ are finitely generated. Hence, $B \cong F_{in} \oplus \tilde{B}$ is almost free.

Proof. Let $A \cong F_A \oplus T_A$ with $F_A = \mathbb{Z}^S$ being a free group and T_A a finite group. Assume the same for C with $F_C = \mathbb{Z}^r$.

For every $c_i \in C_{tor}$, pick a $b_i \in B$ such that $j(b_i) = c_i$. These are finitely many by assumption. Then $\langle b_i \rangle_i$ is a finitely generated subset of B and $\langle b_i \rangle_i \cap F_A$ is a subset of the free group F_A. Therefore, it is finitely generated and free. Let a_i be a basis (of finite length) of the intersection. Define the finite set $S' = \{s \in S \mid \exists i : a_{i,s} \neq 0\}$, the free group $F_{in} = \mathbb{Z}^{S \setminus S'} \subseteq A$ and the finitely generated group $\tilde{A} = T_A \oplus \mathbb{Z}^{S'} \subseteq A$. We obviously have $A \cong \tilde{A} \oplus F_{in}$ and $\langle b_i \rangle_i \cap F_{in} = (0)$.

Division of the original exact sequence by F_{in} gives

$$0 \longrightarrow \tilde{A} \longrightarrow \tilde{B} \longrightarrow C \longrightarrow 0$$

with $\tilde{B} = B/F_{in}$. It is immediate that $\tilde{B}$ is finitely generated because $\tilde{A}$ and C have this property.

The canonical homomorphism $B_{tor} \to \tilde{B}_{tor}$ is an isomorphism because we can construct an inverse $\tilde{B}_{tor} \to B_{tor}$ as follows: Let $x \in \tilde{B}_{tor}$ and pick a lift $\overline{x} \in B$. Then, there exists an n such that $n \cdot \overline{x} \in F_{in}$. Furthermore, $F_{in} \subseteq \ker j$ and hence $n \cdot j(\overline{x}) = 0$. Thus, $j(\overline{x})$ is a torsion element and there exists a b_i and an $a \in A$ such that $\overline{x} = b_i + a$. By potentially choosing another lift, we can assume without loss of generality that $a \in \tilde{A}$. This gives $\overline{x} \in \langle b_i \rangle_i + \tilde{A}$, i.e.

$$n \cdot \overline{x} \in (\langle b_i \rangle_i + \tilde{A}) \cap F_{in} = (0)$$

by construction of F_{in}. Hence, $\overline{x} \in B_{tor}$. This and F_{in} being torsion free proves that $B_{tor} \to \tilde{B}_{tor}$ is an isomorphism.

Using the ker-coker sequence for the first two rows in the following diagram results in

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & B_{tor} \arrow[r, "\sim"] \arrow[d, hook] & \tilde{B}_{tor} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & F_{in} \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{in} \arrow[r] & B_{tor.fr.} \arrow[r] & \tilde{B}_{tor.fr.} \arrow[r] & 0.
\end{tikzcd}$$

$\tilde{B}_{tor.fr.}$ is free and this induces a section which results in an isomorphism

$$B_{tor.fr.} \cong F_{in} \oplus \tilde{B}_{tor.fr.}.$$

Therefore, $B_{tor.fr.}$ is free and again this induces an isomorphism

$$B \cong B_{tor} \oplus B_{tor.fr.} \cong \tilde{B}_{tor} \oplus F_{in} \oplus \tilde{B}_{tor.fr.} \cong \tilde{B} \oplus F_{in}$$

because for the finitely generated group $\tilde{B}$,

$$\tilde{B} \cong \tilde{B}_{tor.fr.} \oplus \tilde{B}_{tor}$$

holds. $\square$

Proposition 1.3.33. *Let K be a number field and $\mathcal{O}_K$ its ring of integers. Then $K^\times$ is almost free.*

Proof. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_K^\times \longrightarrow K^\times \longrightarrow P_K \longrightarrow 0$$

where P_K are fractional principal ideals which are contained in the ideal group J_K. By [Neu99, Cor. I.3.9] we know that J_K is free and therefore P_K is free, too. By choosing a section, we can identify $K^\times$ with $\mathcal{O}_K^\times \oplus P_K$. Furthermore, we know by [Neu99, Thm. I.7.4] that $\mathcal{O}_K^\times$ is finitely generated. □

Proposition 1.3.34. *Let T be a torus over K. Then $T(K)$ is almost free.*

Proof. Let T be split over the finite extension L of K. Then, by definition,

$$T(L) \cong (L^\times)^n$$

is almost free. By remark 1.3.9 we know that $T(K) = T(L)^{\mathrm{Gal}(L/K)}$ is a subgroup of the almost free group $T(L)$ and hence almost free using lemma 1.3.30. □

Lemma 1.3.35. *Let T be a torus over K. Then $H^1(K,T)$ is of finite exponent, i.e. it is annihilated by multiplication with an $n \in \mathbb{N}$.*

Proof. Let T be split over L, a finite extension of K. Remember the exact sequence given in proposition 1.2.13:

$$0 \longrightarrow H^1(\mathrm{Gal}(L/K), T(L)) \longrightarrow H^1(K,T) \longrightarrow H^1(L,T)^{\mathrm{Gal}(L/K)}.$$

Because T splits over L, there is an isomorphism $T \otimes_K L \cong \mathbb{G}_{m,L}^r$ compatible with the action of Gal_L, i.e.

$$H^1(L,T) = H^1(L, T(\overline{K})) = H^1(L, T(\overline{L})) \cong H^1(L, (L^\times))^r.$$

Now this group vanishes by Hilbert's Satz 90 1.2.9. Hence, $H^1(K,T)$ is isomorphic to $H^1(\mathrm{Gal}(L/K), T(L))$ and this group is annihilated by multiplication with $\mathrm{Gal}(L/K)$ due to proposition 1.2.6. □

Remark 1.3.36. Note that the lemma holds for every field.

For abelian varieties stronger results may be proved. Take for example the following theorem:

Theorem 1.3.37 (Mordell-Weil, [Mil08, Thm. 16.7])**.** *Let A be an abelian variety over a number field K. Then $A(K)$ is finitely generated.*

We will now discuss semi-abelian varieties.

Lemma 1.3.38. *For a semi-abelian variety over K there exists an exact sequence*

$$0 \longrightarrow T(K) \longrightarrow G(K) \longrightarrow A(K) \longrightarrow F_G(K) \longrightarrow 0$$

and the image of $A(K)$ in $H^1(K,T)$ called $F_G(K)$ is finite.

Proof. We have an exact sequence

$$0 \longrightarrow T(\overline{K}) \longrightarrow G(\overline{K}) \longrightarrow A(\overline{K}) \longrightarrow 0$$

by proposition 1.3.8. Using the long exact cohomology sequence 1.2.3 yields the desired sequence. $A(K)$ is finitely generated by Mordell-Weil 1.3.37 and $H^1(K,T)$ is torsion by lemma 1.3.35. Hence, the image $F_G(K)$ is finite. □

Theorem 1.3.39. *Let G be a semi-abelian variety over K. Then $G(K)$ is almost free.*

Proof. By lemma 1.3.38 we get a short exact sequence

$$0 \longrightarrow T(K) \longrightarrow G(K) \longrightarrow \ker(A(K) \to F_G(K)) \longrightarrow 0.$$

$\ker(A(K) \to F_G(K))$ is a subset of the finitely generated group $A(K)$ and therefore finitely generated. The claim is then proved using that $T(K)$ is almost free by proposition 1.3.34 together with the result on extensions of almost free groups, 1.3.32. □

1.4 ℓ-adic Completions

There is the well known

Proposition 1.4.1 ([Eis95, Lem. 7.15])**.** *For a noetherian ring R and an ideal $\mathfrak{m}$, the $\mathfrak{m}$-adic completion is exact for finitely generated modules.*

We however need a more precise statement:

Lemma 1.4.2. *Let*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{p} C \longrightarrow 0$$

be a sequence of $\mathbb{Z}$-modules. The following implications hold:

(i) If the sequence is right exact, the ℓ-adic completion of p is surjective again.

(ii) If the ℓ-primary component $C(\ell)$ of C is annihilated by multiplication with ℓ^α, the ℓ-adic completion of a left exact sequence is still left exact.

In particular, if $C(\ell)$ has finite exponent, the ℓ-adic completion is exact.

Proof. Both parts are proved just as proposition 1.4.1 was in Eisenbud's book with the single exception that we need a replacement for the Artin-Rees lemma to compare the topology on A with the topology on A induced by B. The application of the Artin-Rees lemma is the only part where the property "finitely generated" becomes relevant to the proof.

The replacement works as follows: let α be as above and $a \in A \cap \ell^{n+\alpha}B$, i.e. there is a $b \in B$ with $a = \ell^{n+\alpha}b$. Then, we have $[b] \in (B/A)(\ell) \subseteq C(\ell)$ and, by definition of

α, we get $\ell^\alpha[b] = 0$ in B/A and thus $\ell^\alpha b = a' \in A$. Therefore, $a = \ell^{n+\alpha} b = \ell^n a' \in \ell^n A$ holds. In particular, we have shown

$$\ell^n A \subset A \cap \ell^n B \subset \ell^{n-\alpha} A$$

for $n > \alpha$ which is just the statement the Artin-Rees lemma would have yielded. □

Remark 1.4.3. In general, the ℓ-adic completion is neither left nor right exact. Note that part (i) of the previous lemma is no contradiction to that. For an example where the ℓ-adic completion is not right exact see [AM69, Ex. 10.1]. Our result is slightly finer than the result of Breuning in [Bre04, Lem. 4.8].

Lemma 1.4.4. *For a semi-abelian variety G over $K_\mathfrak{p}$, the ℓ-adic completion of the sequence in 1.3.25*

$$0 \longrightarrow T(K_\mathfrak{p})^{\wedge\ell} \longrightarrow G(K_\mathfrak{p})^{\wedge\ell} \longrightarrow A(K_\mathfrak{p})^{\wedge\ell} \longrightarrow F_G(K_\mathfrak{p})^{\wedge\ell} \longrightarrow 0$$

is exact.

Proof. From lemma 1.3.25 we get the exact sequence

$$0 \longrightarrow T(K_\mathfrak{p}) \longrightarrow G(K_\mathfrak{p}) \longrightarrow \ker(A(K_\mathfrak{p}) \to F_G(K_\mathfrak{p})) \longrightarrow 0.$$

The last group has only finite torsion by theorem 1.3.22, hence the previous lemma is applicable.

We are thus left to show that the ℓ-adic completion of

$$0 \longrightarrow \ker(A(K_\mathfrak{p}) \to F_G(K_\mathfrak{p})) \longrightarrow A(K_\mathfrak{p}) \longrightarrow F_G(K_\mathfrak{p}) \longrightarrow 0$$

is exact. Conveniently, the same argument applies here. □

Remark 1.4.5. The analogue of the profinite completion has been shown in [HS05, Lem. 2.2].

The same result holds for the global case.

Lemma 1.4.6. *For a semi-abelian variety G over K, the ℓ-adic completion of 1.3.38 is the exact sequence*

$$0 \longrightarrow T(K)^{\wedge\ell} \longrightarrow G(K)^{\wedge\ell} \longrightarrow A(K)^{\wedge\ell} \longrightarrow F_G(K)^{\wedge\ell} \longrightarrow 0.$$

Proof. Note that we have established an analogous exact sequence as in the previous lemma. Together with the Mordell-Weil theorem, 1.3.37, we can now use the same argumentation as before to obtain the desired result. □

For finitely generated $\mathbb{Z}$-modules, the ℓ-adic completion possesses another important property:

Proposition 1.4.7 ([Eis95, Lem. 7.2a]). *If M is a finitely generated $\mathbb{Z}$-module, then*

$$M^{\wedge\ell} \cong M \otimes \mathbb{Z}_\ell.$$

In particular, the map $F \to F^{\wedge\ell}$ is injective if F is a finitely generated free $\mathbb{Z}$-module because $\mathbb{Z} \to \mathbb{Z}_\ell$ is injective. We want to generalise this statement to free modules:

Lemma 1.4.8. *For a free $\mathbb{Z}$-module F, the canonical map $F \to F^{\wedge\ell}$ is injective.*

Proof. Fix $F = \mathbb{Z}^S$. Let x be an element in the kernel of $F \to F^{\wedge\ell}$. Then, as before, define the finite set $S' = \{s \in S \mid x_s \neq 0\}$. Hence, $x \in \mathbb{Z}^{S'}$. Because $\mathbb{Z}^{S'}$ is a direct summand of F, the map $(\mathbb{Z}^{S'})^{\wedge\ell} \to F^{\wedge\ell}$ is injective. Furthermore, the map $\mathbb{Z}^{S'} \to (\mathbb{Z}^{S'})^{\wedge\ell}$ is injective because $\mathbb{Z}^{S'}$ is a finitely generated free $\mathbb{Z}$-module. Thus, $x = 0$. □

Corollary 1.4.9. *For an almost free group A, the kernel of the canonical map $A \to A^{\wedge\ell}$ is finite.*

Proof. A decomposes into the direct sum $F_A \oplus T_A$ where F_A is free and T_A is finite. The lemma yields that the map

$$A = F_A \oplus T_A \to F_A^{\wedge\ell} \oplus T_A^{\wedge\ell} = A^{\wedge\ell}$$

has finite kernel because $\ker(T_A \to T_A^{\wedge\ell}) \subset T_A$ is finite. □

1.5 Arithmetic of Tori

Let K be a number field and T be a torus over K. We will now define groups associated with a torus that are of arithmetical interest.

Definition 1.5.1 ([Shy77b, p. 366])**.** The *maximal compact subtorus* $T^c_\mathfrak{p}$ of $T(K_\mathfrak{p})$ is defined as

$$\begin{aligned} T^c_\mathfrak{p} &= \bigcap_{\chi \in X^*_{K_\mathfrak{p}}(T)} \ker\left(|\chi|_\mathfrak{p} : T(K_\mathfrak{p}) \xrightarrow{\chi} \mathbb{G}_m(K_\mathfrak{p}) = K_\mathfrak{p}^\times \xrightarrow{|-|_\mathfrak{p}} \mathbb{Z} \right) \\ &= \{ t \in T(K_\mathfrak{p}) \mid \forall \chi \in X^*_{K_\mathfrak{p}}(T) : \chi(t) \in \mathcal{O}^\times_{K_\mathfrak{p}} \} \end{aligned}$$

where $|-|_\mathfrak{p}$ is the $\mathfrak{p}$-adic norm associated with $K_\mathfrak{p}$. It is indeed the unique maximal compact subgroup of $T(K_\mathfrak{p})$, cf. [Ono61, Eq. 2.1.3 and p. 116].

Lemma 1.5.2. *$T^c_\mathfrak{p}$ has a subgroup $\mathbb{Z}_p^a$ of finite index.*

Proof. By proposition 1.3.23 we have

$$0 \longrightarrow \mathbb{Z}_p^a \longrightarrow T(K_\mathfrak{p}) \longrightarrow F \longrightarrow 0$$

where F is finitely generated. $\mathbb{Z}_p^a$ is a subset of $T^c_\mathfrak{p}$ because it is compact. Moreover, the image of $T^c_\mathfrak{p}$ in the discrete group F, namely $T^c_\mathfrak{p}/\mathbb{Z}_p^a$, is compact and hence finite. □

Let S be a finite set of places of K containing the infinite places S_∞ of K.

Definition 1.5.3 ([Shy77b, p. 367]). Define

$$T_{\mathbb{A}}(S) = \prod_{\nu \in S} T(K_\nu) \times \prod_{\mathfrak{p} \notin S} T^c_{\mathfrak{p}}$$

and the *adelic group*

$$T_{\mathbb{A}} = \varinjlim_{S \text{ finite}} T_{\mathbb{A}}(S).$$

Definition 1.5.4 ([Shy77b, p. 367], [Bit11, Def. 5]). The *S-class number of T* is the finite index

$$h_T(S) = \# \frac{T_{\mathbb{A}}}{T(K) \cdot T_{\mathbb{A}}(S)}$$

where $T(K)$ is diagonally embedded into $T_{\mathbb{A}}(S)$. If $S = S_\infty$, we just call it the *class number.*

Definition 1.5.5 ([Shy77b, p. 367]). Define the *S-unit group of T* as a subgroup of $T(K)$ by

$$\begin{aligned} U_T(S) &= T(K) \cap T_{\mathbb{A}}(S) \\ &= \{t \in T(K) \,|\, \forall \mathfrak{p} \notin S : t_{\mathfrak{p}} \in T^c_{\mathfrak{p}}\} \\ &= \{t \in T(K) \,|\, \forall \mathfrak{p} \notin S \; \forall \chi \in X^*_{K_{\mathfrak{p}}}(T) : |\chi(t_{\mathfrak{p}})|_{\mathfrak{p}} = 1\} \end{aligned}$$

where $T(K)$ is diagonally embedded into $T_{\mathbb{A}}(S)$. Let $U_T = U_T(S_\infty)$ denote the *unit group* of T.

Remark 1.5.6. In standard literature, this group is called $T_K(S)$.

Example 1.5.7. Let $T = \mathbb{G}_m$ and $S = S_\infty$. We then get $T^c_{\mathfrak{p}} = \mathcal{O}^\times_{K_{\mathfrak{p}}}$ and hence, the unit group of T is just $\mathcal{O}^\times_K$ and the class number h_T is the usual.

Theorem 1.5.8 (Generalised Dirichlet Unit Theorem, [Shy77a, Main Theorem]). *The S-unit group $U_T(S)$ is the direct sum of the finite group $\mu = \mu_T = T(K) \cap T^c_{\mathbb{A}}$ and a free group $\Gamma = \Gamma_T \cong \mathbb{Z}^{r(S) - r_K}$ where $T^c_{\mathbb{A}}$ is defined as $T^c_{\mathbb{A}} = \prod_{\text{all } \nu} T^c_\nu$, $r_K = \operatorname{rank} X^*_K(T)$ and $r(S) = \sum_{\nu \in S} \operatorname{rank} X^*_{K_\nu}(T)$.*

Remark 1.5.9. Assume $S = S_\infty$. Let $(\chi^\nu_i)^{r_\nu}_{i=1}$ be a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module $X^*_{K_\nu}(T)$ and $(\psi_i)^{r_K}_{i=1}$ be a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module $X^*_K(T)$. Base change induces maps $\iota_\nu : X^*_K(T) \to X^*_{K_\nu}(T)$. Hence, for every ν integers $\alpha^\nu_{i,j} \in \mathbb{Z}$ exist such that $\psi_{i,\nu} := \iota_\nu(\psi_i) = \prod_j (\chi^\nu_j)^{\alpha^\nu_{i,j}}$. Finally, let $(e_i)^{r_\infty - r_K}_{i=1}$ be a $\mathbb{Z}$-basis of $\Gamma \cong \mathbb{Z}^{r_\infty - r_K}$.

Define a homomorphism

$$\begin{array}{rcccccc} \lambda : \; \Gamma \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & T_{\mathbb{A}}(S_\infty)/T^c_{\mathbb{A}} & \longrightarrow & (\mathbb{R}^\times_+)^{r_\infty} & \longrightarrow & \mathbb{R}^{r_\infty} \\ x \otimes r & \longmapsto & r \cdot (x_{\mathfrak{p}})_{\mathfrak{p}} & \longmapsto & r \cdot (|\chi^\nu_i(x_\nu)|_\nu)_{i,\nu} & \longmapsto & r \cdot (\ln |\chi^\nu_i(x_\nu)|_\nu)_{i,\nu} \end{array}$$

as well as a homomorphism

$$\begin{aligned}\alpha:\ \mathbb{R}^{r_\infty} &\longrightarrow \mathbb{R}^{r_K}\\ (x_j^\nu)_{\nu,j} &\longmapsto (\textstyle\sum_{\nu,j}\alpha_{i,j}^\nu x_j^\nu)_i.\end{aligned}$$

Corollary 1.5.10. *Assume the situation above. Then the sequence*

$$0 \longrightarrow \Gamma\otimes_{\mathbb{Z}}\mathbb{R} \xrightarrow{\lambda} \mathbb{R}^{r_\infty} \xrightarrow{\alpha} \mathbb{R}^{r_K} \longrightarrow 0$$

is exact.

Proof. As essentially the derivation of the exact sequence above has already been demonstrated in the proof of theorem 1.5.8 in [Shy77a], we may rephrase Shyr's proof for the case of $S = S_\infty$ to serve our purposes.

Shyr defines a surjective map

$$\begin{aligned}\Lambda_S:\quad T_{\mathbb{A}}(S) &\longrightarrow (\mathbb{R}_+^\times)^{r_K}\\ \text{via } (x_\nu)_\nu &\longmapsto (||\psi_{1,\nu}(x_\nu)_\nu||,\dots,||\psi_{r_K,\nu}(x_\nu)_\nu||)\end{aligned}$$

where $\mathbb{R}_+^\times$ is the multiplicative group of the positive reals and $||x|| = \prod_{\text{all }\nu}|x_\nu|_\nu$. Then Λ_S factors through $T_{\mathbb{A}}(S)/T_{\mathbb{A}}^c$. Denote the surjective map

$$T_{\mathbb{A}}(S)/T_{\mathbb{A}}^c \longrightarrow (\mathbb{R}_+^\times)^{r_K}$$

by φ. We then know that $\Gamma\subset\ker\varphi$ and that, according to Shyr's proof, $\ker\varphi/\Gamma$ is compact.

Furthermore, the maps

$$\begin{aligned}\Lambda_\nu:\quad T(K_\nu) &\longrightarrow (\mathbb{R}_+^\times)^{r_\nu}\\ \text{via } x &\longmapsto (|\chi_1^\nu(x)|_\nu,\dots,|\chi_{r_\nu}^\nu(x)|_\nu)\end{aligned}$$

induce an isomorphism

$$\oplus_\nu\Lambda_\nu:\quad T_{\mathbb{A}}(S)/T_{\mathbb{A}}^c \cong (\mathbb{R}_+^\times)^{r_\infty}\times\mathbb{Z}^{r(S)-r_\infty} = (\mathbb{R}_+^\times)^{r_\infty},$$

cf. [Shy77a, Eq. 3]), since we have assumed $S = S_\infty$.

In particular, we see that $\ker\varphi$ is an $\mathbb{R}$-vector space and that Γ is a lattice of $\ker\varphi$ due to the compactness of the quotient, i.e. $\ker\varphi = \Gamma\otimes\mathbb{R}$.

All this information amounts to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[d, equal] & T_{\mathbb{A}}(S_\infty)/T_{\mathbb{A}}^c \arrow[d, "\wr\ \oplus_\nu\Lambda_\nu"] \arrow[dr, "\Lambda_{S_\infty}"] & & \\
0 \arrow[r] & \Gamma\otimes\mathbb{R} \arrow[r] \arrow[d, equal] & (\mathbb{R}_+^\times)^{r_\infty} \arrow[r] \arrow[d, "\wr\ \ln"] & (\mathbb{R}_+^\times)^{r_K} \arrow[r] \arrow[d, "\wr\ \ln"] & 0 \\
0 \arrow[r] & \Gamma\otimes\mathbb{R} \arrow[r, "\lambda"] & \mathbb{R}^{r_\infty} \arrow[r, "\alpha"] & \mathbb{R}^{r_K} \arrow[r] & 0.
\end{tikzcd}$$

The lower row's second map indeed being α can be seen as follows: by the definition of the $\alpha^{\nu}_{i,j}$, the identity $\psi_{i,\nu} = \prod_j (\chi^{\nu}_j)^{\alpha^{\nu}_{i,j}}$ holds and hence we get that $(x_\nu)_\nu \in T_{\mathbb{A}}(S_\infty)/T^c_{\mathbb{A}}$ maps to

$$\Lambda_S((x_\nu)_\nu) = \ln ||\psi_{i,\nu}(x_\nu)_\nu|| = \sum_{v\in S_\infty} \ln |\psi_{i,\nu}(x_\nu)|_\nu = \sum_{v\in S_\infty} \sum_{i=1}^{r_\nu} \alpha^{\nu}_{i,j} \ln |\chi^{\nu}_i(x_\nu)|_\nu.$$

The second equation holds because for $(x_\nu)_\nu \in T_{\mathbb{A}}(S_\infty)$, we know by definition that $|\chi(x_{\mathfrak{p}})|_{\mathfrak{p}} = 1$ for $\mathfrak{p}$ being a finite place and $\chi \in X^*_{K_{\mathfrak{p}}}(T)$ being arbitrary. □

Definition 1.5.11. When we denote the exact sequence of the previous corollary by $D^\bullet$, the $\mathbb{Z}$-bases above yield a canonical isomorphism

$$\tilde{R}_T : \quad \mathbf{1}_{\mathbb{R}} = \mathbf{d}_{\mathbb{R}}\mathbb{R}^{r_\infty - r_K}\, \mathbf{d}_{\mathbb{R}}\left(\mathbb{R}^{r_\infty}\right)^{-1}\, \mathbf{d}_{\mathbb{R}}\mathbb{R}^{r_K} \xrightarrow[\sim]{can} \mathbf{d}_{\mathbb{R}}D^\bullet \xrightarrow[\sim]{s_D} \mathbf{1}_{\mathbb{R}}$$

where *can* is induced by the choice of bases and s_D is induced by the acyclicity of $D^\bullet$, cf. remark 1.1.4. As $\tilde{R}_T$ is an automorphism of $\mathbf{1}_{\mathbb{R}}$, we can also regard it as a number in $\mathbb{R}^\times$. Define the *regulator* R_T *of* T over K to be the absolute value of the number $\tilde{R}_T$.

Remark 1.5.12. The definition does not depend on the choice of bases because base changes induce a diagram

$$\begin{array}{llcccc}
\tilde{R}_{T,1}: & \mathbf{1}_{\mathbb{R}} = & \mathbf{d}_{\mathbb{R}}\mathbb{R}^{r_\infty - r_K}\, \mathbf{d}_{\mathbb{R}}\left(\mathbb{R}^{r_\infty}\right)^{-1}\, \mathbf{d}_{\mathbb{R}}\mathbb{R}^{r_K} & \xrightarrow[\sim]{can_1} & \mathbf{d}_{\mathbb{R}}D^\bullet & \xrightarrow[\sim]{s_D} & \mathbf{1}_{\mathbb{R}} \\
& \Vert & \Vert & & \wr\downarrow b & & \wr\downarrow \\
\tilde{R}_{T,2}: & \mathbf{1}_{\mathbb{R}} = & \mathbf{d}_{\mathbb{R}}\mathbb{R}^{r_\infty - r_K}\, \mathbf{d}_{\mathbb{R}}\left(\mathbb{R}^{r_\infty}\right)^{-1}\, \mathbf{d}_{\mathbb{R}}\mathbb{R}^{r_K} & \xrightarrow[\sim]{can_2} & \mathbf{d}_{\mathbb{R}}D^\bullet & \xrightarrow[\sim]{s_D} & \mathbf{1}_{\mathbb{R}}
\end{array}$$

where b is the alternating product of the base change matrices' determinants. As the base change matrices are elements of $\mathrm{GL}(\mathbb{Z})$, b has to be ± 1. Therefore, $\tilde{R}_{T,1}$ and $\tilde{R}_{T,2}$ may only differ by a sign.

Remark 1.5.13. We want to explicitly state the transformation matrices of the maps λ and α with respect to the bases chosen above.

Let $S_\infty = \{\nu_1, \ldots, \nu_a\}$. The transformation matrix of the map λ is the r_∞ by $(r_\infty - r_K)$ matrix $X = (X_{\nu_1}|\ldots|X_{\nu_a})^t$ which consists of the $(r_\infty - r_K)$ by r_ν block matrices

$$X_\nu = \begin{pmatrix} \ln |\chi^{\nu}_1(e_1)|_\nu & \ldots & \ln |\chi^{\nu}_{r_\nu}(e_1)|_\nu \\ \vdots & \ddots & \vdots \\ \ln |\chi^{\nu}_1(e_{r_\infty - r_K})|_\nu & \ldots & \ln |\chi^{\nu}_{r_\nu}(e_{r_\infty - r_K})|_\nu \end{pmatrix}.$$

The transformation matrix of the map α is the r_K by r_∞ matrix $A = (A_{\nu_1}|\ldots|A_{\nu_a})$ which consists of the r_K by r_ν block matrices

$$A_\nu = \begin{pmatrix} \alpha^{\nu}_{1,1} & \ldots & \alpha^{\nu}_{1,r_\nu} \\ \vdots & \ddots & \vdots \\ \alpha^{\nu}_{r_K,1} & \ldots & \alpha^{\nu}_{r_K,r_\nu} \end{pmatrix}.$$

Remark 1.5.14. The definition of the regulator reduces to known cases.

For $K = \mathbb{Q}$ this is just [Bit11, Def. 4] which can be seen as follows: the basis $(\chi_i^\infty)_{i=1}^{r_\infty}$ of $X^*_{\mathbb{R}}(T)$ can be chosen such that the first r_K elements are $\psi_{i,\infty} \in X^*_{\mathbb{Q}}(T)$, i.e. the first r_K columns of A_∞ are the identity matrix and the rest is zero. Additionally, using the short exact sequence shown in corollary 1.5.10, we may conclude that X has the form $(0|\tilde{X})^t$ where the matrix 0 spans r_K columns. Thus, the regulator of T is just $|\det \tilde{X}|$, using the canonical split of the short exact sequence in combination with remark 1.1.5.

For $T = \mathbb{G}_m$ this definition reduces to the definition of the regulator of K, cf. [Neu99, Prop. 1.7.5]. Note that we can choose $A_\nu = (1)$ because

$$X^*_K(\mathbb{G}_m) = X^*_{K_\nu}(\mathbb{G}_m) = \mathbb{Z}.$$

Pick a section of $\alpha : \mathbb{R} \to \mathbb{R}^{r_\infty}$ such that $r \to r \cdot \chi_i^\nu$ where χ_i^ν is a basis element of $X^*_{K_\nu}(\mathbb{G}_m)$. Then, the formula stated in remark 1.1.5 together with the Laplace expansion yields exactly the usual form.

2 Mixed Motives

It is not the aim of this thesis to cover the too vast topic of the philosophy and the desired properties of motives. Interested readers may refer to books like [JKS94] as it will be interesting to note that many aspects of the general philosophy echo in this work. We will for now however leave it aside and only make use of the notion $\mathcal{MM}$ to indicate the category of mixed motives over a number field K.

2.1 Category of Mixed Realisations

Definition 2.1.1 ([FPR94, Def. III.2.1.1])**.** An object M of the *category of realisations* MR_K over a number field K, or short MR, consists of

- the objects
  (i) M_{dR}, the *de Rham realisation* which is a finite dimensional K-vector space with a filtration $\mathrm{Fil}^i M_{dR}$ called *Hodge filtration*,
  (ii) $M_{B,\nu}$ for every $\nu \in S_\infty$, the *Betti realisation*, a finite dimensional $\mathbb{Q}$-vector space with an action of the absolute Galois group of K_ν, i.e. in the case of $K_\nu = \mathbb{R}$ there is an involution,
  (iii) M_ℓ for every prime number ℓ, an ℓ-adic pseudo-geometric[1] Galois representation of Gal_K called *ℓ-adic realisation*,
- and comparison isomorphisms, i.e.
  (i) an isomorphism
  $$\iota_{\ell,\nu} : \mathbb{Q}_\ell \otimes_{\mathbb{Q}} M_{B,\nu} \cong M_\ell$$
  of $\mathbb{Q}_\ell$-vector spaces compatible with the action of Gal_{K_ν} for every prime number ℓ and valuation $\nu \in S_\infty$,
  (ii) an isomorphism
  $$\iota_\nu : \mathbb{C} \otimes_{\mathbb{Q}} M_{B,\nu} \cong \mathbb{C} \otimes_K M_{dR}$$
  of $\mathbb{C}$-vector spaces compatible with the action of Gal_{K_ν} for every $\nu \in S_\infty$,
  (iii) an isomorphism
  $$\iota_{\mathfrak{p}} : B_{dR,\mathfrak{p}} \otimes_{\mathbb{Q}_p} M_p \cong B_{dR,\mathfrak{p}} \otimes_{K_{\mathfrak{p}}} M_{dR}$$
  of $B_{dR,\mathfrak{p}}$-vector spaces compatible with the action of $\mathrm{Gal}_{K_{\mathfrak{p}}}$ and the Hodge filtration for every $\mathfrak{p}$ above a prime number p,

[1] An ℓ-adic representation is called *pseudo-geometric* if it is unramified for almost all places and is de Rham for the places which are above ℓ.

- and a finite weight filtration $(W_n M)_{n\in\mathbb{Z}}$ which should obey further compatibility properties, see [FPR94, Def. III.2.1.4].

A morphism in this category consists of a collection of morphisms for every realisation compatible with the comparison isomorphisms.

Remark 2.1.2. The category is abelian. Moreover, it is even a neutral Tannakian category. The unit object $\mathbb{1}_K$ is given by the realisation of $\operatorname{Spec} K$, namely $M_{dR} = K$, $M_{B,\nu} = \mathbb{Q}$ and $M_\ell = \mathbb{Q}_\ell$ with trivial Galois actions. We can define a dual motive M^*, enabling us to establish the Tate object $\mathbb{1}_K(1)$ as the dual motive of $\mathbb{G}_{m,K}$ which gives rise to twists. Denote the *Cartier* or *Kummer dual* $M^*(1)$ by M^D.

2.2 Category of Mixed Motives

Because the category of motives $\mathcal{MM}$ is yet unknown, we define a subcategory of MR which for our purposes serves as a sufficiently good replacement.

"Definition" 2.2.1. The image of $\mathcal{MM}$ in MR induced by the various realisation functors will be called MM.

Remark 2.2.2 ([Jan94]). Canonically, there is a realisation functor h^i from varieties over K to MR induced by taking

- the de Rham cohomology $H^i_{Zar}(X, \Omega^\bullet_{X/K})$,
- the singular cohomology $H^i((X \otimes_{K,\nu} \mathbb{C})(\mathbb{C}), \mathbb{Q})$ and
- the ℓ-adic étale cohomology $H^i_{\text{ét}}(X \otimes_K \overline{K}, \mathbb{Q}_\ell)$.

The image of h^i in the category MR, called MM^{var}, should be contained in MM.

From here on, by *(mixed) motive* we refer to an object in the category MM.

2.3 Motivic Cohomology

Definition 2.3.1 ([FPR94, §III.3.1.2])**.** Define the *motivic cohomology* as

$$H^i_{\mathcal{M}}(K, M) := \operatorname{Ext}_{\mathcal{MM}}(\mathbb{1}, M).$$

Remark 2.3.2. The term *motivic cohomology* is normally reserved for a more complex object in a derived category. Its weight graded parts correspond to the object above, see [Jan94, §3].

2.4 L-functions

Definition 2.4.1 ([Fon92, §3.3])**.** Let $\mathfrak{p}$ be a place of K above p and V be an ℓ-adic representation. Then define

$$P_{\mathfrak{p}}(V, s) = \det_{\mathbb{Q}_{l,\mathfrak{p}}}(1 - \mathrm{Fr}_{\mathfrak{p}}\, s \,|, \mathbf{D}_{\mathfrak{p}}(V)),$$

where $\mathbb{Q}_{l,\mathfrak{p}}$ is either $\mathbb{Q}_\ell$ or $K_{\mathfrak{p},0}$ and where $\mathbf{D}_\mathfrak{p}$ is either the $\mathbb{Q}_{l,\mathfrak{p}}$-vector space $V^{\mathcal{I}_{K_\mathfrak{p}}}$ or $\mathbf{D}_{crys,\mathfrak{p}}V$ for $l \neq p$ respectively $l = p$, and the *L-function*

$$L(M,s) = \prod_{\mathfrak{p}} P_\mathfrak{p}(M_\ell, q_\mathfrak{p}^{-s})^{-1},$$

using $q_\mathfrak{p} = \#\mathcal{O}_{K_\mathfrak{p}}/\mathfrak{p}$, the order of the residue field. It is presumed that the L-function is independent of ℓ.

Conjecture 2.4.2 ([FPR94, Def. III.2.2.2]). *An L-admissable*[2] *object M in the category* MR *possesses an L-function which is absolutely convergent for* $\mathrm{Re}(s)$ *big enough and admits a meromorphic continuation to the whole complex plane.*

[2]a technical property believed to be true for all "true" motives, i.e. elements of MM

3 1-Motives

As we have seen, the general appearance of mixed motives is only conjectured. However, there is a properly defined category which is allegedly constituting motives of "level ≤ 1": 1-motives.

3.1 Definition and Properties

Definition 3.1.1 ([Del74, §10.1.10])**.** A 1-*motive* defined over a field k is a diagram

$$\begin{tikzcd} & & X \arrow[d, "u"] & & \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where

- X is a group scheme which is locally – in the étale topology over k – a constant group scheme given by a finitely generated free $\mathbb{Z}$-module,
- T is a torus over k,
- A is an abelian variety over k,
- G is a semi-abelian variety over k and
- u is a morphism of k-group schemes.

Remark 3.1.2. In our situation, X can also be seen as a finitely generated free $\mathbb{Z}$-module on which Gal_k acts discretely, cf. [Tam94, §II.2].

Remark 3.1.3. For notational convenience, we will often represent a 1-motive by a two-term complex $[X \to G]$ in the obvious way. Note that the torus T and the abelian variety A are uniquely determined by G, see [Con02, Thm. 1.1].

Definition 3.1.4. A morphism of 1-motives $[X_i \to G_i]$ is a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "u_1"] & X_2 \arrow[d, "u_2"] \\ G_1 \arrow[r] & G_2. \end{tikzcd}$$

Definition 3.1.5. Denote the category of 1-motives by $\mathcal{MM}_1$.

In the category of 1-motives exists the notion of a dual which generalises the duality of abelian varieties.

Definition 3.1.6 ([Del74, §10.2])**.** For a 1-motive $M = [X \to G]$, there is a *Cartier dual* M^D,

$$\begin{tikzcd} & & T^D \arrow[d] & & \\ 0 \arrow[r] & X^D \arrow[r] & G^D \arrow[r] & A^D \arrow[r] & 0, \end{tikzcd}$$

where the free $\mathbb{Z}$-module $T^D := \operatorname{Hom}(T, \mathbb{G}_m)$ consists of the characters of T, the algebraic group $X^D := \operatorname{Hom}(X, \mathbb{G}_m)$ is a torus and A^D is the usual dual of A as an abelian variety.

Remark 3.1.7. The construction is accomplished via Grothendieck's biextensions which are too complicated to describe in passing. For more information, refer to [Del74, §10.2] and [BV07, §2.7].

There is the following immediate observation:

Lemma 3.1.8. *The category of* 1*-motives* $\mathcal{MM}_1$ *is additive.*

Definition 3.1.9. Let $\mathcal{C}$ be an additive category. The $\mathbb{Q}$-linearisation of $\mathcal{C}$ is a category called $\mathcal{C}_{\mathbb{Q}}$ with

- the same objects as $\mathcal{C}$ and
- $\operatorname{Hom}_{\mathcal{C}_{\mathbb{Q}}}(A, B) := \mathbb{Q} \otimes \operatorname{Hom}_{\mathcal{C}}(A, B)$ as morphisms.

Lemma 3.1.10 ([Org04, Lem. 3.2.2])**.** *The* $\mathbb{Q}$*-linearisation of* $\mathcal{MM}_1$ *is an abelian category. It is called* 1-motives up to isogeny *and is denoted by* $\mathcal{MM}_{1,\mathbb{Q}}$*.*

We will present a simple but useful lemma for later use:

Lemma 3.1.11. *Let* $\mathcal{C}$ *be an additive category and* $\varphi : A \to B$ *a morphism in this category. If the kernel* $k : \ker \varphi \to A$ *exists, then* $1 \otimes k$ *is the kernel of* $q \otimes \varphi$ *in the category* $\mathcal{C}_{\mathbb{Q}}$ *for all* $q \in \mathbb{Q}^\times$*. The same holds for cokernels.*

Proof. The universal mapping property of the kernel $k : \ker \varphi \to A$ of $\varphi : A \to B$ reads as follows: $\varphi \circ k$ is the zero map and for any map $f : Z \to A$ such that $\varphi \circ f$ is the zero map, the map f factors through k, i.e. there is an $f' : Z \to \ker \varphi$ with $f = k \circ f'$.

Let $q' \otimes f : Z \to A$ be a morphism in $\mathcal{C}_{\mathbb{Q}}$ with $q' \neq 0$ such that $(q \otimes \varphi) \circ (q' \otimes f)$ is the zero map. There then is an $n \neq 0$ with

$$n \cdot \varphi \circ f = \varphi \circ (n \cdot f) = 0$$

in $\operatorname{Hom}_{\mathcal{C}}(Z, B)$ and hence an $f' : Z \to \ker \varphi$ which satisfies $n \cdot f = k \circ f'$. Thus, $\frac{q'}{n} \otimes f' : Z \to \ker \varphi$ satisfies

$$(1 \otimes k) \circ \left(\frac{q'}{n} \otimes f' \right) = \frac{q'}{n} \otimes k \circ f' = q' \otimes f$$

and $1 \otimes k$ fulfils the property of being a kernel. The same reasoning gives the result for cokernels. ☐

Remark 3.1.12. A 1-motive $M = [X \to G]$ has a canonical (increasing) *weight filtration*

$$\begin{aligned} W_i M &= M \text{ for } i \geq 0 \\ W_{-1} M &= [0 \to G] \\ W_{-2} M &= [0 \to T] \\ W_i M &= 0 \text{ for } i \leq -3 \end{aligned}$$

and the sequence

$$0 \longrightarrow W_{i-1} M \longrightarrow W_i M \longrightarrow \mathrm{gr}_i^W M \longrightarrow 0$$

is "exact" in the sense that the induced maps of the underlying free $\mathbb{Z}$-modules and the semi-abelian varieties are exact. In the abelian category of 1-motives up to isogeny, where we have an exact structure, the sequence above is indeed exact, cf. lemma 3.1.11.

3.2 Realisations

This section discusses the definition of a suitable realisation functor $\mathcal{MM}_1 \to \mathrm{MR}$. Consequently, we restrict ourselves to 1-motives over a number field K.

Definition 3.2.1. Define the *Lie algebra* $\mathrm{Lie}\, A$ of a group scheme A over a field k as the kernel of $A(k_\epsilon) \to A(k)$ with $k_\epsilon = k[\epsilon]/\epsilon^2$, see [LLR04, p. 459]. It is a k-vector space and commutes with base change.

Definition 3.2.2 ([Del74, §10.1.7])**.** Define the *de Rham realisation* as the K-vector space

$$\mathrm{T}_{dR} M = \mathrm{Lie}\, G^\natural$$

with the Hodge filtration being described by $\mathrm{Fil}^0\, \mathrm{T}_{dR} M = \ker(\mathrm{Lie}\, G^\natural \twoheadrightarrow \mathrm{Lie}\, G)$. Here, $[X \to G^\natural]$ is the universal vector extension of G by $\mathcal{E}xt(M, \mathbb{G}_a)^\vee$, see [Del74, §10.1.7] and [BV07, §2.5].

Definition 3.2.3 ([Del74, §10.1.3])**.** Define the *integral Betti realisation* $\mathrm{T}_{B,\nu} M$ of $M = [X \xrightarrow{u} G]$ for $\nu \in S_\infty$ as the fibre product $\mathrm{Lie}\, G_\nu \times_{G_\nu(\mathbb{C})} X$ in the category of $\mathbb{Z}$-modules fitting into the diagram of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & H_1(G_\nu, \mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{T}_{B,\nu} M \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & H_1(G_\nu, \mathbb{Z}) \arrow[r] & \mathrm{Lie}\, G_\nu \arrow[r, "\exp"] & G_\nu(\mathbb{C}) \arrow[r] & 0 \end{tikzcd}$$

where G_ν is $G \times_{K,\nu} \mathbb{C}$. There is obviously an action of Gal_{K_ν} on $\mathrm{T}_{B,\nu} M$.

Definition 3.2.4 ([Del74, §10.1.10])**.** Define the *integral ℓ-adic realisation* as the free $\mathbb{Z}_\ell$-module

$$\mathrm{T}_\ell M := \varprojlim_n \mathrm{T}_{\mathbb{Z}/\ell^n\mathbb{Z}} M \text{ with}$$
$$\mathrm{T}_{\mathbb{Z}/n\mathbb{Z}} M := \frac{\{(x,g) \in X \times G(\overline{K}) \mid u(x) = ng\}}{\{(nx, u(x)) \mid x \in X\}}$$

where the projective system is composed of the transfer maps $\varphi_{m,md}((x,g)) = (x, dg)$. The Galois action on $\mathrm{T}_{\mathbb{Z}/n\mathbb{Z}} M$ is the obvious one.

Remark 3.2.5. There are many compatibilities: for one, the realisation functors respect the weight filtration, i.e.

$$\operatorname{gr}_i^W \mathrm{T}_\star M = \mathrm{T}_\star \operatorname{gr}_i^W M$$

where the weight filtration on $\mathrm{T}_\star M$ is $\mathrm{T}_\star W_i M$ and

$$0 \longrightarrow W_{i-1} T_\star M \longrightarrow W_i T_\star M \longrightarrow \operatorname{gr}_i^W T_\star M \longrightarrow 0$$

is still exact.

Furthermore, the realisation functors are compatible with Cartier duality insofar as there are pairings

$$\mathrm{T}_\star M \times \mathrm{T}_\star M^D \to \mathrm{T}_\star[0 \to \mathbb{G}_m]$$

which are natural and non-degenerate in the corresponding category.

Remark 3.2.6. For later use, we want to explicitly state the sequences associated with the weight filtration for the ℓ-adic Tate module: the sequences

$$0 \longrightarrow \mathrm{T}_\ell T \longrightarrow \mathrm{T}_\ell G \longrightarrow \mathrm{T}_\ell A \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{T}_\ell G \longrightarrow \mathrm{T}_\ell M \longrightarrow \mathrm{T}_\ell X \longrightarrow 0$$

are exact. A proof can be found in [Jos09, §2.3.4].

Remark 3.2.7. For convenience, we copy from [Jan94, p. 676] the list of the various realisations of the weight graded parts that can be easily deduced by means of the definitions given above and some extra information provided in [Del74].

	T_{dR}	$\mathrm{T}_{B,\nu}$	T_ℓ
$\operatorname{gr}_0^W M = X$	$X \otimes_{\mathbb{Z}} k$	X	$X^{\wedge\ell}$
$\operatorname{gr}_{-1}^W M = A$	$\operatorname{Lie} A^\natural$	$H_1(A_\nu, \mathbb{Z})$	$\mathrm{T}_\ell A$
$\operatorname{gr}_{-2}^W M = T$	$\operatorname{Lie} T$	$H_1(T_\nu, \mathbb{Z})$	$\mathrm{T}_\ell T$

Be cautious that we *always* use $\mathrm{T}_\ell X$ to denote the ℓ-adic realisation of 1-motives, i.e. the ℓ-adic completion $X^{\wedge\ell}$ of X. For an abelian variety A and a torus T, the object $\mathrm{T}_\ell A$ resp. $\mathrm{T}_\ell T$ is the usual ℓ-adic Tate module $\mathrm{T}_\ell G = \varprojlim_n G[\ell^n]$.

Theorem 3.2.8 (Tate property, [Jos09, Thm. 6.0.1])**.** *The natural map induced by the ℓ-adic realisation*

$$\mathrm{Hom}_{\mathcal{MM}_1}(M_1, M_2) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}_K]}(\mathrm{T}_\ell M_1, \mathrm{T}_\ell M_2)$$

is an isomorphism.

Proposition 3.2.9. *The ℓ-adic representations $\mathrm{V}_\ell M := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{T}_\ell M$ are de Rham.*

Proof. They are potentially semi-stable by [Fon94, §6.3.3].[1] □

Definition 3.2.10. Define the realisation functor

$$\mathcal{H} : \mathcal{MM}_{1,\mathbb{Q}} \longrightarrow \mathrm{MR}$$

consisting of the de Rham realisation $\mathrm{V}_{dR} := \mathrm{T}_{dR}$, the Betti realisations $\mathrm{V}_{B,\nu} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{T}_{B,\nu}$ and the ℓ-adic realisations $\mathrm{V}_\ell := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{T}_\ell$. It is immediate that the objects have the structure definition 2.1.1 requires. Furthermore, the demanded comparison isomorphisms exist according to [Fon94, §6.3.3].[2]

3.3 Properties of the Realisation Functor

Here, we will explore the main properties of the realisation functor.

Proposition 3.3.1. *The realisation functor $\mathcal{H}$ is fully faithful.*

Proof. Elementary using the Tate property 3.2.8 for any prime ℓ. □

Proposition 3.3.2 ([Jan94, Thm. 4.3i)])**.** *The image of $\mathcal{H}$ is contained in* $\mathrm{MM}^{var} \subset \mathrm{MM}$.

Definition 3.3.3. Denote the essential image of $\mathcal{H}$ in MR by MM_1.

Conjecture 3.3.4 ([Fon92, §8.2])**.** MM_1 *is stable under sub-objects, quotients and direct sums. If* MM_1 *contains M, it contains all extensions of M by* $\mathbb{1}$*.*

Remark 3.3.5. The extension property was also conjectured by Deligne in [Del89, §2.4]. The conjecture fosters the philosophy that 1-motives are just all motives of “level ≤ 1”. Therefore, this conjecture has rather the character of a requirement for the category MM. There are two results which need to be mentioned: first, Bertolin proved in [Ber09] that if one replaces MM_1 by the Tannakian subcategory of MR generated by 1-motives, the extension property is true. Another important result is that for Voevodsky's category of motives, the conjecture ascends to being a theorem, cf. [Org04, Thm. 3.4.1].

[1] See next footnote.

[2] Fontaine actually references one of his unpublished articles. Yet there exists an alternative way to obtain the results: due to prop. 3.3.2, we inherit all the good properties from the realisation functor of varieties, see remark 2.2.2.

3.4 Motivic Cohomology

In this section, we want to compute the motivic cohomology of the realisation of 1-motives. To be able to do so, we need to assume conjecture 3.3.4 from here on. Recall the definition of motivic cohomology (3.4),

$$H^i_{\mathcal{M}}(K, M) := \mathrm{Ext}^i_{\mathrm{MR}}(\mathbb{1}, M).$$

Remark 3.4.1. By abuse of notation, we write $H^i_{\mathcal{M}}(K, M)$, $H^i_{\mathcal{M}}(K, X)$ or $H^i_{\mathcal{M}}(K, G)$ instead of $H^i_{\mathcal{M}}(K, \mathcal{H}M)$, $H^i_{\mathcal{M}}(K, \mathcal{H}[X \to 0])$ or $H^i_{\mathcal{M}}(K, \mathcal{H}[0 \to G])$ for reasons of simplicity. Furthermore, we identify $\mathbb{Z}$ with the realisation of the 1-motive $[\mathbb{Z} \to 0]$ which equals $\mathbb{1}$.

Proposition 3.4.2. *For a 1-motive $M = [X \xrightarrow{u} G]$ we get*

$$H^0_{\mathcal{M}}(K, M) = H^0(K, \ker u) \otimes \mathbb{Q}.$$

Proof. Straightforward calculation yields

$$\begin{aligned} H^0_{\mathcal{M}}(K, M) &= \mathrm{Ext}^0_{\mathrm{MM}}(\mathbb{1}, \mathcal{H}M) = \mathrm{Hom}_{\mathrm{MM}}(\mathbb{1}, \mathcal{H}M) \\ &= \mathrm{Hom}_{\mathrm{MM}_1}(\mathbb{1}, \mathcal{H}M) \overset{3.3.1}{=} \mathrm{Hom}_{\mathcal{M}\mathcal{M}_{1,\mathbb{Q}}}(\mathbb{Z}, M) \\ &= \mathbb{Q} \otimes \mathrm{Hom}_{\mathcal{M}\mathcal{M}_1}(\mathbb{Z}, M) \overset{(\star)}{=} \mathbb{Q} \otimes \mathrm{Hom}_{\mathcal{M}\mathcal{M}_1}(\mathbb{Z}, \ker u) \\ &= \mathbb{Q} \otimes \mathrm{Hom}_{\mathbb{Z}[\mathrm{Gal}_K]}(\mathbb{Z}, \ker u) = \mathbb{Q} \otimes H^0(K, \ker u) \end{aligned}$$

where $(\star)$ is a factorisation property directly drawn from the definition of morphisms of 1-motives, cf. definition 3.1.4. ☐

Proposition 3.4.3. *The group $H^1_{\mathcal{M}}(K, X)$ vanishes.*

Proof. Every element of $H^1_{\mathcal{M}}(K, X) = \mathrm{Ext}^1_{\mathrm{MR}}(\mathbb{1}, X)$ corresponds to an extension

$$0 \longrightarrow X \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 0$$

in the category of 1-motives up to isogeny because E is a 1-motive $[X_E \to G_E]$ by conjecture 3.3.4. E can be replaced by the 1-motive $[X_E \to 0]$ because application of the functor W_{-1} to the sequence above is exact and therefore, G_E is isogenous to 0. Hence, the above corresponds to an extension

$$0 \longrightarrow X \xrightarrow{q \otimes i} X_E \xrightarrow{q' \otimes p} \mathbb{Z} \longrightarrow 0$$

in the category of 1-motives up to isogeny. Note that i is injective: the kernel of i exists in the usual category of 1-motives since a submodule of the free $\mathbb{Z}$-module X is still free. By lemma 3.1.11, we have $k := \ker i \cong 0$ in the $\mathbb{Q}$-linearised category, hence there are maps $k \to 0 \to k$ which when composed yield the identity. Consequently, $0 = r \otimes \mathrm{id} \in \mathbb{Q} \otimes \mathrm{Hom}(k, k)$, i.e. there is an n such that $n \cdot \mathrm{id} = 0$. This particularly implies that k must be zero because k is free.

Furthermore, we know $p \circ i$ is zero because $qq' \otimes p \circ i = 0$, thus there is an n such that $n \cdot p \circ i = 0$ in $\mathrm{Hom}(X, \mathbb{Z})$. Note that $\mathbb{Z}$ is torsion free.

Because p maps to $\mathbb{Z}$ and is non-zero, $\operatorname{im} p$ is a non-trivial ideal of $\mathbb{Z}$. Thus, there exists an n with $\operatorname{im} p = (n)$. If we replace $q' \otimes p$ with $(n \cdot q') \otimes \frac{p}{n}$, we may assume p to be surjective.

We therefore have a commutative diagram (for arbitrary n)

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, equal] & d^{-1}(n \cdot s(\mathbb{Z})) \arrow[r, "d"] \arrow[d, "\subset"] & n \cdot s(\mathbb{Z}) \arrow[r] \arrow[d, "\subset"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, equal] & d^{-1}(s(\mathbb{Z})) \arrow[r, "d"] \arrow[d, "\subseteq"] & s(\mathbb{Z}) \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, equal] & X_E \arrow[r, "d"] \arrow[d, equal] & X_E/X \arrow[r] \arrow[d, "f"'] & 0 \\
 & X \arrow[r, hook, "i"] & X_E \arrow[r, two heads, "p"] & \mathbb{Z} \arrow[u, dotted, bend right, "s"'] &
\end{tikzcd}$$

(not exact)

where f exists as p factors through the cokernel X_E/X and s exists because we may choose a splitting due to $\mathbb{Z}$ being free. Hence, s induces an isomorphism $\mathbb{Z} \cong s(\mathbb{Z})$. Furthermore, we know that $d^{-1}(s(\mathbb{Z}))$ as a submodule of the free $\mathbb{Z}$-module X_E is itself free. In the $\mathbb{Q}$-linearised category, the inclusion in the right hand column of the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "q \otimes i"] \arrow[d, equal] & d^{-1}(n \cdot s(\mathbb{Z})) \arrow[r, "q' \otimes d"] \arrow[d, "\subset"] & n \cdot s(\mathbb{Z}) \arrow[r] \arrow[d, "\subset"] & 0 \\
0 \arrow[r] & X \arrow[r, "q \otimes i"] \arrow[d, equal] & d^{-1}(s(\mathbb{Z})) \arrow[r, "q' \otimes d"] \arrow[d, "\subseteq"] & s(\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & X \arrow[r, "q \otimes i"] & X_E \arrow[r, "q' \otimes p"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

becomes an isomorphism almost by definition of the $\mathbb{Q}$-linearisation. The five lemma yields that the inclusions in the central column become isomorphisms, i.e. they all represent the same element in the Ext group.

The second row (in the non-linearised category) corresponds to an element $g \in H^1(K, X)$ with $g : \sigma \mapsto i^{-1}(\sigma(e) - e)$ where $e \in d^{-1}(s(\mathbb{Z}))$ is a lift of $1_s \in s(\mathbb{Z})$, see proposition 1.2.7. Particularly, we may choose the lift $n \cdot e$ of $n \cdot 1_s \in n \cdot s(\mathbb{Z})$. With that, the first row corresponds to the element $n \cdot g$ in $H^1(K, X)$. Note that $H^1(K, X)$ is finite because of lemma 1.2.10, hence for n large enough, the first sequence splits and the associated element in the Ext group is the trivial one. $\square$

Theorem 3.4.4. *There is an isomorphism*

$$H^1_{\mathcal{M}}(K, G) \cong \mathbb{Q} \otimes G(K).$$

Proof. This proof uses the construction given in [Fon92, §8.1]: define the map $\mathbb{Q} \otimes G(K) \to H^1_{\mathcal{M}}(K, G)$ by mapping $q \otimes g$ to the extension

$$0 \longrightarrow G \xrightarrow{q^{-1}\otimes \mathrm{id}_G} [\mathbb{Z} \xrightarrow{u} G] \xrightarrow{1\otimes \mathrm{id}_{\mathbb{Z}}} \mathbb{Z} \longrightarrow 0$$

in the category of 1-motives up to isogeny where $u(n) = n \cdot g$ unless $q \neq 0$. If $q = 0$, use the identity $0 \otimes g = 1 \otimes 0_G$ (where 0_G is the neutral element in G) and then apply the above map to $1 \otimes 0_G$. The map is well-defined: in a first step, we show that $\alpha q \otimes g$ and $q \otimes (\alpha \cdot g)$ for $\alpha \neq 0$ give rise to the same element in $H^1_{\mathcal{M}}(K, G)$. For $q = 0$ this is obvious. Hence, we may assume $q \neq 0$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "(\alpha q)^{-1}\otimes \mathrm{id}_G"] \arrow[d, equal] & {[\mathbb{Z} \xrightarrow{u} G]} \arrow[r, "1\otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[d, "1\otimes \mathrm{id}_{\mathbb{Z}}"', "1\otimes \alpha\, \mathrm{id}_G"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r, "q^{-1}\otimes \mathrm{id}_G"] & {[\mathbb{Z} \xrightarrow{v} G]} \arrow[r, "1\otimes \mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with $u(1) = g$ and $v(1) = \alpha \cdot g$. The diagram implies the equivalence of the extensions. Now, we want to show that if two objects $q \otimes g$ and $q' \otimes g'$ agree in $G(K) \otimes \mathbb{Q}$, they yield the same element in $H^1_{\mathcal{M}}(K, G)$. Without loss of generality, we may assume $q = \frac{1}{p}$ and $q' = \frac{1}{p'}$, and there is the equality

$$1 \otimes (p' \cdot g - p \cdot g') = 1 \otimes 0_G$$

in $\mathbb{Q} \otimes G(K)$. If follows that an $\alpha \neq 0$ exists such that $\alpha p' \cdot g = \alpha p \cdot g'$ in $G(K)$, i.e.

$$\frac{1}{p} \otimes g = \frac{1}{\alpha p p'} \otimes (\alpha p' \cdot g) = \frac{1}{\alpha p p'} \otimes (\alpha p \cdot g') = \frac{1}{p'} \otimes g'.$$

In this situation, our first step's result is applicable and all objects above map to the same element in $H^1_{\mathcal{M}}(K, G)$.

Showing injectivity works as follows: If there is an isomorphism $q' \otimes \varphi$ from $[\mathbb{Z} \xrightarrow{0_G} G]$ to $[\mathbb{Z} \xrightarrow{u} G]$, then the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\varphi_X"] \arrow[d, "0_G"] & \mathbb{Z} \arrow[d, "u"] \\
G \arrow[r, "\varphi_G"] & G
\end{tikzcd}$$

is commutative. Let $n = \varphi_X(1) \neq 0$. Then

$$n \cdot g = u(n) = (u \circ \varphi_X)(1) = 0_G.$$

Hence, g is torsion in $G(K)$.

Surjectivity can be shown as follows: every extension is a 1-motive by conjecture 3.3.4, i.e. the extension corresponds to

$$0 \longrightarrow G \xrightarrow{q\otimes \varphi} [X \xrightarrow{u} G'] \xrightarrow{q'\otimes \rho} \mathbb{Z} \longrightarrow 0.$$

The exactness of the sequence implies $X = \mathbb{Z}$ equipped with the trivial Galois action. A non-trivial homomorphism $\mathbb{Z} \to \mathbb{Z}$ is fully determined by $1 \mapsto n$. Thus, replacing $q' \otimes \rho$ by $nq' \otimes \frac{\rho}{n}$, we may assume $\rho = \mathrm{id}_{\mathbb{Z}}$. Considering

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "q \otimes \varphi"] \arrow[d, equal] & {[\mathbb{Z} \xrightarrow{u} G']} \arrow[r, "q' \otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[d, "q' \otimes \mathrm{id}_{\mathbb{Z}}"', "q' \otimes \mathrm{id}_{G'}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r, "qq' \otimes \varphi"] & {[\mathbb{Z} \xrightarrow{u} G']} \arrow[r, "1 \otimes \mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

we are reduced to the case $q' = 1$.

If $g = u(1)$ itself is not in the image of φ, the exactness of the sequence implies that there must be a $\mu \neq 0$ such that $\mu \cdot g$ lies in the image of φ. The diagram

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "q \otimes \varphi"] \arrow[d, equal] & {[\mathbb{Z} \xrightarrow{u} G']} \arrow[r, "1 \otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[d, "1 \otimes \mathrm{id}_{\mathbb{Z}}"', "1 \otimes \mu\, \mathrm{id}_{G'}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r, "q\mu \otimes \varphi"] & {[\mathbb{Z} \xrightarrow{\mu \cdot u} G']} \arrow[r, "1 \otimes \mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

yields that, without loss of generality, $u(1)$ has a preimage.

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "q \otimes \varphi"] \arrow[d, equal] & {[\mathbb{Z} \xrightarrow{u} G']} \arrow[r, "1 \otimes \mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r, "q \otimes \mathrm{id}_G"] & {[\mathbb{Z} \xrightarrow{v} G]} \arrow[r, "1 \otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[u, "1 \otimes \mathrm{id}_{\mathbb{Z}}", "1 \otimes \varphi"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where $v(1)$ is a preimage of $u(1)$. Note that $v(1)$ has to be invariant under the action of Gal_K because 1 is invariant. Hence, we have transformed the general extension into an extension contained in the image of the map described above, and surjectivity has been proved. $\square$

Theorem 3.4.5. *For a 1-motive $M = [X \xrightarrow{u} G]$ one has*

$$0 \longrightarrow H^0_{\mathcal{M}}(K, M) \longrightarrow X(K) \otimes \mathbb{Q} \xrightarrow{u} G(K) \otimes \mathbb{Q} \longrightarrow H^1_{\mathcal{M}}(K, M) \longrightarrow 0.$$

Proof. Applying the $\mathrm{Hom}_{\mathrm{MR}}(\mathbb{1}, -)$ functor to the short exact weight filtration sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow X \longrightarrow 0$$

yields the long exact Ext sequence up to level one, using that MR is an abelian category, see [Eis95, Ex. A3.26f]. The previous results imply the above sequence. The boundary morphism as defined in [Eis95, Ex. A3.26b] maps $q \otimes x \in \mathrm{Hom}_{\mathcal{MM}_{1,\mathbb{Q}}}(\mathbb{Z}, X) = H^0_{\mathcal{M}}(K, X)$ to the upper row of the diagram

$$
\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "x \otimes q"] & 0 \\
0 \arrow[r] & G \arrow[r] & M \arrow[r, "p \otimes q'"] & X \arrow[r] & 0
\end{tikzcd}
$$

where E is $\ker(-p \otimes q', x \otimes q) : M \oplus \mathbb{Z} \to X$. The latter identifies with $[\mathbb{Z} \xrightarrow{v} G]$ and $v(1) = u(x(1))$ holds. Hence, the boundary morphism is indeed u. $\square$

4 Tamagawa Number Conjecture

This section outlines the Tamagawa Number Conjecture as stated in [FPR94].

4.1 Various Constructions

Similarly to the definition of the global Selmer group $H^1_f(K,-)$, we will proceed to define a motivic subgroup, the finite part of $H^1_{\mathcal{M}}(K,-)$.

Definition 4.1.1 ([FPR94, Def. III.3.1.3])**.** Define

$$H^1_{\mathcal{M},f}(K,M) := \ker\left(H^1_{\mathcal{M}}(K,M) \to \prod_\ell H^1(K,M_\ell)/H^1_f(K,M_\ell)\right)$$

where $H^1_{\mathcal{M}}(K,M) \to H^1(K,M_\ell)$ maps an extension of $\mathbb{1}$ by M onto an extension of $\mathbb{1}_\ell = \mathbb{Q}_\ell$ by M_ℓ via the projection of MR to the ℓ-adic realisation.

Definition 4.1.2. We denote by $t_M = M_{dR}/\operatorname{Fil}^0 M_{dR}$ the *tangent space* of the motive M.

Definition 4.1.3 ([FPR94, Def. III.3.1.5])**.** For a motive M and $\nu \in S_\infty$, the comparison isomorphism

$$\iota_\nu : \mathbb{C} \otimes_{\mathbb{Q}} M_{B,\nu} \cong \mathbb{C} \otimes_K M_{dR},$$

which is compatible with the action of Gal_{K_ν}, induces morphisms

$$\mathbb{R} \otimes_{\mathbb{Q}} M^+_{B,\nu} \longrightarrow (\mathbb{C} \otimes_{\mathbb{Q}} M_{B,\nu})^+ \cong K_\nu \otimes_K M_{dR} \longrightarrow K_\nu \otimes_K t_M$$

where we use $(-)^+$ as $H^0(K_\nu,-)$. Call the composition of the maps $\alpha_{M,\nu}$ and denote $\oplus_\nu \alpha_{M,\nu}$ by α_M. Note that $\oplus_\nu K_\nu \otimes_K t_M = \mathbb{R} \otimes_{\mathbb{Q}} t_M$ is canonically an $\mathbb{R}$-vector space.

Definition 4.1.4 ([FPR94, Def. III.3.1.6])**.** For a motive M and $\nu \in S_\infty$, one has the realisation map $H^i_{\mathcal{M}}(K,M) \to H^i(K_\nu, \mathbb{R}\otimes_{\mathbb{Q}} M_{B,\nu})$ where $H^i(K_\nu,V) = \operatorname{Ext}^i_{SH_{K_\nu}}(\mathbb{1},V)$ is calculated in the category of mixed Hodge structures over K_ν, cf. [FPR94, Def. III.1.1.1 & 1.2.1]. By extension of scalars, one gets an $\mathbb{R}$-linear map $H^i_{\mathcal{M}}(K,M)_{\mathbb{R}} \to H^i(K_\nu, \mathbb{R}\otimes_{\mathbb{Q}} M_{B,\nu})$. Furthermore, an element of $\operatorname{Ext}^0_{SH_{K_\nu}}(\mathbb{1}, \mathbb{R}\otimes_{\mathbb{Q}} M_{B,\nu}) = \operatorname{Hom}(\mathbb{1}, \mathbb{R}\otimes_{\mathbb{Q}} M_{B,\nu})$ gives rise to an element in $\ker \alpha_{M,K_\nu} \subset \mathbb{R}\otimes_{\mathbb{Q}} M^+_{B,\nu}$ by evaluation at 1, see [FPR94, Prop. III.1.1.6 & 1.2.3]. We may therefore define

$$u_M : H^0_{\mathcal{M}}(K,M)_{\mathbb{R}} \to \bigoplus_\nu H^0(K_\nu, \mathbb{R}\otimes_{\mathbb{Q}} M_{B,\nu}) \to \ker \alpha_M.$$

Additionally, it is shown in the same proposition that an extension $0 \to V \to U \to \mathbb{1} \to 0$ in $\operatorname{Ext}^1_{SH_{K_\nu}}(\mathbb{1},V)$ for $V = \mathbb{R}\otimes_{\mathbb{Q}} M_{B,\nu}$ gives rise to an element in $\operatorname{coker} \alpha_{M,\nu}$ via the following construction: the extension induces short exact sequences

$$0 \longrightarrow W_0V^+ \longrightarrow W_0U^+ \longrightarrow \mathbb{R} \longrightarrow 0$$

and

$$0 \longrightarrow W_0V_{\mathbb{C}}^+ \longrightarrow W_0U_{\mathbb{C}}^+ \longrightarrow \mathbb{C}^+ \longrightarrow 0$$

as well as

$$0 \longrightarrow W_0 \operatorname{Fil}^0 V_{\mathbb{C}}^+ \longrightarrow W_0 \operatorname{Fil}^0 U_{\mathbb{C}}^+ \longrightarrow \mathbb{C}^+ \longrightarrow 0.$$

Let $e \in W_0U$ be a lift of $1 \in \mathbb{R}$. Then a $v \in W_0V_{\mathbb{C}}^+$ has to exist such that $e + v \in W_0 \operatorname{Fil}^0 U_{\mathbb{C}}^+$. The image of $v \in V_{\mathbb{C}}^+ = (\mathbb{C} \otimes_{\mathbb{Q}} M_{B,\nu})^+$ in $\operatorname{coker} \alpha_{M,\nu}$ is independent of choices. Altogether, one can define a map

$$v_M : H^1_{\mathcal{M},f}(K, M)_{\mathbb{R}} \hookrightarrow H^1_{\mathcal{M}}(K, M)_{\mathbb{R}} \to \bigoplus_{\nu} H^1(K_\nu, \mathbb{R} \otimes_{\mathbb{Q}} M_{B,\nu}) \to \operatorname{coker} \alpha_M.$$

Note that $\operatorname{coker} \alpha_M$ identifies with $(\ker \alpha_{M^D})^*$ and vice versa. Hence, we get maps

$$\begin{aligned} u_M^* &: \operatorname{coker} \alpha_M \to H^0_{\mathcal{M}}(K, M^D)^*_{\mathbb{R}} \\ v_M^* &: \ker \alpha_M \to H^1_{\mathcal{M},f}(K, M^D)^*_{\mathbb{R}}. \end{aligned}$$

4.2 Tamagawa Number Conjecture: Motivic Version

We are now equipped with the techniques needed to state a conjecture which will enable us to find the value of the L-function's leading coefficient L^* up to a rational multiple.

The basic object which consolidates knowledge of the various different realisations looks as follows:

Definition 4.2.1 ([FPR94, Def. III.4.4.1])**.** Define the *fundamental line* as

$$\begin{aligned} \Delta_{\mathbb{Q}}(M) \coloneqq\ & \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, M)\, \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, M)^{-1} \\ & \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, M^D)^*\, \mathbf{d}_{\mathbb{Q}} (H^1_{\mathcal{M},f}(K, M^D)^*)^{-1} \\ & \mathbf{d}_{\mathbb{Q}} (M_B^+)^{-1}\, \mathbf{d}_{\mathbb{Q}}(t_M) \end{aligned}$$

where $M_B^+ = \oplus_\nu H^0(K_\nu, M_{B,\nu})$.

Conjecture 4.2.2 ([FPR94, Prop. III.3.2.5])**.** *There exists an exact sequence*

$$0 \longrightarrow H^0_{\mathcal{M}}(K, M)_{\mathbb{R}} \xrightarrow{u_M} \ker \alpha_M \xrightarrow{v_M^*} H^1_{\mathcal{M},f}(K, M^D)^*_{\mathbb{R}}$$

$$\xrightarrow{h_M} H^1_{\mathcal{M},f}(K, M)_{\mathbb{R}} \xrightarrow{v_M} \operatorname{coker} \alpha_M \xrightarrow{u_M^*} H^0_{\mathcal{M}}(K, M^D)^*_{\mathbb{R}} \longrightarrow 0.$$

We will denote it by $S_{\mathbb{R}}(M)$.

Definition 4.2.3. The above conjecture induces an isomorphism

$$\vartheta_\infty : \quad S_{\mathbb{R}}(M) \xrightarrow{\sim} 1_{\mathbb{R}}$$

called the *period-regulator map.*

Definition 4.2.4. The map α_M defined in 4.1.3 induces a natural short exact sequence

$$0 \longrightarrow \ker \alpha_M \longrightarrow \bigoplus_\nu \mathbb{R} \otimes_{\mathbb{Q}} M^+_{B,\nu} \xrightarrow{\alpha_M} \mathbb{R} \otimes_{\mathbb{Q}} t_M \longrightarrow \operatorname{coker} \alpha_M \longrightarrow 0$$

which yields an isomorphism

$$s_{\alpha,M} : \bigotimes_\nu \mathbf{d}_{\mathbb{R}} \left(\mathbb{R} \otimes_{\mathbb{Q}} M^+_{B,\nu}\right)^{-1} \mathbf{d}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Q}} t_M) \cong \mathbf{d}_{\mathbb{R}}(\ker \alpha_M)^{-1}\, \mathbf{d}_{\mathbb{R}} \operatorname{coker} \alpha_M.$$

Hence, $s_{\alpha,M}$ induces an isomorphism $\Delta_{\mathbb{Q}}(M)_{\mathbb{R}} \cong \mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(M)$.

Based on the previous definitions, the conjecture can be stated as follows:

Conjecture 4.2.5 ([FPR94, Conj. III.4.4.3])**.** *There is a unique isomorphism*

$$\zeta_{\mathbb{Q}}(M) : \quad \mathbf{1}_{\mathbb{Q}} \xrightarrow{\sim} \Delta_{\mathbb{Q}}(M)$$

such that the composition

$$\mathbf{1}_{\mathbb{C}} \xrightarrow{\zeta_{\mathbb{Q}}(M)_{\mathbb{C}}} \Delta_{\mathbb{Q}}(M)_{\mathbb{C}} \xrightarrow{(s_{\alpha,M})_{\mathbb{C}}} S_{\mathbb{R}}(M)_{\mathbb{C}} \xrightarrow{(\vartheta_\infty)_{\mathbb{C}}} \mathbf{1}_{\mathbb{C}}$$

equals $L^*_K(M)^{-1}$ *in* $\operatorname{Aut}(\mathbf{1}_{\mathbb{C}})$*.*

Remark 4.2.6. There is a slight deviation between standard literature and our definitions as it is $\vartheta_\infty \circ s_{\alpha,M}$ which is usually called the period-regulator map.

4.3 Tamagawa Number Conjecture: ℓ-adic Version

We will now look into conjectures which describe the L-function's leading coefficient L^* up to a factor in $\mathbb{Z}_\ell^\times$.

Definition 4.3.1. Let $S(M)$ be a finite set of places of K containing S_∞ as well as the places of bad reduction of M. Then define $S_\ell(M)$ as $S_\ell \cup S(M)$ and $S_{\ell,f}(M)$ as $S_\ell(M) \cap S_f$.

Proposition 4.3.2 ([BF96, Eq. 1.28])**.** *Let T_ℓ be a Galois stable $\mathbb{Z}_\ell$-structure of M_ℓ (for $\ell \neq 2$), i.e. a Gal_K-invariant lattice of M_ℓ. Then there is a distinguished triangle of perfect complexes*

$$R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, T_\ell) \to R\Gamma_f(K, T_\ell) \to \left(\bigoplus_{\mathfrak{p} \in S_{\ell,f}(M)} R\Gamma_f(K_{\mathfrak{p}}, T_\ell)\right) \oplus \bigoplus_{\nu \in S_\infty} H^0(K_\nu, T_\ell)$$

such that, tensored with $\mathbb{Q}_\ell$, this naturally identifies with the distinguished triangle

$$R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, M_\ell) \to R\Gamma_f(K, M_\ell) \to \left(\bigoplus_{\mathfrak{p} \in S_{\ell,f}(M)} R\Gamma_f(K_{\mathfrak{p}}, M_\ell)\right) \oplus \bigoplus_{\nu \in S_\infty} H^0(K_\nu, M_\ell).$$

Remark 4.3.3. One has to use Burns and Flach's definition in the case of p not dividing $\#\mathrm{Gal}(L/K)$ at the end of section 1.5 in [BF96]. Define $H^i_{BF,f}(K, T_\ell) := H^i R\Gamma_f(K, T_\ell)$. Then $H^i_{BF,f}(K, T_\ell)$ agrees with the Bloch-Kato definition given in 1.2.30 for $i = 0, 1$, i.e. $H^0(K, T_\ell) = H^0_{BF,f}(K, T_\ell)$ and $H^1_f(K, T_\ell) = H^1_{BF,f}(K, T_\ell)$. In case of $i = 2, 3$, $H^i_{BF,f}(K, T_\ell)$ is related to the Bloch-Kato definition via exact sequences. We will explore this relation in section 5.5.

Conjecture 4.3.4 ([FPR94, Conj. III.3.3.1])**.** *There are natural isomorphisms*

$$\begin{aligned} H^0_{\mathcal{M}}(K, M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell &\cong H^0(K, M_\ell) \\ H^1_{\mathcal{M},f}(K, M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell &\cong H^1_f(K, M_\ell). \end{aligned}$$

Proposition 4.3.5 ([Fon92, §4.4])**.** *There are canonical isomorphisms*

$$\begin{aligned} &\eta_{\mathfrak{p}}(M_p): \quad \mathbf{d}_{\mathbb{Q}_p} R\Gamma_f(K_{\mathfrak{p}}, M_p)\, \mathbf{d}_{\mathbb{Q}_p} t_{\mathfrak{p}}(M_p) \longrightarrow \mathbf{1}_{\mathbb{Q}_p} && \text{for } \mathfrak{p} \in S_p \\ &\eta_{\mathfrak{p}}(M_\ell): \quad \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_f(K_{\mathfrak{p}}, M_\ell) \longrightarrow \mathbf{1}_{\mathbb{Q}_\ell} && \text{for } \mathfrak{p} \notin S_\ell \end{aligned}$$

where $t_{\mathfrak{p}}(V) := \mathbf{D}_{dR,\mathfrak{p}}(V)/\mathbf{D}^0_{dR,\mathfrak{p}}(V)$ *is the tangent space of an* ℓ*-adic representation* V*.*

Proof. Due to proposition 1.2.26, we have eq. 7.57 in [Ven07]. Therefore, we can still establish the isomorphisms in the same way as seen in eq. 7.64, [Ven07]. □

Definition and Lemma 4.3.6. *There is a canonical isomorphism called* ℓ*-adic* period-regulator map

$$\vartheta_\ell: \quad \Delta_{\mathbb{Q}}(M)_{\mathbb{Q}_\ell} \cong \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, M_\ell).$$

Proof. We get

$$\begin{aligned} \Delta_{\mathbb{Q}}(M)_{\mathbb{Q}_\ell} = \; & \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, M)_{\mathbb{Q}_\ell}\, \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, M)^{-1}_{\mathbb{Q}_\ell} \\ & \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, M^D)^*_{\mathbb{Q}_\ell}\, \mathbf{d}_{\mathbb{Q}}(H^1_{\mathcal{M},f}(K, M^D)^*)^{-1}_{\mathbb{Q}_\ell} \\ & \bigotimes_{\nu \in S_\infty} \mathbf{d}_{\mathbb{Q}}(M_B^+)^{-1}_{\mathbb{Q}_\ell}\, \mathbf{d}_{\mathbb{Q}}(t_M)_{\mathbb{Q}_\ell} \\ \cong \; & \mathbf{d}_{\mathbb{Q}_\ell} H^0(K, M_\ell)\, \mathbf{d}_{\mathbb{Q}_\ell} H^1_f(K, M_\ell)^{-1} \\ & \mathbf{d}_{\mathbb{Q}_\ell} H^0(K, M_\ell^D)^*\, \mathbf{d}_{\mathbb{Q}_\ell}(H^1_f(K, M_\ell^D)^*)^{-1} \\ & \bigotimes_{\nu \in S_\infty} \mathbf{d}_{\mathbb{Q}_\ell}(M_\ell^+)^{-1} \bigotimes_{\mathfrak{p} \in S_\ell} \mathbf{d}_{\mathbb{Q}_\ell} t_{\mathfrak{p}}(M_\ell) \end{aligned}$$

because of conjecture 4.3.4.

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} M^+_{B,\nu} \cong M^+_\ell$$

and

$$(K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes_K t_M \cong \oplus_{\mathfrak{p} \in S_\ell} K_{\mathfrak{p}} \otimes_K t_M \cong \oplus_{\mathfrak{p} \in S_\ell} t_{\mathfrak{p}}(M_\ell)$$

follow from the compatibility properties of the comparison isomorphisms and [Neu99, Prop. II.8.3], which states that $L \otimes_K K_\nu \cong \prod_{\nu'|\nu} L_{\nu'}$ holds if $L|K$ is a finite separable extension.

By [BF96, p. 74] we get $H^i_f(K,V) \cong H^{3-i}_f(K,V^D)^*$. Hence

$$\begin{aligned}
\Delta_{\mathbb{Q}}(M)_{\mathbb{Q}_\ell} &\cong R\Gamma_f(K,M_\ell) \\
&\qquad \bigotimes_{\nu\in S_\infty} \mathbf{d}_{\mathbb{Q}_\ell}(M_\ell^+)^{-1} \bigotimes_{\mathfrak{p}\in S_\ell} \mathbf{d}_{\mathbb{Q}_\ell} t_{\mathfrak{p}}(M_\ell) \\
&\cong \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, M_\ell) \\
&\qquad \bigotimes_{\mathfrak{p}\in S_{\ell,f}(M)} \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_f(K_{\mathfrak{p}}, M_\ell) \bigotimes_{\nu\in S_\infty} \mathbf{d}_{\mathbb{Q}_\ell} H^0(K_\nu, M_\ell) \\
&\qquad \bigotimes_{\nu\in S_\infty} \mathbf{d}_{\mathbb{Q}_\ell}(M_\ell^+)^{-1} \bigotimes_{\mathfrak{p}\in S_\ell} \mathbf{d}_{\mathbb{Q}_\ell} t_{\mathfrak{p}}(M_\ell) \\
&= \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, M_\ell) \\
&\qquad \bigotimes_{\mathfrak{p}\in S_{\ell,f}(M)} \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_f(K_{\mathfrak{p}}, M_\ell) \bigotimes_{\mathfrak{p}\in S_\ell} \mathbf{d}_{\mathbb{Q}_\ell} t_{\mathfrak{p}}(M_\ell),
\end{aligned}$$

using proposition 4.3.2. Then, by the canonical map $\oplus_{\mathfrak{p}\in S_{\ell,f}(M)}\eta_{\mathfrak{p}}$, we get the desired canonical isomorphism. □

The ℓ-adic version of the Tamagawa Number Conjecture then reads as:

Conjecture 4.3.7 ([FPR94, Conj. III.4.5.2])**.** *Assuming conjecture 4.2.5. Let T_ℓ be a Galois stable $\mathbb{Z}_\ell$-lattice of M_ℓ. Then there exists a unique isomorphism*

$$\zeta_{\mathbb{Z}_\ell}(T_\ell):\ \mathbf{1}_{\mathbb{Z}_\ell} \longrightarrow \mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_c(U,T_\ell)$$

which by base change induces the map

$$\zeta_{\mathbb{Z}_\ell}(T_\ell)_{\mathbb{Q}_\ell}:\ \mathbf{1}_{\mathbb{Q}_\ell} \xrightarrow{\zeta_{\mathbb{Q}}(M)_{\mathbb{Q}_\ell}} \Delta_{\mathbb{Q}}(M)_{\mathbb{Q}_\ell} \xrightarrow{\vartheta_\ell} \mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_c(U,M_\ell).$$

Remark 4.3.8. It can be shown that this conjecture is independent of both T_ℓ and $S(M)$.

5 Tamagawa Number Conjecture for Semi-Abelian Varieties

In this section, we will explore the implications of the Tamagawa Number Conjecture for the L-function of the motive $h^1(G)(1)$.

5.1 Reformulation

First, we note that the L-function of the motive M only depends on the ℓ-adic realisation of M. Hence, we need precise knowledge of the ℓ-adic realisation of our motive $h^1(G)(1)$.

There is a convenient theorem which transforms étale cohomology to group cohomology:

Theorem 5.1.1 ([Mil12, p. 125])**.** *For a locally constant étale $\mathbb{Z}_\ell$-sheaf $\mathcal{M}$ on a connected variety X, there is the following canonical isomorphism between the étale and the group cohomology groups:*

$$H^1_{ét}(X, \mathcal{M}) \cong H^1(\pi_{ét}(X, \overline{x}), M),$$

where $\pi_{ét}(X, \overline{x})$ is the étale fundamental group of X with base point $\overline{x}$ and M is the finitely generated $\mathbb{Z}_\ell$-module with continuous action of $\pi_{ét}(X, \overline{x})$ associated with $\mathcal{M}$.

The next step is to determine the étale fundamental group of a semi-abelian variety for which we need the following generalisation of a theorem of Lang and Serre:

Proposition 5.1.2 ([Sti12, §13.1, Prop. 170])**.** *Let G be a geometrically connected algebraic group over k. Any connected étale cover $h : X \to G$ together with a lift $e' \in X(k)$ of $e \in G(k)$ canonically has the structure of an étale isogeny, i.e. X is an algebraic group and h is a surjective homomorphism with finite kernel.*

Now we are able to actually deduce the form of the étale fundamental group for a general (abelian) algebraic group:

Theorem 5.1.3. *For an (abelian) algebraic group G over k there is*

$$\pi_{ét}(\overline{G}, \overline{e}) = \varprojlim_n G[n](\overline{k}) := \mathrm{T}G = \prod_\ell \mathrm{T}_\ell G$$

with $\overline{G} = G \otimes_k \overline{k}$.

Proof. Define the fibre functor $F_{\overline{e}}(Y) := \mathrm{Hom}_{\overline{G}}(\overline{e}, Y)$ where Y is a finite étale X-scheme. We say Y is Galois if $\mathrm{Aut}_X(Y) \to F_{\overline{e}}(Y)$ is bijective and call it a Galois covering if $Y \to X$ is surjective. For additional information see [Mil80, §I.5].

The étale fundamental group satisfies the equality

$$\pi_{\text{ét}}(\overline{G}, \overline{e}) = \varprojlim_{H \text{ open}} \pi_{\text{ét}}(\overline{G}, \overline{e})/H$$

because it is a profinite group. H corresponds to a Galois covering $h : Y_H \to \overline{G}$ by [Sza09, Thm. 5.4.2b))] with $\mathrm{Aut}_X(Y_H) = \pi_{\text{ét}}(\overline{G}, \overline{e})/H$.

By the previous proposition we may deduce that h is an étale isogeny. Furthermore, $\mathrm{Aut}_X(Y_H) = \mathrm{Hom}_{\overline{G}}(\overline{e}, Y)$ may be identified with $\ker(h)(\overline{k})$. Hence, we have established

$$\pi_{\text{ét}}(\overline{G}, \overline{e}) = \varprojlim_{h \text{ étale isogeny}} \ker(h)(\overline{k})$$

because all étale isogenies are Galois coverings.

For $n > 0$ denote the multiplication with n on G by $[n]_G$. $[n]_{\overline{G}}$ are étale isogenies and we want to show that they form a cofinal system in the index category of the projective limit above, i.e. for every étale isogeny $h : X \to \overline{G}$ exists an $n \in \mathbb{Z} \setminus \{0\}$ and an $f : \overline{G} \to X$ such that $h \circ f = [n]_{\overline{G}}$. f is called the dual isogeny of h and is constructed as follows: $\ker h$ is finite and we have an $n \neq 0$ such that $n \cdot \ker h = 0$. Hence, $\ker h \subset \ker[n]_X$. $\overline{G} = X/\ker h$ has the universal property of a cokernel. Thus, $[n]_X$ factors as seen here:

$$\begin{tikzcd}
\ker h \arrow[r] & X \arrow[r, two heads, "h"] \arrow[d, "{[n]_X}"] & \overline{G} \arrow[dl, dotted, "f"] \\
& X. &
\end{tikzcd}$$

In order to show that $\alpha \coloneqq h \circ f$ actually is $[n]_{\overline{G}}$, consider

$$[n]_{\overline{G}} \circ h = h \circ [n]_X = h \circ f \circ h = \alpha \circ h$$

where the first equality holds due to h being a group homomorphism. The claim follows because h is an epimorphism.

We conclude with

$$\pi_{\text{ét}}(\overline{G}, \overline{e}) = \varprojlim_{\text{n}} G[n](\overline{k}).$$

□

Remark 5.1.4. The construction of the dual étale isogeny is done analogously to the abelian variety case, see [Sza09, Cor. 5.6.9]. Indeed, the whole proof is analogous to the abelian variety case.

Therefore, we can prove

Theorem 5.1.5. *For a semi-abelian variety G, the ℓ-adic étale realisation of $h^1(G)(1)$ is the Cartier dual of the (rational) Tate module of G.*

Proof. The above theorems imply:

$$\begin{aligned}
h^1(G)(1)_{\text{ét}} &= H^1_{\text{ét}}(\overline{G}, \mathbb{Q}_\ell)(1) = H^1_{\text{ét}}(\overline{G}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(1) \\
&\overset{5.1.1}{=} H^1(\pi_{\text{ét}}(\overline{G}, \overline{e}), \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(1) = \mathrm{Hom}(\pi_{\text{ét}}(\overline{G}, \overline{e}), \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(1) \\
&\overset{5.1.3}{=} \mathrm{Hom}(\mathrm{T}G, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(1) = \mathrm{Hom}(\mathrm{T}_\ell G, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(1) \\
&= \mathrm{V}_\ell G^D.
\end{aligned}$$

□

Remark 5.1.6. Because the 1-motive $[0 \to G]^D$ has exactly the same ℓ-adic realisation as $h^1(G)(1)$, their L-functions coincide. Thus, it is sufficient to understand the implications of the Tamagawa Number Conjecture in the case of the 1-motive $[0 \to G]^D$.

Remark 5.1.7. The semi-abelian variety G will be regarded as a 1-motive in the following sections. In particular, the realisations of G are then the realisations of $[0 \to G]$. As stated above, the étale realisation of G^D coincides with the étale realisation of the motive $h^1(G)(1)$. Whether a similar statement holds for the de Rham and the Betti realisation is not known to the author. While the final result remains unaffected by this uncertainty, there are intermediate results in the final two sections which are sensitive to this issue. Be mindful of this before transferring conclusions.

5.2 Preliminaries

Lemma 5.2.1. *Let G be a semi-abelian variety over k. Then the exact Kummer sequence*[1]

$$0 \longrightarrow G[\ell^n] \longrightarrow G \xrightarrow{[\ell^n]} G \longrightarrow 0$$

induces exact sequences functorial in k and G

$$0 \longrightarrow H^0(k, \mathrm{T}_\ell G) \longrightarrow \mathrm{T}_\ell G(k) \longrightarrow 0$$

and

$$0 \longrightarrow G(k)^{\wedge\ell} \longrightarrow H^1(k, \mathrm{T}_\ell G) \longrightarrow \mathrm{T}_\ell H^1(k, G) \longrightarrow 0.$$

Proof. Applying cohomology to the Kummer sequence, we get via the long exact cohomology sequence the isomorphism

$$0 \longrightarrow H^0(k, G[\ell^n]) \longrightarrow G(k)[\ell^n] \longrightarrow 0$$

and the short exact sequence

$$0 \longrightarrow G(k)/\ell^n G(k) \xrightarrow{\delta_{Kummer}} H^1(k, G[\ell^n]) \longrightarrow H^1(k, G)[\ell^n] \longrightarrow 0.$$

[1] We wrote G instead of $G(\overline{k})$ to keep the notation simple. This consequently results in us understanding exactness as in the category of $\mathbb{Z}[\mathrm{Gal}_k]$-modules.

The projective limit of these sequences yields the desired result because the transition maps of the projective system $(G(k)/\ell^n G(k))_n$ are by construction just the projections. Hence, all transition maps are surjective and the system has the Mittag-Leffler property. [Neu99, §II.7] □

Corollary 5.2.2. *For a torus T over k, there is an isomorphism $T(k)^{\wedge\ell} \cong H^1(k, \mathrm{T}_\ell T)$ induced by the Kummer boundary map.*

Proof. $H^1(k,T)$ is annihilated by multiplication with an integer according to remark 1.3.36. Hence, $\mathrm{T}_\ell H^1(k,G)$ vanishes as the transition maps are multiplication with ℓ and the above lemma implies the desired result. □

Corollary 5.2.3. *For a semi-abelian variety over a field k, which is either a number field or a p-adic local field, the group $H^0(k, \mathrm{T}_\ell G)$ vanishes.*

Proof. By the lemma we have $H^0(k, \mathrm{T}_\ell G) = \mathrm{T}_\ell G(k)$ and $G(k)$ has only finite torsion as by the structure theory of semi-abelian varieties, see proposition 1.3.23 and theorem 1.3.39. □

Lemma 5.2.4. *Let X be a finitely generated free $\mathbb{Z}[\mathrm{Gal}_k]$-module regarded as a 1-motive. Then the Kummer sequence*

$$0 \longrightarrow X \xrightarrow{[\ell^n]} X \longrightarrow X/\ell^n X \longrightarrow 0$$

induces exact sequences functorial in k and X

$$0 \longrightarrow H^i(k,X)^{\wedge\ell} \longrightarrow H^i(k,\mathrm{T}_\ell X) \xrightarrow{\delta_{Kummer}} \mathrm{T}_\ell H^{i+1}(k,X) \longrightarrow 0$$

for $i \geq 0$.

Proof. Applying cohomology to the Kummer sequence, we get via the long exact cohomology sequence the short exact sequence

$$0 \longrightarrow H^i(k,X)/\ell^n H^i(k,X) \longrightarrow H^i(k,X/\ell^n X) \longrightarrow H^{i+1}(k,X)[\ell^n] \longrightarrow 0$$

for all $i \geq 0$. Again, the transition maps of $H^i(k,X)/\ell^n H^i(k,X)$ are surjective and the desired result follows by taking the projective limit. □

Remark 5.2.5. Recall the warning issued in remark 3.2.7. $\mathrm{T}_\ell X$ denotes the ℓ-adic completion of X because in this case we think of T_ℓ as the realisation functor of a 1-motive. In all other cases the group $\mathrm{T}_\ell H$ is the usual ℓ-adic Tate module of an abelian group H.

Corollary 5.2.6. *For a finitely generated free $\mathbb{Z}[\mathrm{Gal}_k]$-module X regarded as a 1-motive there is an isomorphism $H^0(k, \mathrm{T}_\ell X) \cong H^0(k,X)^{\wedge\ell}$.*

Proof. Lemma 1.2.10 implies that $H^1(k,X)$ is finite. The claim follows using the previous lemma. □

Lemma 5.2.7. *The aforementioned decompositions are natural in the sense that for a* 1*-motive* $M = [X \xrightarrow{u} G]$*, the associated weight filtration of the Tate modules*

$$0 \longrightarrow \mathrm{T}_\ell G \longrightarrow \mathrm{T}_\ell M \longrightarrow \mathrm{T}_\ell X \longrightarrow 0$$

induces via the long exact cohomology sequence a commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & \\
 & \mathrm{T}_\ell H^1(k,X) \arrow[u] \arrow[r] & \mathrm{T}_\ell H^1(k,G) \arrow[u] & \\
H^0(k,\mathrm{T}_\ell M) \arrow[r] & H^0(k,\mathrm{T}_\ell X) \arrow[u, "\delta_{Kummer}"] \arrow[r, "\delta"] & H^1(k,\mathrm{T}_\ell G) \arrow[u] \arrow[r] & H^1(k,\mathrm{T}_\ell M) \\
 & H^0(k,X)^{\wedge\ell} \arrow[u] \arrow[r, "H^0(u)^{\wedge\ell}"] & H^0(k,G)^{\wedge\ell} \arrow[u, "\delta_{Kummer}"'] & \\
 & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd}$$

Proof. The above has to be proved using the explicit description of the first boundary morphism stated in remark 1.2.5. Let $x = (x_n)_n \in H^0(k, \mathrm{T}_\ell X)$ with $x_n \in X/\ell^n X$. Due to surjectivity, x comes from an element $m = (x_n, g_n)_n \in \mathrm{T}_\ell M$ with

$$(x_n, g_n)_n \in \mathrm{T}_{\mathbb{Z}/\ell^n\mathbb{Z}} M = \frac{\{(x,g) \in X \times G(\overline{K}) \,|\, u(x) = \ell^n g\}}{\{(\ell^n x, u(x)) \,|\, x \in X\}}.$$

$\delta(x)$ is then the function which sends $\sigma \in \mathrm{Gal}_k$ to

$$\sigma m - m = (0, \sigma g_n - g_n)_n = (\sigma g_n - g_n)_n \in \mathrm{T}_\ell G.$$

Hence, the image of $\delta(x)$ in $\mathrm{T}_\ell H^1(k,G)$ is a coboundary and

$$H^0(k,X)^{\wedge\ell} \to H^0(k,\mathrm{T}_\ell X) \to H^1(k,\mathrm{T}_\ell G) \to \mathrm{T}_\ell H^1(k,G)$$

is the zero map. The desired factorisation follows as the map from $H^0(k,X)^{\wedge\ell}$ to $H^1(k,\mathrm{T}_\ell G)$ factors through the Kummer map.

Straightforward calculation yields the image of $(x_n)_n$ in $H^0(k,G)^{\wedge\ell}$ under the map $\delta_{Kummer} \circ H^0(u)^{\wedge\ell}$ which matches the formula above. Thus, the morphism between the ℓ-adic completions in the diagram is indeed $H^0(u)^{\wedge\ell}$ □

Remark 5.2.8. Lemmas 5.2.1 and 5.2.4 are specialisations of [Jos09, Prop. 2.2.9] which is a combination of both. Jossen's proposition heavily depends on the construction of T_ℓ which uses complexes of ℓ-adic sheaves. We refrained from presenting this theory due to its highly technical nature. However, Jossen's constructions following the philosophy already outlined by Deligne are essential in order to understand Tate modules, their decomposition and their compatibilities.

5.3 Determination of H^1_f

Here, we will explicitly derive formulas for the local, global and geometric H^1_f groups.

5.3.1 ℓ-adic representations over p-adic fields

In this subsection, all algebraic groups are defined over a number field K. Recall remark 1.3.26.

The case $\ell \neq p$

Lemma 5.3.1. *For a torus T the group $H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell T)$ vanishes.*

Proof. In [HK11, Lem. 2.3.3] the claim is proved for $K_\mathfrak{p} = \mathbb{Q}_p$ and generalises to our situation. □

Lemma 5.3.2. *For an abelian variety A the group $H^1(K_\mathfrak{p}, \mathrm{V}_\ell A)$ vanishes. Hence, $H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell A)$ vanishes, too.*

Proof. Using Mattuck-Tate, $A(K_\mathfrak{p}) \cong \mathcal{O}^d_{K_\mathfrak{p}} \oplus finite$ (1.3.22), it becomes apparent that the ℓ-adic completion of $A(K_\mathfrak{p})$ is finite as $\mathcal{O}_{K_\mathfrak{p}}$ is a finitely generated $\mathbb{Z}_p$-module. By local Tate duality (1.3.21) we get

$$H^1(K_\mathfrak{p}, A)[\ell^n] = \mathrm{Hom}(A^D(K_\mathfrak{p})/\ell^n A^D(K_\mathfrak{p}), \mathbb{Q}/\mathbb{Z}).$$

Again by Mattuck-Tate, this group is finite and stabilises for $n \gg 0$ as ℓ is invertible in $\mathbb{Z}_p$. Thus, $\mathrm{T}_\ell H^1(K_\mathfrak{p}, A)$ vanishes.

It follows that $H^1(K_\mathfrak{p}, \mathrm{T}_\ell A)$ must be finite by the Kummer sequence 5.2.1. □

Lemma 5.3.3. *For a semi-abelian variety G the group $H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G)$ vanishes.*

Proof. Applying $H^0(K_\mathfrak{p}, -)$ to the exact sequence seen in remark 3.2.6 we get an exact sequence

$$0 \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_\ell T) \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_\ell G) \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_\ell A).$$

Both the left and right hand side vanish using the previous results and the fact that $\dim_{\mathbb{Q}_\ell} H^0(K_\mathfrak{p}, V) = \dim_{\mathbb{Q}_\ell} H^1_f(K_\mathfrak{p}, V)$ as shown in lemma 1.2.24. The latter also implies the result. □

Lemma 5.3.4. *For the Cartier dual G^D (in the category of 1-motives) of the semi-abelian variety G, the group $H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G^D)$ is $H^1(K_\mathfrak{p}, \mathrm{V}_\ell G^D)$.*

Proof. By theorem 1.2.25 we know that $H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G)$ and $H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G^D)$ are orthogonal. □

The case $\ell = p$

Lemma 5.3.5. *For a torus T the group $\mathbf{D}^i_{dR}\mathrm{V}_pT$ vanishes for $i \geq 0$.*

Proof. By [FO08, p. 148], there is an injection

$$\mathrm{gr}^i\, \mathbf{D}_{dR}(V) = \mathbf{D}^i_{dR}(V)/\mathbf{D}^{i+1}_{dR}(V) \hookrightarrow H^0(K_{\mathfrak{p}}, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)$$

where $\mathbb{C}_p$ is the completion of the algebraic closure of $K_{\mathfrak{p}}$. Furthermore, we know $H^0(K_{\mathfrak{p}}, \mathbb{C}_p(i))$ vanishes for $i \neq 0$.

Let L be the splitting field of T, i.e. a finite extension of K such that $T_L \cong \mathbb{G}^r_{m,L}$. Consequently, we have the isomorphism $\mathrm{V}_pT \cong \mathbb{Q}_p(1)^r$ of $\mathbb{Q}_p[\mathrm{Gal}_L]$-modules. Let $L_{\mathfrak{p}}$ be the corresponding extension of $K_{\mathfrak{p}}$ with Galois group $H = \mathrm{Gal}(L_{\mathfrak{p}}/K_{\mathfrak{p}})$ and $\mathrm{V}_pT \cong \mathbb{Q}_p(1)^r$ as a $\mathbb{Q}_p[\mathrm{Gal}_{L_{\mathfrak{p}}}]$-module. Then

$$\begin{aligned} H^0(K_{\mathfrak{p}}, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} \mathrm{V}_pT) &= H^0(L_{\mathfrak{p}}, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} \mathrm{V}_pT)^H = H^0(L_{\mathfrak{p}}, \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1)^r)^H \\ &= \left(H^0(L_{\mathfrak{p}}, \mathbb{C}_p(i+1))^r\right)^H . \end{aligned}$$

The above-mentioned result on $H^0(K_{\mathfrak{p}}, \mathbb{C}_p(i))$ can still be applied to finite extensions of $K_{\mathfrak{p}}$ as they yield the same $\mathbb{C}_p$ (and the according theory applied to $L_{\mathfrak{p}}$ instead of $K_{\mathfrak{p}}$). We see that the term vanishes if $i \neq -1$, i.e. $\mathrm{gr}^i\, \mathbf{D}_{dR}(\mathrm{V}_pT) = 0$ for $i \neq -1$. If $\mathbf{D}^i_{dR}(\mathrm{V}_pT) \neq 0$ for $i \geq 0$, there must be a $j \geq i \geq 0$ such that $\mathrm{gr}^j\, \mathbf{D}_{dR}(\mathrm{V}_pT) \neq 0$. However, we just proved that this cannot be possible. □

Lemma 5.3.6. *Let T be a torus and $\nu_{\mathfrak{p}} : K^\times_{\mathfrak{p}} \to \mathbb{Z}$ the $\mathfrak{p}$-adic valuation. Then there is the exact sequence*

$$0 \longrightarrow (T^c_{\mathfrak{p}})^{\wedge p} \longrightarrow T(K_{\mathfrak{p}})^{\wedge p} \xrightarrow{(\nu_{\mathfrak{p}} \circ \chi_i)_i} \mathbb{Z}_p^{r_{K_{\mathfrak{p}}}} \longrightarrow 0$$

where $T^c_{\mathfrak{p}}$ is the maximal compact subtorus of T as defined in 1.5.1 and where $(\chi_i)_i$ constitute a basis of the character group $X^*_{K_{\mathfrak{p}}}(T)$ which is of rank $r_{K_{\mathfrak{p}}}$.

Proof. The proof is analogous to the proof of [HK11, Lem. 2.3.4]. The sequence

$$0 \longrightarrow T^c_{\mathfrak{p}} \longrightarrow T(K_{\mathfrak{p}}) \xrightarrow{(\nu_{\mathfrak{p}} \circ \chi_i)_i} \mathbb{Z}^{r_{K_{\mathfrak{p}}}} \longrightarrow 0$$

is exact on the left and in the centre due to the definition of $T^c_{\mathfrak{p}}$ as the kernel of the map. It is exact on the right due to the surjectivity shown in [Shy77a, Eq. 3]. The last term in the sequence is free and hence the p-adic completion of the sequence exact, see lemma 1.4.2. □

Proposition 5.3.7. *Let T be a torus and $\nu_{\mathfrak{p}} : K^\times_{\mathfrak{p}} \to \mathbb{Z}$ the $\mathfrak{p}$-adic valuation. Then*

$$H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pT) = \bigcap_{\chi \in X^*_{K_{\mathfrak{p}}}(T)} \ker\left(H^1(K_{\mathfrak{p}}, \mathrm{V}_pT) \xrightarrow{\nu_{\mathfrak{p}} \circ \chi} \mathbb{Q}_p\right) = (T^c_{\mathfrak{p}})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof. We generalise the proof of [HK11, Lem. 2.3.5] to our situation.

The equality on the right hand side follows directly from the previous lemma by tensoring the sequence with $\mathbb{Q}_p$.

The inclusion of H^1_f into the middle term Huber and Kings conclude as follows: first, observe that by lemma 1.2.22, which reads as

$$H^1_f(K_\mathfrak{p}, \mathrm{V}_p\mathbb{G}_m) = (\mathcal{O}^\times_{K_\mathfrak{p}})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset (K^\times_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1(K_\mathfrak{p}, \mathrm{V}_p\mathbb{G}_m),$$

the prototype

$$H^1_f(K_\mathfrak{p}, \mathrm{V}_p\mathbb{G}_m) = \ker\left((K^\times_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\nu_\mathfrak{p}} \mathbb{Q}_p\right)$$

of the desired result is true.

Every $\chi \in X^*_{K_\mathfrak{p}}(T)$ defines a morphism

$$H^1(K_\mathfrak{p}, \mathrm{V}_pT) \xrightarrow{\chi} H^1(K_\mathfrak{p}, \mathrm{V}_p\mathbb{G}_m),$$

in particular a morphism of subspaces

$$H^1_f(K_\mathfrak{p}, \mathrm{V}_pT) \xrightarrow{\chi} H^1_f(K_\mathfrak{p}, \mathrm{V}_p\mathbb{G}_m),$$

because H^1_f is defined in a functorial manner. Combining both results yields the inclusion.

As for the other direction, they argue via dimension counting: we have

$$\begin{aligned}\dim_{\mathbb{Q}_p}(T^c_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &= \dim_{\mathbb{Q}_p} H^1(K_\mathfrak{p}, \mathrm{V}_pT) - r_{K_\mathfrak{p}} \\ &= [K_\mathfrak{p} : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} \mathrm{V}_pT + 0 + \dim_{\mathbb{Q}_p} H^2(K_\mathfrak{p}, \mathrm{V}_pT) - r_{K_\mathfrak{p}}\end{aligned}$$

using the exact sequence of the previous lemma tensored with $\mathbb{Q}_p$ and the Euler characteristic formula 1.2.16. The local duality 1.2.25 gives us

$$H^2(K_\mathfrak{p}, \mathrm{V}_pT) \cong H^0(K_\mathfrak{p}, \mathrm{V}_pT^D)^* = \mathrm{Hom}_{\mathbb{Z}_p}\left(X^*_{K_\mathfrak{p}}(T), \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

and thus we see that the last two terms in the above formula cancel out.

An identity for $H^1_f(K_\mathfrak{p}, \mathrm{V}_pT)$ is given by corollary 1.2.27:

$$\begin{aligned}\dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pT) &= \dim_{\mathbb{Q}_p} \mathbf{D}_{dR}\mathrm{V}_pT - \dim_{\mathbb{Q}_p} \mathbf{D}^0_{dR}\mathrm{V}_pT + \dim_{\mathbb{Q}_p} H^0(K_\mathfrak{p}, \mathrm{V}_pT) \\ &= [K_\mathfrak{p} : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} \mathrm{V}_pT - 0 + 0.\end{aligned}$$

This holds because V_pT is a de Rham representation and $\mathbf{D}^0_{dR}\mathrm{V}_pT = 0$ due to lemma 5.3.5. We conclude that $H^1_f(K_\mathfrak{p}, \mathrm{V}_pT)$ is a subvector space of $(T^c_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and their dimensions coincide, hence both are equal and the claim is proved. □

Proposition 5.3.8 ([BK90, Ex. 3.11]). *For an abelian variety A we have*

$$H^1_f(K_\mathfrak{p}, \mathrm{V}_pA) = A(K_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \overset{\delta_{Kummer}}{\hookrightarrow} H^1(K_\mathfrak{p}, \mathrm{V}_pA)$$

where δ_{Kummer} is the boundary morphism defined in lemma 5.2.1.

Lemma 5.3.9. *Let*

$$0 \longrightarrow V_1 \xrightarrow{d_1} V_2 \longrightarrow \dots \longrightarrow V_n$$

be an exact sequence of k-vector spaces. Let $U_1 \to \dots \to U_n$ be a subsequence which satisfies

$$\sum_{i=1}^{n} (-1)^i \dim U_i = 0.$$

Then the sequence

$$0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow \dots \longrightarrow U_n \longrightarrow 0$$

is exact.

Proof. Proof by induction. For $i = 1$ and $i = 2$ the claim holds because we find $U_1 = 0$ respectively $U_1 \cong U_2$.

Let $U_1 \to U_2 \to \dots \to U_{n+1}$ be the subsequence, then $U_2/U_1 \to \dots \to U_{n+1}$ is a subsequence of the exact sequence $0 \to V_2/V_1 \to \dots \to V_{n+1}$ of length n. Furthermore, the alternating sum of dimensions is given by

$$-\dim(U_2/U_1) + \sum_{i=2}^{n} (-1)^i \dim U_{i+1} = -\sum_{i=1}^{n+1} (-1)^i \dim U_i = 0$$

because $\dim(U_2/U_1) = \dim U_2 - \dim U_1$ by the injectivity of d_1. The subcomplex is exact by induction hypothesis. In particular, $0 \to U_2/U_1 \to U_3$ is exact which again accounts for the exactness of $0 \to U_1 \to U_2 \to U_3$. Therefore, the whole sequence must be exact. □

Proposition 5.3.10. *Let G be a semi-abelian variety. Then it holds that*

$$0 \longrightarrow H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pT) \longrightarrow H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pG) \longrightarrow H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pA) \longrightarrow 0$$

is an exact sequence. In particular, there is $H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pG) \subset G(K_{\mathfrak{p}})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. The dimension of $H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pG)$ can be calculated by means of corollary 1.2.27 as follows:

$$\dim_{\mathbb{Q}_p} H^1_f(K_{\mathfrak{p}}, \mathrm{V}_pG) = \dim_{\mathbb{Q}_p} \mathbf{D}_{dR}\mathrm{V}_pG - \dim_{\mathbb{Q}_p} \mathbf{D}^0_{dR}\mathrm{V}_pG + \dim_{\mathbb{Q}_p} H^0(K_{\mathfrak{p}}, \mathrm{V}_pG).$$

Besides, we know that

$$0 \longrightarrow \mathrm{V}_pT \longrightarrow \mathrm{V}_pG \longrightarrow \mathrm{V}_pA \longrightarrow 0 \tag{5.1}$$

is an exact sequence of de Rham representations. By theorem 1.2.18, we see that applying $\mathbf{D}_{dR}$ to such a sequence results in an exact one, i.e.

$$\dim_{\mathbb{Q}_p} \mathbf{D}_{dR}\mathrm{V}_pG = \dim_{\mathbb{Q}_p} \mathbf{D}_{dR}\mathrm{V}_pT + \dim_{\mathbb{Q}_p} \mathbf{D}_{dR}\mathrm{V}_pA.$$

Furthermore, $\mathbf{D}_{dR}$ is exact as a functor whose target is filtered vector spaces and we can thus conclude

$$\dim_{\mathbb{Q}_p} \mathbf{D}^0_{dR}\mathrm{V}_pG = \dim_{\mathbb{Q}_p} \mathbf{D}^0_{dR}\mathrm{V}_pT + \dim_{\mathbb{Q}_p} \mathbf{D}^0_{dR}\mathrm{V}_pA.$$

Together with $H^0(K_\mathfrak{p}, \mathrm{V}_pG) = 0$ which holds by corollary 5.2.3, we can deduce that

$$\dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pG) = \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pT) + \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pA). \tag{5.2}$$

Applying the long exact cohomology sequence to the exact sequence (5.1) yields

$$0 \longrightarrow H^1(K_\mathfrak{p}, \mathrm{V}_pT) \longrightarrow H^1(K_\mathfrak{p}, \mathrm{V}_pG) \longrightarrow H^1(K_\mathfrak{p}, \mathrm{V}_pA).$$

Now, lemma 5.3.9 establishes the desired result for the subsequence of $H^1_f(K_\mathfrak{p}, -)$ groups using the additivity of the dimensions as seen in (5.2).

In order to show $H^1_f(K_\mathfrak{p}, \mathrm{V}_pG) \subset G(K_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, it is sufficient to realise that the image of $H^1_f(K_\mathfrak{p}, \mathrm{V}_pG)$ in $\mathrm{V}_pH^1(K_\mathfrak{p}, G)$ vanishes, see lemma 5.2.1. $\mathrm{V}_pH^1(K_\mathfrak{p}, G) \to \mathrm{V}_pH^1(K_\mathfrak{p}, A)$ is injective because $H^1(K_\mathfrak{p}, T)$ is finite and T_ℓ is left exact. By proposition 5.3.8, $H^1_f(K_\mathfrak{p}, \mathrm{V}_pA)$ vanishes in $\mathrm{V}_pH^1(K_\mathfrak{p}, A)$ and the same must thus hold for $H^1_f(K_\mathfrak{p}, \mathrm{V}_pG)$. □

Proposition 5.3.11. *Let G be a semi-abelian variety. Then there is an exact sequence*

$$\begin{array}{l} 0 \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_pG^D) \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_pT^D) \xrightarrow{\delta} \\ H^1_f(K_\mathfrak{p}, \mathrm{V}_pA^D) \longrightarrow H^1_f(K_\mathfrak{p}, \mathrm{V}_pG^D) \longrightarrow H^1_f(K_\mathfrak{p}, \mathrm{V}_pT^D) \longrightarrow 0. \end{array}$$

Proof. Applying the long exact cohomology sequence to the exact sequence $0 \to \mathrm{V}_pA^D \to \mathrm{V}_pG^D \to \mathrm{V}_pT^D \to 0$ yields

$$\begin{array}{l} 0 \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_pG^D) \longrightarrow H^0(K_\mathfrak{p}, \mathrm{V}_pT^D) \xrightarrow{\delta} \\ H^1(K_\mathfrak{p}, \mathrm{V}_pA^D) \longrightarrow H^1(K_\mathfrak{p}, \mathrm{V}_pG^D) \longrightarrow H^1(K_\mathfrak{p}, \mathrm{V}_pT^D) \longrightarrow 0 \end{array} \tag{5.3}$$

due to $H^0(K_\mathfrak{p}, \mathrm{V}_pG) = 0$ by corollary 5.2.3 together with local duality 1.2.25.

Then, the claimed exact sequence being a sequence is almost obvious with the exception of the map δ. By proposition 5.3.8, we now simply have to show that δ factors through δ_{Kummer}. This however follows from $H^0(K_\mathfrak{p}, \mathrm{V}_pX) \cong H^0(K_\mathfrak{p}, X)^{\wedge p}$, which holds by corollary 5.2.6, and by using the compatibility shown in lemma 5.2.7.

In a second step, we need to compute the dimensions of the involved vector spaces. By orthogonality in the local duality, cf. 1.2.25, there are the following identities:

$$\begin{aligned} \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pA^D) &= \dim_{\mathbb{Q}_p} H^1(K_\mathfrak{p}, \mathrm{V}_pA^D) - \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pA) \\ \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pG^D) &= \dim_{\mathbb{Q}_p} H^1(K_\mathfrak{p}, \mathrm{V}_pG^D) - \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pG) \\ \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pT^D) &= \dim_{\mathbb{Q}_p} H^1(K_\mathfrak{p}, \mathrm{V}_pT^D) - \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_pT). \end{aligned}$$

Hence, the alternating sum of the dimensions of the vector spaces of the desired sequence is easily calculated as zero using that the alternating sum of the dimensions of the vector spaces in sequence (5.3) is zero together with the orthogonality relations and the equality

$$\dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_p G) = \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_p T) + \dim_{\mathbb{Q}_p} H^1_f(K_\mathfrak{p}, \mathrm{V}_p A)$$

from the previous proposition. Thus, applying lemma 5.3.9 to the claimed exact sequence yields the proof. $\square$

Remark 5.3.12. Knowing that

$$0 \longrightarrow \mathbf{D}_{crys}\mathrm{V}_p T \longrightarrow \mathbf{D}_{crys}\mathrm{V}_p G \longrightarrow \mathbf{D}_{crys}\mathrm{V}_p A \longrightarrow 0$$

and its dual are exact sequences, one can obtain more conceptual proofs of the previous propositions using proposition 1.2.26 and the ker-coker exact sequence.

5.3.2 ℓ-adic representations over number fields

We have already defined $H^1_f(K, V)$ as

$$\ker\left(H^1(K,V) \to \prod_{\mathfrak{p}} H^1(K_\mathfrak{p}, V)/H^1_f(K_\mathfrak{p}, V)\right)$$

in 1.2.28.

Lemma 5.3.13. *From the Tate-Shafarevich conjecture 0.0.1 it follows that*

$$\text{Ш}^1(K,G) \coloneqq \ker\left(H^1(K,G) \longrightarrow \prod_{\textit{all } \mathfrak{p}} H^1(K_\mathfrak{p}, G)\right)$$

is finite. On the right hand side, the local Galois cohomology groups are implicitly calculated using the invariants of G under the Galois group $\mathrm{Gal}(\overline{K}/\overline{K_\mathfrak{p}})$.

Proof. This is a specalisation of [Jos09, Prop. 4.3.11]. $\square$

Lemma 5.3.14. *We have*

$$(\prod_i \mathbb{Z}_\ell^{r_i}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \subset \prod_i (\mathbb{Z}_\ell^{r_i} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

Proof. Tensoring the injection

$$\prod_i \mathbb{Z}_\ell^{r_i} \longrightarrow \prod_i \mathbb{Q}_\ell^{r_i}$$

with $\mathbb{Q}_\ell$ proves the claim. $\square$

Lemma 5.3.15. *For a semi-abelian variety G there is the equality*

$$H^1_f(K, \mathrm{V}_\ell G) = \ker\left(G(K)^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to \prod_{\mathfrak{p}} \frac{G(K_\mathfrak{p})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell}{H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G)}\right).$$

The right hand side is here regarded as a subset of $H^1(K, \mathrm{V}_\ell G)$ via the boundary morphism δ_{Kummer} defined in lemma 5.2.1.

Proof. We know the inclusion

$$H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G) \subset G(K_\mathfrak{p})^{\wedge p} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \overset{\delta_{Kummer}}{\hookrightarrow} H^1(K_\mathfrak{p}, \mathrm{V}_\ell G)$$

exists by proposition 5.3.10. Hence, there is the diagram

$$\begin{tikzcd}
0 \arrow[r] & G(K)^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \arrow[r] \arrow[d] & H^1(K, \mathrm{V}_\ell G) \arrow[r] \arrow[d] & \mathrm{V}_\ell H^1(K, G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{\mathfrak{p}} \frac{G(K_\mathfrak{p})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell}{H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G)} \arrow[r] & \prod_{\mathfrak{p}} \frac{H^1(K_\mathfrak{p}, \mathrm{V}_\ell G)}{H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G)} \arrow[r] & \prod_{\mathfrak{p}} \mathrm{V}_\ell H^1(K_\mathfrak{p}, G) \arrow[r] & 0.
\end{tikzcd}$$

Furthermore,

$$\prod_{\mathfrak{p}\in S_f} \mathrm{V}_\ell H^1(K_\mathfrak{p}, G) = \prod_{\text{all } \mathfrak{p}} \mathrm{V}_\ell H^1(K_\mathfrak{p}, G) \supset \mathrm{V}_\ell \prod_{\text{all } \mathfrak{p}} H^1(K_\mathfrak{p}, G)$$

holds. The equality is due to $\mathrm{T}_\ell H^1(K_\nu, G) = 0$ for $\nu \in S_\infty$ because Gal_{K_ν} is finite. The inclusion follows from the fact that $\mathrm{T}_\ell = \varprojlim_n (-)[\ell^n]$ commutes with infinite projective limits together with lemma 5.3.14. Note that the map on the right hand side of the diagram factors through $\mathrm{V}_\ell \prod_{\text{all } \mathfrak{p}} H^1(K_\mathfrak{p}, G)$ by construction. Hence, the left exactness of T_ℓ gives that the kernel of the right hand side map is

$$\mathrm{V}_\ell \ker\left(H^1(K, G) \to \prod_{\text{all } \mathfrak{p}} H^1(K_\mathfrak{p}, G)\right) = \mathrm{V}_\ell Ш^1(K, G) = 0,$$

additionally using lemma 5.3.13. The desired result then follows from the ker-coker sequence noting that the kernel of the middle vertical arrow is $H^1_f(K, G)$ by definition. □

Proposition 5.3.16. *Let T be a torus. Then*

$$H^1_f(K, \mathrm{V}_\ell T) = U_T^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \subset T(K)^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \overset{\delta_{Kummer}}{\hookrightarrow} H^1(K, \mathrm{V}_\ell T).$$

Proof. Because $T^c_\mathfrak{p}$ possesses a subgroup of the form $\mathbb{Z}_p^d$ with finite index according to lemma 1.5.2, the group $(T^c_\mathfrak{p})^{\wedge\ell} = (T^c_\mathfrak{p}/\mathbb{Z}_p^d)^{\wedge\ell}$ is finite for $\mathfrak{p} \notin S_\ell$. Therefore,

$$(T(K_\mathfrak{p})/T^c_\mathfrak{p})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = T(K_\mathfrak{p})^{\wedge\ell}/(T^c_\mathfrak{p})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong T(K_\mathfrak{p})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

using that $T(K_\mathfrak{p})/T^c_\mathfrak{p}$ is a free $\mathbb{Z}$-module as seen in the proof of lemma 5.3.6.

By definition of U_T in 1.5.5, there is an exact sequence

$$0 \longrightarrow U_T \longrightarrow T(K) \longrightarrow \prod_{\mathfrak{p}} T(K_{\mathfrak{p}})/T^c_{\mathfrak{p}}.$$

Note that the right hand side is componentwisely free and that by lemma 1.4.2 the completion of the sequence must therefore be exact. Furthermore, tensoring this sequence with $\mathbb{Q}_\ell$ and employing lemma 5.3.14 yields the exact sequence

$$0 \to U_T^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to T(K)^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to \prod_{\mathfrak{p}\notin S_\ell} T(K_{\mathfrak{p}})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \times \prod_{\mathfrak{p}\in S_\ell} T(K_{\mathfrak{p}})^{\wedge\ell}/(T^c_{\mathfrak{p}})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

This implies the desired result using lemmas 5.3.15 and 5.3.1 together with proposition 5.3.7. ☐

Proposition 5.3.17. *For an abelian variety A there is the identity*

$$H^1_f(K, \mathrm{V}_\ell A) = A(K)^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \overset{\delta_{Kummer}}{\hookrightarrow} H^1(K, \mathrm{V}_\ell A).$$

Proof. From proposition 5.3.8 and lemma 5.3.2, we know $\prod_{\mathfrak{p}} \frac{G(K_{\mathfrak{p}})^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell}{H^1_f(K_{\mathfrak{p}}, \mathrm{V}_\ell G)} = 0$ as every term individually equals zero. We may thus deduce the identity by means of lemma 5.3.15. ☐

Proposition 5.3.18. *Let X be a finitely generated free $\mathbb{Z}$-module regarded as a 1-motive. Then $H^1_f(K, \mathrm{V}_\ell X) = 0$.*

Proof. Denote by T the torus X^D. By employing proposition 1.2.29, we get

$$\begin{aligned}
\dim H^1_f(K, \mathrm{V}_\ell X) &= \dim H^1_f(K, T) \\
&\qquad + \dim H^0(K, \mathrm{V}_\ell X) - \underbrace{\dim H^0(K, T)}_{\text{vanishes using cor. 5.2.3}} \\
&\qquad + \underbrace{\sum_{\mathfrak{p}\in S_\ell} \Big(\dim \mathbf{D}_{dR} \mathrm{V}_\ell X - \dim \mathbf{D}^0_{dR} \mathrm{V}_\ell X\Big)}_{\text{vanishes using lem. 5.3.5 and prop. 1.2.19}} \\
&\qquad - \sum_{\nu\in S_\infty} \dim H^0(K_\nu, \mathrm{V}_\ell X) \\
&\overset{5.2.6}{=} \mathrm{rk}\, U_T + r_K - r_\infty \\
&= 0
\end{aligned}$$

where the last equality holds due to theorem 1.5.8. ☐

Remark 5.3.19. It would be interesting to see a direct proof of the previous proposition.

Proposition 5.3.20. *Let G be a semi-abelian variety. Then there is an exact sequence*

$$0 \longrightarrow H^0(K, \mathrm{V}_\ell G^D) \longrightarrow H^0(K, \mathrm{V}_\ell T^D) \xrightarrow{\delta} H^1_f(K, \mathrm{V}_\ell A^D) \longrightarrow H^1_f(K, \mathrm{V}_\ell G^D) \longrightarrow 0.$$

Proof. The long exact cohomology sequence yields

$$\begin{tikzcd} 0 \arrow[r] & H^0(K, \mathrm{V}_\ell G^D) \arrow[r] & H^0(K, \mathrm{V}_\ell T^D) \arrow[r, "\delta"] & {} \\ H^1(K, \mathrm{V}_\ell A^D) \arrow[r] & H^1(K, \mathrm{V}_\ell G^D) \arrow[r] & H^1(K, \mathrm{V}_\ell T^D) \arrow[r] & 0. \end{tikzcd}$$

Furthermore, there is the $\mathfrak{p}$-adic analogue of the above sequence with the subsequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(K_\mathfrak{p}, \mathrm{V}_\ell G^D) \arrow[r] & H^0(K_\mathfrak{p}, \mathrm{V}_\ell T^D) \arrow[r, "\delta"] & {} \\ H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell A^D) \arrow[r] & H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G^D) \arrow[r] & H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell T^D) \arrow[r] & 0 \end{tikzcd}$$

which is exact for all $\mathfrak{p}$ due to proposition 5.3.11 and lemma 5.3.4.

Hence, the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \frac{H^1(K, \mathrm{V}_\ell A^D)}{H^0(K, \mathrm{V}_\ell T^D)} \arrow[r] \arrow[d] & H^1(K, \mathrm{V}_\ell G^D) \arrow[r] \arrow[d] & H^1(K, \mathrm{V}_\ell T^D) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \prod_\mathfrak{p} \frac{H^1(K_\mathfrak{p}, \mathrm{V}_\ell A^D)}{H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell A^D)} \arrow[r] & \prod_\mathfrak{p} \frac{H^1(K_\mathfrak{p}, \mathrm{V}_\ell G^D)}{H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell G^D)} \arrow[r] & \prod_\mathfrak{p} \frac{H^1(K_\mathfrak{p}, \mathrm{V}_\ell T^D)}{H^1_f(K_\mathfrak{p}, \mathrm{V}_\ell T^D)} \arrow[r] & 0. \end{tikzcd}$$

It is well-defined because the map δ factors through $H^1_f(K, \mathrm{V}_\ell A^D)$ due to proposition 5.3.17, corollary 5.2.6 and lemma 5.2.7, and because the same holds for the local case due to proposition 5.3.8. The sequences are exact due to the above-mentioned results.

The ker-coker sequence then implies the exactness of

$$\begin{tikzcd} 0 \arrow[r] & \frac{H^1_f(K, \mathrm{V}_\ell A^D)}{H^0(K, \mathrm{V}_\ell T^D)} \arrow[r] & H^1_f(K, \mathrm{V}_\ell G^D) \arrow[r] & H^1_f(K, \mathrm{V}_\ell T^D) \end{tikzcd}$$

as the kernels of the sequence are by definition the $H^1_f(K, -)$ groups. In this situation, the previous proposition yields the desired result. □

Lemma 5.3.21. *We have an equality*

$$\dim H^1_f(K, \mathrm{V}_\ell G) = \dim H^1_f(K, \mathrm{V}_\ell T) + \dim H^1_f(K, \mathrm{V}_\ell A).$$

Proof. Remember that $H^0(K, \mathrm{V}_\ell G) = 0$ by theorem 5.2.3. By employing proposition 1.2.29, we then get

$$\begin{aligned} &\dim H^1_f(K, \mathrm{V}_\ell T) + \dim H^1_f(K, \mathrm{V}_\ell A) \\ &\quad = \underbrace{\dim H^1_f(K, \mathrm{V}_\ell T^D) + \dim H^1_f(K, \mathrm{V}_\ell A^D)}_{= \dim H^1_f(K, \mathrm{V}_\ell G^D) + \dim H^0(K, \mathrm{V}_\ell T^D) - \dim H^0(K, \mathrm{V}_\ell G^D) \text{ due to prop. 5.3.20}} \\ &\qquad - \dim H^0(K, \mathrm{V}_\ell T^D) \\ &\qquad + \sum_{\mathfrak{p} \in S_\ell} \underbrace{\left(\dim \mathbf{D}_{dR} \mathrm{V}_\ell T^D - \dim \mathbf{D}^0_{dR} \mathrm{V}_\ell T^D + \dim \mathbf{D}_{dR} \mathrm{V}_\ell A^D - \dim \mathbf{D}^0_{dR} \mathrm{V}_\ell A^D\right)}_{= \dim \mathbf{D}_{dR} \mathrm{V}_\ell G^D - \dim \mathbf{D}^0_{dR} \mathrm{V}_\ell G^D \text{ due to thm. 1.2.18}} \\ &\qquad - \sum_{\nu \in S_\infty} \left(\dim H^0(K_\nu, \mathrm{V}_\ell T^D) + \dim H^0(K_\nu, \mathrm{V}_\ell A^D)\right) \end{aligned}$$

$$
\begin{aligned}
&= \dim H^1_f(K, \mathrm{V}_\ell G^D) \\
&\quad - \dim H^0(K, \mathrm{V}_\ell G^D) \\
&\quad + \sum_{\mathfrak{p}\in S_\ell} \left(\dim \mathbf{D}_{dR}\mathrm{V}_\ell G - \dim \mathbf{D}^0_{dR}\mathrm{V}_\ell G^D\right) \\
&\quad - \sum_{\nu\in S_\infty} \dim H^0(K_\nu, \mathrm{V}_\ell G^D) \\
&= \dim H^1_f(K, \mathrm{V}_\ell G)
\end{aligned}
$$

because $H^0(K_\nu, -)$ is exact for archimedean valuations.[2] □

Proposition 5.3.22. *Let G be a semi-abelian variety. Then there is an exact sequence*

$$0 \longrightarrow H^1_f(K, \mathrm{V}_\ell T) \longrightarrow H^1_f(K, \mathrm{V}_\ell G) \longrightarrow H^1_f(K, \mathrm{V}_\ell A) \longrightarrow 0.$$

Proof. The claimed exact sequence is indeed a sequence due to functoriality. We are left to calculate the alternating sum of the dimensions of the involved vector spaces which is zero by the previous lemma. Hence, the result follows from lemma 5.3.9. □

5.3.3 Geometric representations

Lemma 5.3.23. *Let G be a semi-abelian variety. The realisation map $H^1_{\mathcal{M}}(K, G) \to H^1(K, \mathrm{V}_\ell G)$ factors through the Kummer map δ_{Kummer}.*

Proof. Let $g \otimes q$ be an element of $G(K) \otimes \mathbb{Q}$. Then this corresponds to an element in $\mathrm{Ext}^1_{\mathrm{MM}}(\mathbb{1}, \mathcal{H}G)$ due to

$$G(K) \otimes \mathbb{Q} = H^1_{\mathcal{M}}(K, G) = \mathrm{Ext}^1_{\mathrm{MM}}(\mathbb{1}, \mathcal{H}G)$$

as seen in theorem 3.4.4. Let this element be the extension

$$0 \longrightarrow G \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 0$$

with $E = [\mathbb{Z} \xrightarrow{u} G]$ and $u(1) = g$. Then the associated element in $\mathrm{Ext}^1_{\mathbb{Q}_p[\mathrm{Gal}_K]}$ is

$$0 \longrightarrow \mathrm{V}_\ell G \longrightarrow \mathrm{V}_\ell E \longrightarrow \mathrm{V}_\ell \mathbb{Z} \longrightarrow 0.$$

In proposition 1.2.7 we described how to derive an element of $H^1(K, \mathrm{V}_\ell G)$ from the extension: choose a lift $e \in E$ of $1 \in \mathbb{Z}$ which gives rise to a cocycle $\mathrm{Gal}_K \to \mathrm{V}_\ell G$ defined by $\sigma \mapsto \sigma(e) - e$. The definition of the Tate module given in 3.2.4 yields $e = (1, g_n)_n \otimes q'$ such that $\ell^n \cdot g_n = u(1) = g$. Thus, $\sigma(e) - e$ is just $(\sigma g_n - g_n)_n \otimes q'$.

Furthermore, the map

$$G(K) \otimes \mathbb{Q} \to G(K)^{\wedge\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\delta} H^1(K, \mathrm{V}_\ell G)$$

sends $g \otimes q''$ to $(g)_n \otimes q''$ and then to $(\sigma g_n - g_n)_n \otimes q''$ where g_n is any element with $\ell^n \cdot g_n = g$. Consequently, both maps coincide. □

[2] This is trivial for $K_\nu = \mathbb{C}$. For $K_\nu = \mathbb{R}$, the non-trivial element in Gal_{K_ν} is an involution. Taking fixed points with respect to an involution is always exact if 2 is invertible.

Proposition 5.3.24. *Let T be a torus. Then $H^1_{\mathcal{M},f}(K,T) = U_T \otimes \mathbb{Q}$.*

Proof. The exact sequence

$$0 \longrightarrow U_T \longrightarrow T \longrightarrow T/U_T \longrightarrow 0$$

induces an exact sequence after ℓ-adic completion because the last group is almost free due to lemma 1.3.31. Lemma 1.4.9 states that the kernel of the right hand map in the diagram

$$\begin{tikzcd} 0 \arrow[r] & U_T \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & T/U_T \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & U_T^{\wedge\ell} \arrow[r] & T^{\wedge\ell} \arrow[r] & (T/U_T)^{\wedge\ell} \arrow[r] & 0 \end{tikzcd}$$

is finite.

Using the definition of $H^1_{\mathcal{M},f}(K,M)$, 4.1.1, we see

$$\begin{aligned} H^1_{\mathcal{M},f}(K,T) &= \ker\left(H^1_{\mathcal{M}}(K,T) \to \prod_\ell \frac{H^1(K,\mathrm{V}_\ell T)}{H^1_f(K,\mathrm{V}_\ell T)}\right) \\ &= \ker\left(T(K)\otimes\mathbb{Q} \to \prod_\ell \frac{T(K)^{\wedge\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell}{U_T^{\wedge\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell} \overset{\delta_{Kummer}}{\hookrightarrow} \prod_\ell \frac{H^1(K,\mathrm{V}_\ell T)}{U_T^{\wedge\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell}\right) \\ &= \ker\left(T(K)\otimes\mathbb{Q} \to \prod_\ell (T(K)/U_T)^{\wedge\ell}\otimes\mathbb{Q}\right) \\ &= U_T \otimes \mathbb{Q} \end{aligned}$$

employing $H^1_f(K,\mathrm{V}_\ell T) = U_T^{\wedge\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ taken from proposition 5.3.16. The last equality holds because the kernel of

$$T(K)\otimes\mathbb{Q} \longrightarrow (T(K)/U_T)^{\wedge\ell}\otimes\mathbb{Q}$$

is $U_T\otimes\mathbb{Q}$ for all ℓ individually as seen above. □

Proposition 5.3.25. *Let A be an abelian variety. Then*

$$H^1_{\mathcal{M},f}(K,A) = H^1_{\mathcal{M}}(K,A) = A(K)\otimes\mathbb{Q}.$$

Proof. We get

$$\begin{aligned} H^1_{\mathcal{M},f}(K,A) &= \ker\left(H^1_{\mathcal{M}}(K,A) \to \prod_\ell \frac{H^1(K,\mathrm{V}_\ell A)}{H^1_f(K,\mathrm{V}_\ell A)}\right) \\ &= \ker\left(H^1_{\mathcal{M}}(K,M) \to \prod_\ell \frac{A(K)^{\wedge\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell}{H^1_f(K,\mathrm{V}_\ell A)} \overset{\delta_{Kummer}}{\hookrightarrow} \prod_\ell \frac{H^1(K,\mathrm{V}_\ell A)}{H^1_f(K,\mathrm{V}_\ell A)}\right) \\ &= \ker\left(H^1_{\mathcal{M}}(K,A) \to 0\right) \\ &= H^1_{\mathcal{M}}(K,A) \end{aligned}$$

using $H^1_f(K,\mathrm{V}_\ell A) = A(K)^{\wedge\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ as established in proposition 5.3.17. □

Proposition 5.3.26. *Let G be a semi-abelian variety. Then there is an exact sequence*

$$0 \longrightarrow H^1_{\mathcal{M},f}(K,T) \longrightarrow H^1_{\mathcal{M},f}(K,G) \longrightarrow H^1_{\mathcal{M},f}(K,A) \longrightarrow 0.$$

Proof. Due to functoriality, the above is certainly a sequence. Furthermore, we know

$$\dim_{\mathbb{Q}} H^1_{\mathcal{M},f}(K,G) = \dim_{\mathbb{Q}_\ell} H^1_f(K, \mathrm{V}_\ell G)$$

as conjecture 4.3.4 has been proven correct for the case of 1-motives, see [Fon92, Prop. in §8.2]. Hence, the sequence is exact as its global analogue is exact, cf. proposition 5.3.22. □

Proposition 5.3.27. *Let G be a semi-abelian variety. Then there is an exact sequence*

$$0 \longrightarrow H^0_{\mathcal{M}}(K,G^D) \longrightarrow H^0_{\mathcal{M}}(K,T^D) \longrightarrow H^1_{\mathcal{M},f}(K,A^D) \longrightarrow H^1_{\mathcal{M},f}(K,G^D) \longrightarrow 0.$$

Proof. Proposition 3.4.5 stated that there is an exact sequence

$$0 \longrightarrow H^0_{\mathcal{M}}(K,G^D) \longrightarrow H^0_{\mathcal{M}}(K,T^D) \longrightarrow H^1_{\mathcal{M}}(K,A^D) \longrightarrow H^1_{\mathcal{M}}(K,G^D) \longrightarrow 0.$$

Using the equality $H^1_{\mathcal{M},f}(K,A) = H^1_{\mathcal{M}}(K,A)$ shown in proposition 5.3.25, we can easily see that $H^1_{\mathcal{M},f}(K,G^D) = H^1_{\mathcal{M}}(K,G^D)$ due to the functoriality of $H^1_{\mathcal{M},f}$ and the surjectivity of the map $H^1_{\mathcal{M}}(K,A^D) \to H^1_{\mathcal{M}}(K,G^D)$. Thus, the desired sequence is just the sequence above. □

5.4 The Leading Coefficient of the L-function up to a Rational Multiple

Lemma 5.4.1. *The tangent space of a 1-motive $M = [X \to G]$ is* $\operatorname{Lie} G$.

Proof. Taking definition 3.2.2 into account, it follows that

$$t_M = M_{dR}/M^0_{dR} = \operatorname{Lie} G^\natural / \ker(\operatorname{Lie} G^\natural \twoheadrightarrow \operatorname{Lie} G) = \operatorname{Lie} G.$$

□

Lemma 5.4.2. *Let G be a semi-abelian variety over K. Then, the fundamental line for the 1-motive G^D reads as follows:*

$$\Delta_{\mathbb{Q}}(G^D) \overset{\omega}{\cong} \Delta_{\mathbb{Q}}(T^D) \otimes \Delta_{\mathbb{Q}}(A^D)$$

with $\omega = s^{-1}_{G^D} \cdot s_G \cdot s^{-1}_{B,G^D}$ defined in the proof below (where $s^{-1} : A^{-1} \xrightarrow{\sim} B^{-1}$ is induced by $s : A \xrightarrow{\sim} B$ via the dual of the inverse with respect to composition),[3]

$$\Delta_{\mathbb{Q}}(T^D) = \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K,T^D)\, \mathbf{d}_{\mathbb{Q}} \left((T^D)^+_B\right)^{-1} \mathbf{d}_{\mathbb{Q}} (H^1_{\mathcal{M},f}(K,T)^*)^{-1}$$

and

$$\begin{aligned}\Delta_{\mathbb{Q}}(A^D) = {} & \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K,A^D)^{-1}\, \mathbf{d}_{\mathbb{Q}} (H^1_{\mathcal{M},f}(K,A)^*)^{-1} \\ & \mathbf{d}_{\mathbb{Q}} \left((A^D)^+_B\right)^{-1} \mathbf{d}_{\mathbb{Q}} (\operatorname{Lie} A^D).\end{aligned}$$

[3]for a more general definition, see [Ven07, Rem. 1.2]

Proof. By definition 4.2.1 we have

$$\begin{aligned}\Delta_{\mathbb{Q}}(G^D) = \; & \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, G^D)\, \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, G^D)^{-1} \\ & \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, G)^* \,\mathbf{d}_{\mathbb{Q}}(H^1_{\mathcal{M},f}(K, G)^*)^{-1} \\ & \mathbf{d}_{\mathbb{Q}} \left((G^D)^+_B\right)^{-1} \mathbf{d}_{\mathbb{Q}}(t_{G^D}).\end{aligned}$$

$H^0_{\mathcal{M}}(K, G) = 0$ as stated in proposition 3.4.2, $H^1_{\mathcal{M},f}(K, T^D) = 0$ as stated in proposition 3.4.3 and the previous lemma together give the identities for $\Delta_{\mathbb{Q}}(T^D)$ and $\Delta_{\mathbb{Q}}(A^D)$.

Almost by definition (cf. 3.2.3), there is an exact sequence

$$0 \longrightarrow H_1(G \times_{K,\nu} \mathbb{C}, \mathbb{Z}) \longrightarrow \mathrm{T}_{B,\nu} M \longrightarrow X \longrightarrow 0.$$

Because tensoring with $\mathbb{Q}$ and, as mentioned before, the functor $(-)^+$ are both exact, we get an exact sequence

$$0 \longrightarrow G^+_{B,\nu} = H_1(G \times_{K,\nu} \mathbb{C}, \mathbb{Z})^+ \otimes \mathbb{Q} \longrightarrow M^+_{B,\nu} \longrightarrow X^+_{B,\nu} = X^+ \otimes \mathbb{Q} \longrightarrow 0$$

which induces the isomorphism

$$s_{B,G^D} : \mathbf{d}_{\mathbb{Q}} \left((G^D)^+_B\right) \cong \mathbf{d}_{\mathbb{Q}} \left((T^D)^+_B\right) \mathbf{d}_{\mathbb{Q}} \left((A^D)^+_B\right).$$

Additionally, the exact sequences

$$0 \longrightarrow H^1_{\mathcal{M},f}(K, T) \longrightarrow H^1_{\mathcal{M},f}(K, G) \longrightarrow H^1_{\mathcal{M},f}(K, A) \longrightarrow 0$$

and

$$0 \longrightarrow H^0_{\mathcal{M}}(K, G^D) \longrightarrow H^0_{\mathcal{M}}(K, T^D) \longrightarrow H^1_{\mathcal{M},f}(K, A^D) \longrightarrow H^1_{\mathcal{M},f}(K, G^D) \longrightarrow 0,$$

taken from propositions 5.3.26 and 5.3.27, yield the result because they induce isomorphisms

$$s_G : \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, G) \cong \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, T)\, \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, A)$$

and

$$s_{G^D} : \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, G^D)\, \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, G^D)^{-1} \cong \mathbf{d}_{\mathbb{Q}} H^0_{\mathcal{M}}(K, T^D)\, \mathbf{d}_{\mathbb{Q}} H^1_{\mathcal{M},f}(K, A^D)^{-1},$$

respectively. □

Proposition 5.4.3. *There is a canonical isomorphism*

$$\mathbf{d}_{\mathbb{R}} \left(\ker \alpha_{G^D}\right)^{-1} \overset{\psi}{\cong} \mathbf{d}_{\mathbb{R}} \left(\ker \alpha_{T^D}\right)^{-1}$$

with $\psi = (s_{B,G^D})^{-1}_{\mathbb{R}} \cdot \overline{s}_{\alpha,G^D} \cdot s_{\alpha,T^D} \cdot s_{\alpha,A^D}$*, taking* $\overline{s}_{\alpha,G^D}$ *as the inverse map with respect to composition of* s_{α,G^D}*. The inverse exists because the map is an isomorphism.*[4]

[4] For a more general definition, see [Ven07, Rem. 1.2].

Proof. There is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{R} \otimes_{\mathbb{Q}} (A^D)^+_B \arrow[r, "\alpha_{A^D}"] \arrow[d] & \mathbb{R} \otimes_{\mathbb{Q}} t_{A^D} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \alpha_{G^D} \arrow[r] \arrow[d, "\wr"] & \mathbb{R} \otimes_{\mathbb{Q}} (G^D)^+_B \arrow[r, "\alpha_{G^D}"] \arrow[d] & \mathbb{R} \otimes_{\mathbb{Q}} t_{G^D} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \alpha_{T^D} \arrow[r, equal] \arrow[d] & \mathbb{R} \otimes_{\mathbb{Q}} (T^D)^+_B \arrow[r, "\alpha_{T^D}"] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0. &
\end{tikzcd}$$

The two middle columns are exact because the comparison isomorphisms and the Hodge filtrations respect the weight filtration and $(-)^+$ is exact. Note that α_{A^D} is an isomorphism because h_{A^D} is (the inverse of) the Néron-Tate height pairing and, in particular, an isomorphism, cf. [Fon92, §7.1 and §8.3D]. The diagram is then fully established by the ker-coker sequence and we hence get

$$\begin{aligned}
\mathbf{d}_{\mathbb{R}} \left(\ker \alpha_{G^D}\right)^{-1} &\overset{\overline{s}_{\alpha,G^D}}{\cong} \mathbf{d}_{\mathbb{R}} \left(\mathbb{R} \otimes_{\mathbb{Q}} (G^D)^+_B\right)^{-1} \mathbf{d}_{\mathbb{R}} \left(\mathbb{R} \otimes_{\mathbb{Q}} t_{G^D}\right) \\
&\overset{(s_{B,G^D})^{-1}_{\mathbb{R}}}{\cong} \mathbf{d}_{\mathbb{R}} \left(\mathbb{R} \otimes_{\mathbb{Q}} (T^D)^+_B\right)^{-1} \mathbf{d}_{\mathbb{R}} \left(\mathbb{R} \otimes_{\mathbb{Q}} (A^D)^+_B\right)^{-1} \mathbf{d}_{\mathbb{R}} \left(\mathbb{R} \otimes_{\mathbb{Q}} t_{A^D}\right) \\
&\overset{s_{\alpha,T^D} \cdot s_{\alpha,A^D}}{\cong} \mathbf{d}_{\mathbb{R}} \left(\ker \alpha_{T^D}\right)^{-1}.
\end{aligned}$$

□

Corollary 5.4.4. *For the exact sequence in 4.2.2, the same reasoning as above yields*

$$\mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(G^D) \overset{\phi}{\cong} \mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(T^D) \otimes \mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(A^D)$$

with $\phi = (s_{G^D})^{-1}_{\mathbb{R}} \cdot (s_G)_{\mathbb{R}} \cdot \psi$.

Definition 5.4.5. Denote the 1-motive G^D by $[T^D \xrightarrow{u} A^D]$. Define R_u such that the diagram

$$\begin{tikzcd}
\mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(G^D) \arrow[r, "\vartheta_{\infty,G^D}"] \arrow[d, equal, "\wr"', "\phi"] & \mathbf{1}_{\mathbb{R}} \arrow[d, equal, "\wr"', "R_u"] \\
\mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(T^D) \arrow[r, "\vartheta_{\infty,T^D}"] & \mathbf{1}_{\mathbb{R}} \\[-15pt]
\bigotimes & \bigotimes \\[-15pt]
\mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(A^D) \arrow[r, "\vartheta_{\infty,A^D}"] & \mathbf{1}_{\mathbb{R}}
\end{tikzcd}$$

is commutative. Then, R_u is an automorphism of $\mathbf{1}_{\mathbb{R}}$ due to

$$R_u : \quad \mathbf{1}_{\mathbb{R}} \xrightarrow{\sim} \mathbf{1}_{\mathbb{R}} \cong \mathbf{1}_{\mathbb{R}} \otimes \mathbf{1}_{\mathbb{R}}.$$

Hence, we may regard it as an element of $\mathbb{R}^\times$ and represent its numerical value by abuse of notation by R_u as well. Call R_u the *regulator of* u.

Remark 5.4.6. Consider the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & (\star) \arrow[d, "u"] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H^1_{\mathcal{M},f}(K,A)^*_{\mathbb{R}} \arrow[r] \arrow[d] & H^1_{\mathcal{M},f}(K,A^D)_{\mathbb{R}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_{\mathcal{M}}(K,G^D)_{\mathbb{R}} \arrow[r] \arrow[d] & \ker \alpha_{G^D} \arrow[r] \arrow[d, "\wr"] & H^1_{\mathcal{M},f}(K,G)^*_{\mathbb{R}} \arrow[r] \arrow[d] & H^1_{\mathcal{M},f}(K,G^D)_{\mathbb{R}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_{\mathcal{M}}(K,T^D)_{\mathbb{R}} \arrow[r] \arrow[d, "u"] & \ker \alpha_{T^D} \arrow[r] \arrow[d] & H^1_{\mathcal{M},f}(K,T)^*_{\mathbb{R}} \arrow[r] \arrow[d] & 0 & \\
 & (\star) & 0 & 0 & &
\end{tikzcd}$$

where the rows are the sequences $S_{\mathbb{R}}(A^D)$, $S_{\mathbb{R}}(G^D)$ and $S_{\mathbb{R}}(T^D)$, respectively.

If u vanishes, we are within the scope of proposition 1.1.3 implying that R_u is just 1. In [FPR94, §4.4.4b)], Fontaine and Perrin-Riou state that diagrams of the form of the one in the previous definition are always commutative if one uses certain exact sequences of motives and multiplication as the right hand side morphism. This suggests that R_u might always be 1. However, this cannot be validated as there is no reasoning given. Furthermore, if one allows arbitrary groups in the above diagram, there are many examples which yield an R_u that differs from 1.

Definition 5.4.7. Denote the free $\mathbb{Z}$-module T^D by Y. We will now choose a

$\mathbb{Z}$-basis	of a $\mathbb{Z}$-lattice	in the object below:
$(y_i)_i$	$Y(K)$	$H^0_{\mathcal{M}}(K,T^D) = Y(K) \otimes \mathbb{Q}$
$(y_i^\nu)_{i,\nu}$	$\oplus_\nu Y(K_\nu)$	$(T^D)^+_{B,\nu} = \oplus_\nu Y(K_\nu) \otimes \mathbb{Q}$
$(\gamma_i)_i$	Γ	$H^1_{\mathcal{M},f}(K,T) = U_T \otimes \mathbb{Q}$

Γ is the free part of U_T.

Define

$$\Delta_{\mathbb{Z}}(T^D) \coloneqq \mathbf{d}_{\mathbb{Z}} Y(K) \bigotimes_\nu \mathbf{d}_{\mathbb{Z}}((T^D)^+)^{-1}\, \mathbf{d}_{\mathbb{Z}}(\Gamma^*)^{-1}$$

which is a $\mathbb{Z}$-lattice in $\Delta_{\mathbb{Q}}(T^D) = \mathbb{Q} \otimes_{\mathbb{Z}} \Delta_{\mathbb{Z}}(T^D)$. Additionally, the above choice of bases induces an isomorphism

$$can_{\mathbb{Z},T^D} : \mathbf{1}_{\mathbb{Z}} = \mathbf{d}_{\mathbb{Z}} \mathbb{Z}^{\operatorname{rk} Y(K)} \bigotimes_\nu \mathbf{d}_{\mathbb{Z}}(\mathbb{Z}^{\operatorname{rk} Y(K_\nu)})^{-1}\, \mathbf{d}_{\mathbb{Z}}(\mathbb{Z}^{\operatorname{rk}\Gamma})^{-1} \xrightarrow{\sim} \Delta_{\mathbb{Z}}(T^D).$$

Approaching $\Delta_{\mathbb{Z}}(A^D)$ in the same way leads to a

$\mathbb{Z}$-basis	of a $\mathbb{Z}$-lattice	in the object below:
$(a_i)_i$	$A^D(K)_{free}$	$H^1_{\mathcal{M},f}(K,A^D) = A^D(K) \otimes \mathbb{Q}$
$(\tilde{a}_i)_i$	$A(K)^* = \operatorname{Hom}(A(K),\mathbb{Z})$	$H^1_{\mathcal{M},f}(K,A)^* = A(K)^* \otimes \mathbb{Q}$
$(b_i^\nu)_{i,\nu}$	$\oplus_\nu H_1(G \times_{K,\nu} \mathbb{C},\mathbb{Z})^+$	$(A^D)^+_B = \oplus_\nu H_1(G \times_{K,\nu} \mathbb{C},\mathbb{Z})^+ \otimes \mathbb{Q}$
$(l_i)_i$	$\operatorname{Lie}_{\mathcal{O}_K}(A^D)$	$\operatorname{Lie} A^D$

Define $\mathrm{Lie}_{\mathcal{O}_K}(A^D)$ as $\mathrm{Hom}_{\mathcal{O}_K}(\Omega^1_{\mathcal{A}/\mathcal{O}_K}(\mathcal{A}), \mathcal{O}_K)$ for a Néron model $\mathcal{A}$ of A^D over $\mathcal{O}_K$. It is a $\mathbb{Z}$-lattice of $\mathrm{Lie}\, A^D$, see [Ven07, Ex. 2.1B)].

Then define

$$\Delta_{\mathbb{Z}}(A^D) \coloneqq \mathbf{d}_{\mathbb{Z}}(A^D(K)_{free})^{-1}\, \mathbf{d}_{\mathbb{Z}}(A(K)^*)^{-1} \bigotimes_{\nu} \mathbf{d}_{\mathbb{Z}} H_1(A^D \times_{K,\nu} \mathbb{C}, \mathbb{Z})^+\, \mathbf{d}_{\mathbb{Z}} \mathrm{Lie}_{\mathcal{O}_K} A^D.$$

This is an integral structure of $\Delta_{\mathbb{Q}}(A^D)$ which again with the bases above induces an isomorphism $can_{\mathbb{Z},A^D} : \mathbf{1}_{\mathbb{Z}} \xrightarrow{\sim} \Delta_{\mathbb{Z}}(A^D)$.

Moreover,

$$\Delta_{\mathbb{Z}}(G^D) \coloneqq \Delta_{\mathbb{Z}}(T^D)\, \Delta_{\mathbb{Z}}(A^D)$$

is a $\mathbb{Z}$-lattice in $\Delta_{\mathbb{Q}}(G^D) \overset{\omega}{\cong} \Delta_{\mathbb{Q}}(T^D)\, \Delta_{\mathbb{Q}}(A^D)$ and, once more, the choices above lead to a map

$$can_{\mathbb{Z},G^D} : \mathbf{1}_{\mathbb{Z}} \xrightarrow{\sim} \Delta_{\mathbb{Z}}(G^D).$$

Finally, define $can_{\mathbb{Q},-}$ as the base change $(can_{\mathbb{Z},-})_{\mathbb{Q}}$.

Remark 5.4.8. The previous results and definitions imply the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{1}_{\mathbb{R}} \arrow[r, "(can_{\mathbb{Z},G^D})_{\mathbb{R}}"] \arrow[d, equal, "\wr", "mult."'] & \mathbf{d}_{\mathbb{R}}\Delta_{\mathbb{Q}}(G^D)_{\mathbb{R}} \arrow[r, "s_{\alpha,G^D}"] \arrow[d, equal, "\wr"', "\omega_{\mathbb{R}}"] & \mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(G^D) \arrow[r, "\vartheta_{\infty,G^D}"] \arrow[d, equal, "\wr"', "\phi"] & \mathbf{1}_{\mathbb{R}} \arrow[d, equal, "\wr"', "R_u"] \\
\mathbf{1}_{\mathbb{R}} \arrow[r, "(can_{\mathbb{Z},T^D})_{\mathbb{R}}"] & \mathbf{d}_{\mathbb{R}}\Delta_{\mathbb{Q}}(T^D)_{\mathbb{R}} \arrow[r, "s_{\alpha,T^D}"] & \mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(T^D) \arrow[r, "\vartheta_{\infty,T^D}"] & \mathbf{1}_{\mathbb{R}} \\[-20pt]
\otimes & \otimes & \otimes & \otimes \\[-20pt]
\mathbf{1}_{\mathbb{R}} \arrow[r, "(can_{\mathbb{Z},A^D})_{\mathbb{R}}"] & \mathbf{d}_{\mathbb{R}}\Delta_{\mathbb{Q}}(A^D)_{\mathbb{R}} \arrow[r, "s_{\alpha,A^D}"] & \mathbf{d}_{\mathbb{R}} S_{\mathbb{R}}(A^D) \arrow[r, "\vartheta_{\infty,A^D}"] & \mathbf{1}_{\mathbb{R}}.
\end{tikzcd}$$

Proposition 5.4.9. *We have*

$$(\vartheta_{\infty,A^D})_{\mathbb{C}} \circ (s_{\alpha,A^D})_{\mathbb{C}} \circ (can_{\mathbb{Q},A^D})_{\mathbb{C}} = \frac{1}{\Omega^+_{\infty}(A) \cdot R_A}$$

regarded as elements of $\mathrm{Aut}(\mathbf{1}_{\mathbb{C}}) = \mathbb{C}^\times$ *where the period* $\Omega^+_\infty(A)$ *is the determinant of*

$$\alpha_{A^D} : \quad \mathbb{R} \otimes_{\mathbb{Q}} (A^D)^+_B \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Q}} t_{A^D}$$

and the regulator R_A *is the determinant of the Néron-Tate height pairing*

$$H^1_{\mathcal{M},f}(K, A^D)_{\mathbb{R}} \xrightarrow{\sim} H^1_{\mathcal{M},f}(K, A)^*_{\mathbb{R}},$$

each taken with respect to the bases chosen above.

Proof. The sequence 4.2.2 in our case collapses to

$$H^1_{\mathcal{M},f}(K,A)^*_{\mathbb{R}} \xrightarrow[\sim]{h_{A^D}} H^1_{\mathcal{M},f}(K,A^D)_{\mathbb{R}}.$$

We have already seen that h_{A^D} is the inverse of the Néron-Tate height pairing and that h_{A^D} and α_{A^D} are isomorphisms. Thus

$$(\vartheta_\infty)_{\mathbb{C}} \circ (s_{\alpha,A^D})_{\mathbb{C}} \circ (can_{\mathbb{Q}})_{\mathbb{C}} = \frac{\det h_{A^D}}{\det \alpha_{A^D}} = \frac{1}{\Omega^+_\infty(A) \cdot R_A}.$$

□

Proposition 5.4.10. *The exact sequence 4.2.2 corresponds for $M = Y = T^D$ to the dual of the sequence in the Dirichlet Unit Theorem 1.5.10 for the torus T.*

Proof. The dualised exact sequence 4.2.2 reads as

$$0 \longrightarrow \Gamma \otimes \mathbb{R} \xrightarrow{v_T} \bigoplus_\nu H^0(K_\nu, Y)^* \otimes \mathbb{R} = \mathbb{R}^{r_\infty} \xrightarrow{u_Y^*} H^0_{\mathcal{M}}(K,Y)^* \otimes \mathbb{R} = \mathbb{R}^{r_K} \longrightarrow 0.$$

Recall definition 4.1.4 of $u_{Y,\nu}$, in our case

$$\begin{array}{ccccccc} \mathbb{R}^{r_K} \xrightarrow{\sim} H^0_{\mathcal{M}}(K,Y)_{\mathbb{R}} & \longrightarrow & \mathrm{Hom}_{SH_{K_\nu}(\mathbb{R})}(\mathbb{R}, \mathbb{R} \otimes_{\mathbb{Z}} Y) & \longrightarrow & \ker \alpha_{M,\nu} \xrightarrow{\sim} \mathbb{R}^{r_\nu} \\ (\mathbb{Z} \xrightarrow{\theta} Y) \otimes r & \longmapsto & (\mathbb{R} \xrightarrow{r\cdot\theta} \mathbb{R} \otimes_{\mathbb{Z}} Y) & \longmapsto & r \cdot \theta(1). \end{array}$$

Using remark 1.5.12, it is evident that u_Y and the dual of α in 1.5.9 coincide.

Comparing the map v_T to λ is more complicated. To keep notation simple, let T_ν be the $\mathbb{C}$-valued points of $T \times_{K,\nu} \mathbb{C}$. Note that $W_0 M = M$.

An extension $0 \to T \to E \to \mathbb{Z} \to 0$ in $H^1_{\mathcal{M},f}(K,T) \cong \Gamma \otimes \mathbb{Q}$ corresponds to $\gamma \otimes q$ such that $E = [\mathbb{Z} \xrightarrow{u} T]$ with $u(1) = \gamma$. For $\nu \in S_\infty$, the extension is assigned by definition 4.1.4 to the upper row of the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1(T_\nu,\mathbb{Z}) \arrow[r] \arrow[d, equal] & (E)_{B,\nu} \arrow[r] \arrow[d, "\rho"] & \mathbb{Z} \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & H_1(T_\nu,\mathbb{Z}) \arrow[r] \arrow[d, "\chi"] & \mathrm{Lie}\, T_\nu \arrow[r, "\exp"] \arrow[d, "\chi"] & T_\nu \arrow[r] \arrow[d, "\chi"] & 0 \\ & \mathbb{Z} \arrow[r, "x \mapsto 2\pi i x"] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^\times. & \end{tikzcd}$$

The middle row is taken from the definition of the Betti realisation, cf. 3.2.3. The lower row holds for every $\chi \in H^0(\mathbb{C}, Y)$ and relies on the results $\mathrm{Lie}\, \mathbb{C}^\times = \mathbb{C}$ and $H_1(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z}$. Note that the exponential function in the lower row is the usual one.

Now, choose a lift $e \in (\mathbb{R} \otimes (E)_{B,\nu})^+$ of $1 \in \mathbb{R} \otimes \mathbb{Z}$. Since the upper right rectangle is a fibre product, the image of e in $\mathbb{C}$ is $\ln \chi(\gamma) + 2\pi i n_\chi$ due to $u(1) = \gamma$. The next step involves choosing a $v \in (H_1(T_\nu, \mathbb{Z}) \otimes \mathbb{C})^+$ such that

$$e + v \in \mathrm{Fil}^0 \left((E)_{B,\nu} \otimes \mathbb{C}\right)^+ .$$

We have

$$\mathrm{Fil}^0 \left((E)_{B,\nu} \otimes \mathbb{C}\right) = \ker(\rho_{\mathbb{C}} : (E)_{B,\nu} \otimes \mathbb{C} \to \mathrm{Lie}\, T_\nu)$$

which is the induced filtration by $(E)_{dR}$, see [Del74, §10.1.3]. Hence, the image of v in $\mathrm{Lie}\, T_\nu$ obviously is $-\rho_{\mathbb{R}}(e)$.

$\alpha_{T,\nu}$ is by definition 4.1.3 the composition of the maps

$$\mathbb{R} \otimes H_1(T_\nu, \mathbb{Z})^+ \longrightarrow (\mathbb{C} \otimes H_1(T_\nu, \mathbb{Z}))^+ \cong K_\nu \otimes_K \mathrm{Lie}\, T$$

using that $(T)_{dR} = \mathrm{Lie}\, T = t_T$ which can be read off the table in remark 3.2.7.

Consider the following commutative diagram for $\chi \in H^0(K_\nu, Y)$:

$$\begin{tikzcd}
\alpha_{T,\nu} : \mathbb{R} \otimes H_1(T_\nu, \mathbb{Z})^+ \arrow[r] \arrow[d, "\chi"] & (\mathbb{C} \otimes H_1(T_\nu, \mathbb{Z}))^+ \arrow[r, "\sim"] \arrow[d, "\chi"] & K_\nu \otimes_K \mathrm{Lie}\, T \arrow[d, "\chi"] \\
\mathbb{R} \arrow[r] & \mathbb{C} \arrow[r, "x \mapsto 2\pi i x"] & \mathbb{C}
\end{tikzcd}$$

where we used the results $H_1(T_\nu, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Lie}\, K_\nu^\times = K_\nu$ as well as the inclusion $K_\nu \subset \mathbb{C}$. We can deduce that the composition $\mathrm{Re} \circ \chi \circ \alpha_{T,\nu}$ vanishes. Hence, there is the factorisation

$$\begin{tikzcd}
\mathbb{R} \otimes H_1(T_\nu, \mathbb{Z})^+ \arrow[r, "\alpha_{T,\nu}"] & K_\nu \otimes_K \mathrm{Lie}\, T \arrow[r, two heads] \arrow[d, "\bigoplus_j \mathrm{Re} \circ \chi_j"] & \mathrm{coker}\, \alpha_{T,\nu} \arrow[dl, dotted, "\varphi"] \\
& \mathbb{R}^{r_\nu} &
\end{tikzcd}$$

where χ_j is a $\mathbb{Z}$-basis of $H^0(K_\nu, Y)$. φ is certainly surjective. However, since $\mathrm{coker}\, \alpha_{T,\nu}$ is of dimension r_ν, φ is even an isomorphism. We can now determine the image of v in $\mathbb{R}^{r_\nu}$ as follows: the image of v in $\mathrm{Lie}\, T_\nu$ is $-\rho_{\mathbb{R}}(e)$ as we have seen above. Furthermore, the image of e in $\mathrm{Lie}\, \mathbb{C}^\times$ is $\ln \chi(\gamma) + 2\pi i n_\chi$. Hence, we can conclude

$$(\mathrm{Re} \circ \chi_j(v))_j = -\mathrm{Re} \left(\ln \chi_j(\gamma) + 2\pi i n_{\chi_j}\right)_j = -\left(\ln |\chi_j(\gamma)|_\nu\right)_j .$$

This coincides with the definition of λ in 1.5.9 when the bases are chosen accordingly. □

Corollary 5.4.11. *We have*

$$(\vartheta_{\infty,T^D})_{\mathbb{C}} \circ (s_{\alpha,T^D})_{\mathbb{C}} \circ (can_{\mathbb{Q},T^D})_{\mathbb{C}} = \frac{1}{R_T}$$

regarded as an element of $\mathrm{Aut}(\mathbf{1}_{\mathbb{C}}) = \mathbb{C}^\times$ *where* R_T *is the regulator of the torus* T*, see 1.5.11.*

Proof. Straightforward noting that α_{T^D} is zero because $t_Y = 0$ and employing the definition of the regulator as well as the fact that dualising leads to inversion of determinants. □

Proposition 5.4.12. *Conjecture 4.2.5 implies the equality*

$$\zeta_{\mathbb{Q}}(G^D) = \frac{\Omega_\infty^+(A) \cdot R_A \cdot R_T \cdot R_u}{L^*(G^D)} \cdot can_{\mathbb{Q},G^D}.$$

Proof. Due to the commutative diagram in 5.4.8, we may compute

$$(\vartheta_\infty)_{\mathbb{C}} \circ (s_{\alpha,G^D})_{\mathbb{C}} \circ (can_{\mathbb{Q},G^D})_{\mathbb{C}}$$

for T^D and A^D separately, multiply the results and add a factor R_u which then reads as follows:

$$(\vartheta_{\infty,G^D})_{\mathbb{C}} \circ (s_{\alpha,G^D})_{\mathbb{C}} \circ (can_{\mathbb{Q},G^D})_{\mathbb{C}} = \frac{1}{R_T} \cdot \frac{1}{\Omega_\infty^+(A) \cdot R_A} \cdot \frac{1}{R_u}.$$

Using this result, we see that the function

$$\frac{\Omega_\infty^+(A) \cdot R_A \cdot R_T \cdot R_u}{L^*(G^D)} \cdot can_{\mathbb{Q},G^D}$$

possesses the property which, according to the conjecture, uniquely identifies $\zeta_{\mathbb{Q}}(G^D)$. Hence the desired equality holds. □

5.5 The Leading Coefficient of the L-function up to Sign and a Power of Two

We can only deduce the exact value of L^* up to a power of two, the reason being that proposition 4.3.2 is only applicable in case of $\ell \neq 2$.

Definition 5.5.1 ([BF96, p. 86])**.** We define *Bloch-Kato's Tate-Shafarevich group* for a finitely generated $\mathbb{Z}_\ell$-module T as

$$Ш_{BK}^{(l)}(T) := \ker\left(\frac{H^1(\mathcal{O}_{K,S_\ell(M)}, V/T)}{H_f^1(K,T) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \longrightarrow \bigoplus_{\mathfrak{p} \in S_{\ell,f}(M)} \frac{H^1(K_{\mathfrak{p}}, V/T)}{H_f^1(K_{\mathfrak{p}},T) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell}\right).$$

This group is independent of $S_\ell(M)$. For a finitely generated $\mathbb{Z}$-module Λ, we define

$$Ш_{BK}(\Lambda) := \prod_\ell Ш_{BK}^{(l)}(\Lambda \otimes \mathbb{Z}_\ell).$$

Lemma 5.5.2. *The groups possess two important properties:*

- $Ш_{BK}^{(l)}(T)$ *is finite and*
- *the ℓ-primary component of* $Ш_{BK}(\Lambda)$ *is* $Ш_{BK}^{(l)}(\Lambda \otimes \mathbb{Z}_\ell)$.

Proof. The first assertion we find in [BK90, p. 377]. The statement on the ℓ-primary component is then obvious because $Ш^{(l)}_{BK}(\Lambda \otimes \mathbb{Z}_\ell)$ is a $\mathbb{Z}_\ell$-module and due to it being finite, it has only ℓ-torsion. $\square$

Definition 5.5.3. Let T be a finitely generated $\mathbb{Z}_\ell$-module. Then we define the *Pontryagin dual* $T^\vee$ of T as $\mathrm{Hom}(T, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ where we use the continuous homomorphisms as we did before.

Proposition 5.5.4 ([BF96, p. 86])**.** *For T as above there are short exact sequences*

$$0 \longrightarrow Ш^{(\ell)}_{BK}(T^D)^\vee \longrightarrow H^2_{BF,f}(K,T) \longrightarrow H^1_{BF,f}(K,T^D)^* \longrightarrow 0$$

and

$$0 \longrightarrow (H^1(\mathcal{O}_{K,S_\ell(M)}, T^D)_{tor})^\vee \longrightarrow H^3_{BF,f}(K,T) \longrightarrow H^0_{BF,f}(K,T^D)^* \longrightarrow 0.$$

Remark 5.5.5. Our choice for the ℓ-adic integral structure of $\mathrm{V}_\ell G^D$ is the canonical one: we choose the integral realisations of a 1-motive, e.g. the ℓ-adic Tate module $\mathrm{T}_\ell G^D$, as an integral structure of $\mathrm{V}_\ell G^D$.

Definition 5.5.6. Define

$$\Delta_{\mathbb{Z}_\ell}(G^D) \coloneqq \Delta_{\mathbb{Z}}(G^D)_{\mathbb{Z}_\ell}.$$

Lemma 5.5.7. *There is an isomorphism*

$$\begin{aligned}\Delta_{\mathbb{Z}_\ell}(G^D) \cong\ & \mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_f(K, \mathrm{T}_\ell G^D) \\ & \bigotimes_{\nu \in S_\infty} \mathbf{d}_{\mathbb{Z}_\ell}((\mathrm{T}_\ell G^D)^+)^{-1} \bigotimes_{\mathfrak{p} \in S_\ell} \mathbf{d}_{\mathbb{Z}_\ell} \mathrm{Lie}_{\mathcal{O}_{K_\mathfrak{p}}} A^D \\ & \bigotimes_{i=0}^{3} \mathbf{d}_{\mathbb{Z}_\ell} H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor}^{(-1)^{i+1}}\end{aligned}$$

where $\mathrm{Lie}_{\mathcal{O}_{K_\mathfrak{p}}} A^D \coloneqq (\mathrm{Lie}_{\mathcal{O}_K} A^D) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}}$.

Proof. Using that the comparison isomorphisms for 1-motives are compatible with the integral structures of their realisations yields $(\mathrm{T}_{B,\nu} G^D)^+ \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong (\mathrm{T}_\ell G^D)^+$. Furthermore, consider

$$(\mathrm{Lie}_{\mathcal{O}_K} A^D) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = (\mathrm{Lie}_{\mathcal{O}_K} A^D) \otimes_{\mathcal{O}_K} \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \bigoplus_{\mathfrak{p} \in S_\ell} (\mathrm{Lie}_{\mathcal{O}_K} A^D) \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}}.$$

We know by conjecture 4.3.4 that there are isomorphisms[5]

$$\begin{aligned} H^0_{\mathcal{M}}(K,-) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell &\cong H^0(K, \mathrm{V}_\ell(-)) \\ \text{and } H^1_{\mathcal{M},f}(K,-) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell &\cong H^1_f(K, \mathrm{V}_\ell(-)) \end{aligned}$$

[5] note that this conjecture can be proved in the case of 1-motives, cf. [Fon92, Prop. in §8.2]

and that therefore a $\mathbb{Z}$-lattice of $H^1_{\mathcal{M},f}(K,-)$ yields a $\mathbb{Z}_\ell$-lattice in $H^1_f(K,\mathrm{V}_\ell(-))$. The same holds true for $i=0$. Hence, we can obtain induced $\mathbb{Z}_\ell$-lattices in $H^i_{BF,f}(K,\mathrm{T}_\ell G^D)$ for $i=0,1$ as here the Burns-Flach definition coincides with the Block-Kato definition, see remark 4.3.3. Furthermore, we get $\mathbb{Z}_\ell$-lattices in $H^i_{BF,f}(K,\mathrm{T}_\ell G^D)$ for $i=2,3$ using proposition 5.5.4 with the left side of the sequences being torsion and the above argument.

Note that the above construction ensures that the $\mathbb{Z}_\ell$-lattices are isomorphic to $H^i_{BF,f}(K,\mathrm{T}_\ell G^D)_{tor.fr.}$. Hence, the formula follows from $\mathbf{d}_R A = \mathbf{d}_R A_{tor}\, \mathbf{d}_R A_{tor.fr.}$. □

Lemma 5.5.8. *We have*

$$\begin{aligned} H^0_{BF,f}(K,\mathrm{T}_\ell G^D)_{tor} &= 0\\ H^1_{BF,f}(K,\mathrm{T}_\ell G^D)_{tor} &= \mathbb{H}^0(K,[T^D\to A^D])(\ell)\\ H^2_{BF,f}(K,\mathrm{T}_\ell G^D)_{tor} &= Ш_{BK}(\Lambda)(\ell)^\vee\\ H^3_{BF,f}(K,\mathrm{T}_\ell G^D)_{tor} &= G(K)(\ell)^\vee \end{aligned}$$

using the Galois hypercohomology of the complex $[T^D\to A^D] := [T^D\to A^D(\overline{K})]$ *where* $A^D(\overline{K})$ *is placed in degree zero and* $\Lambda := \mathrm{T}_{B,\nu}G$ *for a fixed* $\nu\in S_\infty$.

Proof. Case $i=0$: due to remark 4.3.3, we know that the Burns-Flach definition coincides with the Block-Kato definition and consequently

$$H^0_{BF,f}(K,\mathrm{T}_\ell G^D)\subseteq H^0(K,\mathrm{T}_\ell G^D)=0$$

as seen in corollary 5.2.3.

Case $i=1$: the torsion group of $H^1_f(K,\mathrm{T}_\ell G^D)$ is by definition just the torsion group of $H^1(K,\mathrm{T}_\ell G^D)$. Hence, using [Jos09, Rem. 2.2.10], we see that the torsion group of $H^1(K,\mathrm{T}_\ell G^D)$ equals the torsion group of $\mathbb{H}^0(K,[T^D\to A^D])^{\wedge\ell}$. The spectral sequence for the Galois hypercohomology

$$E_2^{p,q}=H^p(H^q(K,A^\bullet))\Rightarrow \mathbb{H}^{p+q}(K,A^\bullet),$$

cf. [CE56, Chap. XVII], degenerates and yields

$$0\to E_2^{0,0}=\operatorname{coker} H^0(u)\to E^0_\infty=\mathbb{H}^0(K,[T^D\to A^D])\to E_2^{-1,1}=\ker H^1(u)\to 0.$$

Hence, the sequence

$$(T^D)(K)\longrightarrow A^D(K)\longrightarrow \mathbb{H}^0(K,[T^D\to A^D])\longrightarrow H^1(K,T^D)\longrightarrow H^1(K,A^D)$$

is exact, compare [HS05, §2, formula (6)]. $A^D(K)$ is finitely generated and $H^1(K,T^D)$ is finite, hence $\mathbb{H}^0(K,[T^D\to A^D])$ is a finitely generated $\mathbb{Z}$-module and

$$(\mathbb{H}^0(K,[T^D\to A^D])^{\wedge\ell})_{tor}=\mathbb{H}^0(K,[T^D\to A^D])(\ell).$$

Case $i=2$: according to the first sequence in 5.5.4, $(Ш^{(\ell)}_{BK}(\mathrm{T}_\ell G)^\vee)_{tor}$ is the torsion group of $H^2_{BF,f}(K,\mathrm{T}_\ell G^D)_{tor}$ as the last term in the sequence has no torsion. We conclude that there is an equality of finite groups

$$Ш_{BK}(\mathrm{T}_{B,\nu}G)(\ell)=Ш^{(l)}_{BK}(\mathrm{T}_{B,\nu}G\otimes\mathbb{Z}_\ell)=Ш^{(l)}_{BK}(\mathrm{T}_\ell G)$$

using the comparison isomorphism $\mathrm{T}_\ell G \cong \mathrm{T}_{B,\nu} G \otimes \mathbb{Z}_\ell$.

Case $i = 3$: the second sequence in 5.5.4 implies that the torsion of $H^3_{BF,f}$ is the torsion of $(H^1(\mathcal{O}_{K,S_\ell(M)}, \mathrm{T}_\ell G)_{tor})^\vee$. It follows from the proof of [Mil86, Prop. II.2.9] that we have

$$H^1(\mathcal{O}_{K,S_\ell(M)}, \mathrm{T}_\ell G) = H^1(\pi_{\text{ét}}(\mathcal{O}_{K,S_\ell(M)}), \mathrm{T}_\ell G)$$

where the right hand side is the (continuous) group cohomology of $\pi_{\text{ét}}(\mathcal{O}_{K,S_\ell(M)})$. The fundamental group is the Galois group $G(S_\ell(M)) := \mathrm{Gal}(K_{S_\ell(M)}/K)$ of the maximal field extension $K_{S_\ell(M)}$ which is unramified outside $S_\ell(M)$. The first part of the five-term sequence 1.2.13 reads as follows:

$$0 \longrightarrow H^1(K_{S_\ell(M)}/K, \mathrm{T}_\ell G) \longrightarrow H^1(K, \mathrm{T}_\ell G) \longrightarrow H^1(K_{S_\ell(M)}, \mathrm{T}_\ell G)^{G(S_\ell(M))}.$$

The absolute Galois group of $K_{S_\ell(M)}$ operates trivially on $\mathrm{T}_\ell G$ due to the definition of $S_\ell(M)$. Thus, $\mathrm{Hom}(\mathrm{Gal}_{K_{S_\ell(M)}}, \mathrm{T}_\ell G)$ is torsion free because $\mathrm{T}_\ell G$ also is torsion free. We may then deduce that the last group in the sequence is torsion free, too, which yields an isomorphism of the torsion groups of the first and second entry. Using lemma 5.2.1, we see that the torsion group of $H^1(K, \mathrm{T}_\ell G)$ coincides with the torsion group of $G(K)^{\wedge\ell}$ and hence with the finite group $G(K)(\ell)$. □

Remark 5.5.9. The group $Ш_{BK}(\Lambda)$ is conjectured to be finite, see [BK90, Conj. 5.15].

Lemma 5.5.10. *There are integral versions $\psi_\mathfrak{p}$ of $\eta_\mathfrak{p}$, which have been introduced in 4.3.5, as follows:*

$$\begin{aligned} &\psi_\mathfrak{p}(\mathrm{T}_p G^D): \quad \mathbf{d}_{\mathbb{Z}_p} R\Gamma_f(K_\mathfrak{p}, \mathrm{T}_p G^D)\, \mathbf{d}_{\mathbb{Z}_p} \mathrm{Lie}_{\mathcal{O}_{K_\mathfrak{p}}} A^D \longrightarrow \mathbf{1}_{\mathbb{Z}_p} && \textit{for } \mathfrak{p} \in S_p \\ &\psi_\mathfrak{p}(\mathrm{T}_\ell G^D): \quad \mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_f(K_\mathfrak{p}, \mathrm{T}_\ell G^D) \longrightarrow \mathbf{1}_{\mathbb{Z}_\ell} && \textit{for } \mathfrak{p} \notin S_\ell \end{aligned}$$

satisfying the equation

$$\psi_\mathfrak{p}(\mathrm{T}_\ell G^D)_{\mathbb{Q}_\ell} = c_\mathfrak{p}(\mathrm{T}_\ell G^D) \cdot \eta_\mathfrak{p}(\mathrm{T}_\ell G^D)$$

where $c_\mathfrak{p}(T_\ell)$ is called the Tamagawa number of T_ℓ at $\mathfrak{p}$ *and will be defined in the proof below.*

Proof. $\mathrm{Lie}_{\mathcal{O}_{K_\mathfrak{p}}} A^D$ is a lattice in $t_\mathfrak{p}(\mathrm{V}_p G^D)$ because it is a lattice in

$$\begin{aligned} \mathrm{Lie}_{\mathcal{O}_{K_\mathfrak{p}}} A^D \otimes \mathbb{Q} &= \mathrm{Lie}_{\mathcal{O}_K} A^D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}} \otimes \mathbb{Q} = \mathrm{Lie}\, A^D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}} \\ &= \mathrm{Lie}\, A^D \otimes_K K \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}} = \mathrm{Lie}\, A^D \otimes_K K_\mathfrak{p} \\ &\cong t_\mathfrak{p}(\mathrm{V}_p G^D) \end{aligned}$$

where the last equality is justified by the compatibility of the comparison isomorphisms.

Hence, the left hand sides of the morphisms above describe lattices in

$$\begin{aligned} &\mathbf{d}_{\mathbb{Q}_p} R\Gamma_f(K_\mathfrak{p}, \mathrm{V}_p G^D)\, \mathbf{d}_{\mathbb{Q}_p} t_\mathfrak{p}(\mathrm{V}_p G^D) \quad \text{resp.} \\ &\mathbf{d}_{\mathbb{Q}_\ell} R\Gamma_f(K_\mathfrak{p}, \mathrm{V}_\ell G^D). \end{aligned}$$

Via $\eta_\mathfrak{p}(\mathrm{T}_\ell G^D)$, they become a lattice in $\mathbf{1}_{\mathbb{Q}_\ell}$. Using the canonical basis e of $\mathbf{1}_{\mathbb{Q}_\ell}$, we see that the lattice can be described by $\mathbb{Z}_\ell \cdot c \cdot e$. We may choose c such that it be a power of ℓ. This convention fixes c uniquely and we name it $c_\mathfrak{p}(\mathrm{T}_\ell G^D)$. Additionally, the construction yields the desired morphisms $\psi_\mathfrak{p}(\mathrm{T}_\ell G^D)$. □

Remark 5.5.11. Our definition of the Tamagawa numbers is compatible with the definition in [FPR94, §4.1.2]. In particular, assuming $\ell \neq p$, we have $c_{\mathfrak{p}}(T_\ell) = 1$ if $V_\ell = T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is unramified, or, more generally, $c_{\mathfrak{p}}(T_\ell) = \# H^1(\mathcal{I}_{K_{\mathfrak{p}}}, T_\ell)_{tor}^{\mathrm{Gal}_{K_{\mathfrak{p}}}}$ by [FPR94, Prop. 4.2.2].

It is conjectured that the Tamagawa numbers vanish for almost all ℓ and $\mathfrak{p}$.

Theorem 5.5.12. *Conjecture 4.3.7 implies*

$$\frac{L^*(G^D)}{\Omega^+_\infty(A) \cdot R_A \cdot R_T \cdot R_u} = \alpha \cdot \frac{\#\text{Ш}_{BK}(\Lambda)}{\#G(K)_{tor} \cdot \#\mathbb{H}^0(K, [T^D \to A^D])_{tor}} \cdot \prod_{\ell, \mathfrak{p}} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D)$$

where α is of the form $\pm 2^n$.

Proof. In complete analogy to the definition of the ℓ-adic period-regulator map ϑ_ℓ in 4.3.6 we define

$$\kappa_\ell : \quad \Delta_{\mathbb{Z}_\ell}(G^D) \to \mathbf{d}_{\mathbb{Z}_p} R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, \mathrm{T}_\ell G^D)$$

by

$$\begin{aligned}
\Delta_{\mathbb{Z}_\ell}(G^D) \overset{5.5.7}{\cong}\ & \mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_f(K, \mathrm{T}_\ell G^D) \\
& \bigotimes_{\nu \in S_\infty} \mathbf{d}_{\mathbb{Z}_\ell}((\mathrm{T}_\ell G^D)^+)^{-1} \bigotimes_{\mathfrak{p} \in S_\ell} \mathbf{d}_{\mathbb{Z}_\ell} \mathrm{Lie}_{\mathcal{O}_{K_{\mathfrak{p}}}} A^D \\
& \bigotimes_{i=0}^{3} \mathbf{d}_{\mathbb{Z}_\ell} H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor}^{(-1)^{i+1}} \\
\overset{4.3.2}{\cong}\ & \mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, \mathrm{T}_\ell G) \\
& \bigotimes_{\mathfrak{p} \in S_{\ell,f}(M)} \mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_f(K_{\mathfrak{p}}, \mathrm{T}_\ell G) \bigotimes_{\mathfrak{p} \in S_\ell} \mathbf{d}_{\mathbb{Z}_\ell} \mathrm{Lie}_{\mathcal{O}_{K_{\mathfrak{p}}}} A^D \\
& \bigotimes_{i=0}^{3} \mathbf{d}_{\mathbb{Z}_\ell} H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor}^{(-1)^{i+1}}.
\end{aligned}$$

This is isomorphic to $\mathbf{d}_{\mathbb{Z}_\ell} R\Gamma_c(\mathcal{O}_{K,S_\ell(M)}, \mathrm{T}_\ell G)$ using the $\psi_{\mathfrak{p}}(\mathrm{T}_\ell G^D)$ defined in lemma 5.5.10 for all $\mathfrak{p} \in S_{\ell,f}(M)$ and trivialising the finite groups in accordance with remark 1.1.2 (viii).

The isomorphism κ_ℓ is an integral version of the ℓ-adic period-regulator map ϑ_ℓ such that

$$(\kappa_\ell)_{\mathbb{Q}_\ell} = \left(\epsilon \prod_{i=0}^{3} \left(\# H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor} \right)^{(-1)^i} \prod_{\mathfrak{p} \in S_{\ell,f}(M)} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D) \right) \cdot \vartheta_\ell$$

holds. Here, the extra terms $H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor}$ are trivialised by the identity, see remark 1.1.2 (viii), i.e. transformed to the inverse of their cardinality modulo $\mathbb{Z}_\ell^\times$, hence $\epsilon \in \mathbb{Z}_\ell^\times$.

We may calculate

$$\begin{aligned}
(\kappa_\ell \circ can_{\mathbb{Z}_\ell})_{\mathbb{Q}_\ell} &= (\kappa_\ell)_{\mathbb{Q}_\ell} \circ (can_{\mathbb{Q}})_{\mathbb{Q}_\ell} \\
&= \left(\epsilon \prod_{i=0}^{3} \left(\# H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor} \right)^{(-1)^i} \prod_{\mathfrak{p} \in S_{\ell,f}(M)} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D) \right) \cdot \vartheta_\ell
\end{aligned}$$

$$\circ \left(\frac{L^*(G^D)}{\Omega_\infty^+(A) \cdot R_A \cdot R_T \cdot R_u} \right)^{-1} \cdot \zeta_{\mathbb{Q}}(G^D)_{\mathbb{Q}_\ell}$$
$$= \left(\epsilon \prod_{i=0}^{3} \left(\# H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor} \right)^{(-1)^i} \prod_{\mathfrak{p} \in S_{\ell,f}(M)} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D) \right)$$
$$\cdot \left(\frac{L^*(G^D)}{\Omega_\infty^+(A) \cdot R_A \cdot R_T \cdot R_u} \right)^{-1} \cdot \zeta_{\mathbb{Z}_\ell}(G^D)_{\mathbb{Q}_\ell}$$

using the main result of last section, proposition 5.4.12, for the second equality and conjecture 4.3.7 for the third equality.

Being automorphisms of $\mathbf{1}_{\mathbb{Z}_\ell}$, $\zeta_{\mathbb{Z}_\ell}(G^D)$ and $\kappa_\ell \circ can_{\mathbb{Z}_\ell}$ may only differ by an element α' of $\mathrm{Aut}(\mathbf{1}_{\mathbb{Z}_\ell}) = \mathbb{Z}_\ell^\times$ and thus

$$\frac{L^*(G^D)}{\Omega_\infty^+(A) \cdot R_A \cdot R_T \cdot R_u} = \alpha' \cdot \epsilon \prod_{i=0}^{3} \left(\# H^i_{BF,f}(K, \mathrm{T}_\ell G^D)_{tor} \right)^{(-1)^i} \cdot \prod_{\mathfrak{p} \in S_{\ell,f}(M)} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D)$$
$$= \alpha \cdot \frac{\# Ш_{BK}(\Lambda)}{\# G(K)_{tor} \cdot \# \mathbb{H}^0(K, [T^D \to A^D])_{tor}} \cdot \prod_{\ell, \mathfrak{p}} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D).$$

We see that $\alpha \in \mathbb{Z}_\ell^\times$, knowing that $c_{\mathfrak{p}}(\mathrm{T}_\ell G^D)$ vanishes for places of good reduction and that $S_{\ell,f}(M)$ contains all places with bad reduction.

We can go through this construction for all $\ell \neq 2$, hence $\alpha \in \mathbb{Z}_\ell^\times$ for all $\ell \neq 2$. The claim follows. □

Due to remark 5.1.6, we may deduce

Corollary 5.5.13. *For a semi-abelian variety G, the Tamagawa Number Conjecture implies that*

$$\frac{L^*(h^1(G)(1))}{\Omega_\infty^+(A) \cdot R_A \cdot R_T \cdot R_u} = \alpha \cdot \frac{\# Ш_{BK}(\Lambda)}{\# G(K)_{tor} \cdot \# \mathbb{H}^0(K, [T^D \to A^D])_{tor}} \cdot \prod_{\ell, \mathfrak{p}} c_{\mathfrak{p}}(\mathrm{T}_\ell G^D)$$

where α is of the form $\pm 2^n$.

Corollary 5.5.14. *For an abelian variety A over K, the conjecture implies the one of Birch and Swinnerton-Dyer (up to sign and a power of two):*

$$\frac{L^*(h^1(A)(1))}{\Omega_\infty^+(A) \cdot R_A} = \alpha \cdot \frac{\# Ш(A)}{\# A(K)_{tor} \cdot \# A^D(K)_{tor}} \cdot \prod_{\ell, \mathfrak{p}} c_{\mathfrak{p}}(\mathrm{T}_\ell A^D)$$

Note that the L-function in this case is just the usual Hasse-Weil L-function of A.

Proof. It holds that

$$Ш^{(\ell)}_{BK}(\mathrm{T}_{B,\nu} A \otimes \mathbb{Z}_\ell) = Ш^{(\ell)}_{BK}(\mathrm{T}_\ell A) = Ш(A)(\ell)$$

if $Ш(A)$ is finite, see [BF96, p. 95]. □

Lemma 5.5.15. *Let T be a torus. Then there is a short exact sequence of finite groups for $\ell \neq 2$*

$$0 \longrightarrow Cl(T)(\ell) \longrightarrow Ш_{BK}(\mathrm{T}_{B,\nu}T)(\ell) \longrightarrow Ш(T)(\ell) \longrightarrow 0$$

where $Cl(T)$ is the class group of T *(with respect to $S = S_\infty$) defined as*

$$\frac{T_{\mathbb{A}}}{T(K) \cdot T_{\mathbb{A}}(S_\infty)},$$

cf. definition 1.5.4.

Proof. We essentially verify the proof of [HK11, Prop. 2.6.3] for our situation.

By abuse of notation, we set $T_\nu^c = T(K_\nu)$ for $\nu \in S_\infty$. Then there is the isomorphism

$$T_{\mathbb{A}}/\prod_{\text{all } \mathfrak{p}} T_{\mathfrak{p}}^c = \varinjlim_{S \text{ finite}} \left(T_{\mathbb{A}}(S)/\prod_{\text{all } \mathfrak{p}} T_{\mathfrak{p}}^c \right) \cong \varinjlim_{S \text{ finite}} \bigoplus_{\mathfrak{p} \in S} T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c = \bigoplus_{\text{all } \mathfrak{p}} T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c.$$

Due to the definition of U_T, 1.5.5, and $T_\nu^c = T(K_\nu)$, we obtain the exact sequence

$$0 \longrightarrow U_T \longrightarrow T(K) \longrightarrow \bigoplus_{\mathfrak{p}} T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c \longrightarrow Cl(T) \longrightarrow 0.$$

Hence, we get the following commutative diagram for every $n \in \mathbb{Z}$:

$$\begin{tikzcd}
0 \arrow[r] & T(K)/U_T \arrow[r] \arrow[d, "{[\ell^n]}"] & \bigoplus_{\mathfrak{p}} T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c \arrow[r] \arrow[d, hook, "{[\ell^n]}"] & Cl(T) \arrow[r] \arrow[d, "{[\ell^n]}"] & 0 \\
0 \arrow[r] & T(K)/U_T \arrow[r] & \bigoplus_{\mathfrak{p}} T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c \arrow[r] & Cl(T) \arrow[r] & 0
\end{tikzcd}$$

where the morphism in the centre is injective because $T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c \cong \mathbb{Z}^{r_{K_{\mathfrak{p}}}}$ is free as we have seen in the proof of 5.3.6.

The resulting ker-coker sequence then reads as

$$0 \to Cl(T)[\ell^n] \to (T(K)/U_T) \otimes \mathbb{Z}/\ell^n\mathbb{Z} \to \left(\bigoplus_{\mathfrak{p}} T(K_{\mathfrak{p}})/T_{\mathfrak{p}}^c \right) \otimes \mathbb{Z}/\ell^n\mathbb{Z} \to Cl(T)/\ell^n \to 0.$$

Taking the direct limit establishes the exact sequence

$$0 \longrightarrow Cl(T)(\ell) \longrightarrow \frac{T(K) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell}{U_T \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \longrightarrow \bigoplus_{\mathfrak{p}} \frac{T(K_{\mathfrak{p}}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell}{T_{\mathfrak{p}}^c \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \longrightarrow 0$$

because $Cl(T)$ is finite.

In lemma 5.2.1 we have seen the exactness of

$$0 \longrightarrow T(k) \otimes \mathbb{Z}/\ell^n\mathbb{Z} \longrightarrow H^1(k, T[\ell^n]) \longrightarrow H^1(k, T)[\ell^n] \longrightarrow 0.$$

Taking the direct limit yields

$$0 \longrightarrow T(k)\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^1(k,T(\ell)) \longrightarrow H^1(k,T)(\ell) \longrightarrow 0$$

and we find

$$0 \longrightarrow Ш(T)(\ell) \longrightarrow \frac{H^1(K,T(\ell))}{T(K)\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \longrightarrow \prod_{\text{all } \mathfrak{p}} \frac{H^1(K_\mathfrak{p},T(\ell))}{T(K_\mathfrak{p})\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell}$$

because taking the ℓ component is left exact.

A combination of the exact sequences above together with the definition of $Ш^{(\ell)}_{BK}$, cf. 5.5.1, leads us to the commutative diagram

$$\begin{tikzcd}
 & Cl(T)(\ell) \arrow[d, hook] & Ш^{(\ell)}_{BK}(\mathrm{T}_\ell T) \arrow[d, hook] & Ш(T)(\ell) \arrow[d, hook] & \\
0 \arrow[r] & \frac{T(K)\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell}{U_T\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \frac{H^1(K,T(\ell))}{U_T\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \arrow[r] \arrow[d] & \frac{H^1(K,T(\ell))}{T(K)\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{\mathfrak{p}} \frac{T(K_\mathfrak{p})\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell}{T^c_\mathfrak{p}\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \arrow[r] & \bigoplus_{\mathfrak{p}} \frac{H^1(K_\mathfrak{p},T(\ell))}{T^c_\mathfrak{p}\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} \arrow[r] & \prod_{\text{all } \mathfrak{p}} \frac{H^1(K_\mathfrak{p},T(\ell))}{T(K_\mathfrak{p})\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell} &
\end{tikzcd}$$

in the case of $\ell \neq 2$ because then the group $H^1(K_\nu, T)(\ell)$ vanishes for $\nu \in S_\infty$. We have also used $\mathrm{V}_\ell T/\mathrm{T}_\ell T = T(\ell)$. The ker-coker sequence now yields the desired result. □

Corollary 5.5.16. *For a torus T over K, the conjecture implies that*

$$\frac{L^*(h^1(T)(1))}{R_T} = \alpha \cdot \frac{h_T \cdot \#Ш(T)}{\#\mu_T \cdot \#H^1(K,T^D)} \cdot \prod_{\ell,\mathfrak{p}} c_\mathfrak{p}(\mathrm{T}_\ell T^D)$$

where μ_T was defined in theorem 1.5.8. In particular, for $T = \mathbb{G}_m$ we obtain the analytic class number formula (up to sign and a power of to):

$$L^*(h^0(\operatorname{Spec} K)) = -\frac{R_K \cdot h_K}{\#\mu_K}.$$

Proof. The formula

$$\#Ш_{BK}(\mathrm{T}_{B,\nu}T) = h_T \cdot \#Ш(T)$$

modulo 2 follows from the previous lemma. The second statement follows because $H^1(K,\mathbb{Z})$ and $Ш(\mathbb{G}_m)$ vanish due to lemma 1.2.10 respectively theorem 1.2.9, because all Tamagawa numbers equal 1 and because the equality of motives

$$h^1(\mathbb{G}_m)(1) = \mathbb{1}(-1)(1) = \mathbb{1} = h^0(\operatorname{Spec} K)$$

holds. □

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Erklärung

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