On the Main Conjecture for p-adic Lie Extensions of Heisenberg Type of Totally Real Number Fields

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Introduction

The main conjecture of Iwasawa theory studies the mysterious relationship between purely arithmetic problems and special values of p-adic zeta functions. Here, p denotes an odd prime number. Let A be the ideal class group of $\mathbb{Q}(\mu_p)$, where μ_p is the group of p-th roots of unity. We call p regular if p does not divide the order of A. One of the first results leading to this relationship is a theorem by E. E. Kummer, which states that p is irregular if and only if p divides the numerator of at least one of the rational numbers $\zeta(-1), \zeta(-3), \dots, \zeta(4-p)$. More generally, J. Herbrand and K. A. Ribet found that for $n = 3, 5, \dots, p-2$, the idempotent of $\mathbb{Z}_p[G(\mathbb{Q}(\mu_p)|\mathbb{Q})]$ associated to the n-th power of the cyclotomic character annihilates $A(p)$ if and only if p divides the numerator of $\zeta(n+1-p)$. Probably one of the most important results concerning this relationship is the proof of the main conjecture for abelian extensions of totally real number fields by K. Iwasawa, B. Mazur, A. Wiles and others. This is a far-reaching generalisation of the results of Herbrand and Ribet.

We quickly recall this theorem here. Let $F_\infty|F$ be a $\mathbb{Z}_p$-extension of totally real number fields, where $[F:\mathbb{Q}]$ is finite, with Galois group $G = G(F_\infty|F)$. Let X be the Galois group of the maximal abelian p-extension of F_∞, unramified outside p. Then X carries a natural $\Lambda(G)$-module structure, and X is a finitely generated torsion $\Lambda(G)$-module. Let $Q(G)$ be the total ring of quotients of $\Lambda(G)$. We need a certain interpolation property of p-adic zeta functions. This will be expressed by an element of $Q(G)$. We want to compare X with elements of $Q(G)$. Classically, this is done by the construction of a characteristic element $F_X \in \Lambda(G)$. The structure theory of finitely generated torsion $\Lambda(G)$-modules tells us that there is an exact sequence

$$0 \to \bigoplus_{i=1}^{r} \Lambda(G)/\Lambda(G)f_i \to X \to D \to 0,$$

where D is a $\Lambda(G)$-module of finite cardinality. We define the characteristic element $F_X := f_1 \cdots f_r$. Let $I(G)$ be the augmentation ideal of $\Lambda(G)$. Then the main conjecture of commutative Iwasawa theory says that the p-adic zeta function $\xi \in Q(G)$ exists and that

$$F_X \Lambda(G) = \xi I(G).$$

Now assume that G is a (not necessarily commutative) p-adic Lie group which can be written as a semidirect product $G = H \rtimes \Gamma$, with $\Gamma \cong \mathbb{Z}_p$. Put

$$\begin{aligned} S &:= S(G) \\ &:= \{f \in \Lambda(G) | \Lambda(G)/\Lambda(G)f \text{ is finitely generated as } \Lambda(H)\text{-module}\}. \end{aligned}$$

Following Kato [**25**], we will define a complex $C^\bullet$ that encodes the arithmetic information of the module X. We will show that this complex defines an element $[C^\bullet]$ of the group $K_0(\Lambda(G), \Lambda(G)_S)$. (If G is p-torsion free, we get $[C^\bullet] = [\mathbb{Z}_p] - [X]$.) The p-adic zeta function will be an element of $K_1(\Lambda(G)_S)$. We use the connecting homomorphism

$$\partial : K_1(\Lambda(G)_S) \to K_0(\Lambda(G), \Lambda(G)_S)$$

of K-theory to compare elements of $K_1(\Lambda(G)_S)$ with $[C^\bullet]$. This seems reasonable since in the commutative case we have $\partial(F_X) = [X]$. In our more general setting the main conjecture (as formulated by Kato) says that the p-adic zeta function $\xi \in K_1(\Lambda(G)_S)$ exists and that

$$\partial(\xi) = -[C^\bullet].$$

The aim of this paper is to work out the details of [**25**]. The basic strategy of the proof of the main conjecture given in this paper was developed by D. Burns and K. Kato [**25**]. Let $\mathcal{I}$ be a set of pairs (U, V), where U is an open normal subgroup of G and V is an open subgroup of H, such that V is a normal subgroup of U and U/V is commutative. There are natural maps

$$\begin{aligned} \theta : K_1(\Lambda(G)) &\to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times \\ \theta_S : K_1(\Lambda(G)_S) &\to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times_S. \end{aligned}$$

We will show that we can always find $\mathcal{I}$ such that θ becomes injective. Assume that there is a group Ψ_S with

$$\operatorname{im}\theta_S \subset \Psi_S \subset \textstyle\prod_{\mathcal{I}} \Lambda(U/V)^\times_S \qquad \text{and}$$
$$\Psi_S \cap \textstyle\prod_{\mathcal{I}} \Lambda(U/V)^\times = \operatorname{im}\theta.$$

Theorem (Theorem 2.40). *Assume that θ is injective. Let $\xi_{(U,V)} \in Q(U/V)$ be the p-adic zeta functions for U/V. Assume that $(\xi_{U,V})_{\mathcal{I}} \in \Psi_S$. Under further assumptions, see property 2.39, we get that the main conjecture is true for $F_\infty|F$.*

We will show that the assumptions of the above theorem are satisfied when G is a pro-p p-adic Lie group that is a quotient of the product of the p-adic Heisenberg group and a commutative p-adic Lie group. The proof of the assumptions splits in the algebraic problem to determine the kernel and the image of θ and to prove the inclusion $\operatorname{im}\theta_S \subset \Psi_S$, and the analytic problem to show $(\xi_{U,V})_{\mathcal{I}} \in \Psi_S$.

We will prove the algebraic part for groups without any arithmetic structure. We will construct the integral logarithms defined by R. Oliver (cf. [**34**]),

$$\begin{aligned}\mathscr{L} = \mathscr{L}_{P_1} &: K_1(\Lambda(P_1)) \to \mathbb{Z}_p[\![\mathrm{Conj}(P_1)]\!]\\ \mathscr{L}_S = \mathscr{L}_{P_2,S} &: K_1((\Lambda(P_2)_S)^\wedge) \to \mathbb{Z}_p[\![\mathrm{Conj}(P_2)]\!]_S^\wedge,\end{aligned}$$

where P_1, P_2 are pro-p p-adic Lie groups, where P_2 contains a subgroup H_2 such that $P_2/H_2 \cong \mathbb{Z}_p$, where $S := S(P_2)$, where $(\Lambda(P_2)_S)^\wedge$ denotes a completion of $\Lambda(P_2)_S$ (see definition 3.10) and where $\mathbb{Z}_p[\![\mathrm{Conj}(P_1)]\!]$ and $\mathbb{Z}_p[\![\mathrm{Conj}(P_2)]\!]_S^\wedge$ are certain quotient $\mathbb{Z}_p$-modules of $\Lambda(P_1)$ and $(\Lambda(P_2)_S)^\wedge$, respectively. We will use the integral logarithm to transfer the multiplicative homomorphisms θ and θ_S to the additive homomorphisms

$$\begin{aligned}\tau &: \mathbb{Z}_p[\![\mathrm{Conj}(P_1)]\!] \to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)\\ \tau_S &: \mathbb{Z}_p[\![\mathrm{Conj}(P_2)]\!]_S^\wedge \to \prod_{(U,V)\in\mathcal{I}} (\Lambda(U/V)_S)^\wedge .\end{aligned}$$

Assume $\mathcal{I} = \{(U_n, V_n) | n \in \underline{\underline{c}}\}$, where $\underline{\underline{c}} = \{0, 1, \dots, c\}$ if c is finite or $\underline{\underline{c}} = \mathbb{N}$ otherwise. Let I_n be the image of τ composed with the projection $\prod_m \Lambda(U_m/V_m) \to \Lambda(U_n/V_n)$. We define $I_{n,S} \subset \Lambda(U_n/V_n)_S$ similarly. We will show that τ and τ_S are injective and see that the images Ω, Ω_S of τ, τ_S are the sets of elements of $\prod_n I_n$, $\prod_n I_{n,S}$, respectively that satisfy certain natural compatibility relations with respect to the trace map. This allows us to construct a candidate Ψ for the image of θ as follows: We will show that $\sigma \mapsto \sigma^p$ for $\sigma \in U_n/V_n$ ($n \in \underline{\underline{c}}$) induces the ring homomorphisms

$$\begin{aligned}\varphi &: \Lambda(U_n/V_n) \to \Lambda(U_{n+1}/V_{n+1})\\ \varphi &: \Lambda(U_n/V_n)_S \to \Lambda(U_{n+1}/V_{n+1})_S.\end{aligned}$$

Let Ψ be the set of elements $(x_n)_n \in \prod_n \Lambda(U_n/V_n)$ that satisfy certain natural compatibility relations with respect to the norm map such that $x_n\varphi(x_{n-1})^{-1} \in I_n$ for all $n \geq 1$. We define $\Psi_S \subset \prod_n \Lambda(U_n/V_n)_S$ similarly. The following theorem is the main point of our proof that indeed $\Psi = \operatorname{im}\theta$. (The inclusion $\operatorname{im}(\theta) \subset \Psi$ follows from a similar diagram and then, by the five lemma, this inclusion is an identity.)

Theorem (Theorem 3.47). *The diagram*

$$\begin{tikzcd}
1 \arrow[r] & \mu_{p-1} \times (P_1)^{ab} \arrow[r] \arrow[d, equal] & K_1(\Lambda(P_1)) \arrow[r, "\mathscr{L}"] \arrow[d, "\theta"] & \mathbb{Z}_p[\![\mathrm{Conj}(P_1)]\!] \arrow[r] \arrow[d, "\tau", "\cong"'] & (P_1)^{ab} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mu_{p-1} \times (P_1)^{ab} \arrow[r] & \Psi \arrow[r, "\widetilde{\mathscr{L}}"] & \Omega \arrow[r] & (P_1)^{ab} \arrow[r] & 1
\end{tikzcd}$$

is commutative with exact rows, and hence $K_1(\Lambda(P_1)) \cong \Psi$.

We will use a similar construction as above to prove $\mathrm{im}\,(\theta_S) \subset \Psi_S$, but we will not need to calculate the kernel and cokernel of $\mathscr{L}_S$. The identity $\Psi_S \cap \prod_{n \in \underline{\underline{c}}} \Lambda(U_n/V_n)^\times = \Psi$ will be a simple corollary of the general theory.

We will now address the technical difficulties that arise in the above argument more closely. Our construction of the integral logarithm $\mathscr{L}$ starts with the investigation of the p-adic power series

$$\log(1-x) := \sum_{i \geq 1} \frac{x^i}{i} \in \mathbb{Q}_p[\![\mathrm{Conj}(P_1)]\!] \quad \text{for } x \in J(\Lambda(P_1))$$

$$\exp(x) := \sum_{i \geq 0} \frac{x^i}{i!} \in K_1(\Lambda(P_1), I) \quad \text{for } x \in I/[\Lambda(P_1), I],$$

where $J(\Lambda(P_1))$ is the Jacobson radical of $\Lambda(P_1)$ and I is a two sided ideal such that $I^p \subset pIJ(\Lambda(P_1))$. Convergence of these series follows similarly to the case of the usual p-adic logarithm and exponential map. The proof of the homomorphism property does not easily generalise to our situation. We will prove it in a more general setting. Let A be the ring of non-commutative power series in two indeterminates over a divisible commutative topological ring R, and let U be the subset of power series with constant term 1. Then U is a multiplicative group.

Proposition (Corollaries 3.31 and 3.34). *If the power series* $\log(x)$ *and* $\exp(x)$ *converge for* $x \in U$ *and* $x \in A$*, respectively, then the maps*

$$\log : U \to A/\overline{[A, A]} \quad \textit{and} \quad \exp : A \to U/\overline{[U, U]}$$

are homomorphisms.

We will then apply this to our case. Since $\Lambda(P_1)$ and $(\Lambda(P_2)_S)^\wedge$ are semi-local (cf. [**13**], [**8**], [**41**]), the K_1-groups of these rings are generated by their respective groups of units. We will define

$$\mathscr{L}_{P_1} := (1 - \frac{1}{p}\varphi) \circ \log$$

and show that the image of $\mathscr{L}_{P_1}$ is integral. We put $\mathscr{L}_{P_2,S}(x) := \mathscr{L}_{P_2}(x)$ for $x \in \Lambda(P_2)^\times$ and

$$\mathscr{L}_{P_2/W,S(P_2/W)}(x) := \frac{1}{p}\log(x^p\varphi(x)^{-1}),$$
$$x \in \left(\left(\Lambda(P_2/W)_{S(Z(P_2/W))}\right)^\wedge\right)^\times$$

for subgroups $W \subset P_2$ such that the centre of P_2/W is open. We show that this uniquely defines $\mathscr{L}_{P_2,S}$.

The fact that τ and τ_S are injective with image Ω and $\widehat{\Omega_S}$, respectively, and that the sets Ψ and Ψ_S are groups follow from explicit calculations with the generators of the Heisenberg group. A main ingredient in the proof of $\operatorname{im}\theta \subset \Psi$ and $\operatorname{im}\theta_S \subset \Psi_S$ is the equivalence

$$\tau \circ \mathscr{L}_{P_1}(x) \in \Omega \Leftrightarrow \theta(x) \in \Psi,$$

see proposition 3.57.

Let $F_n = F_\infty^{U_n}$ be the fixed field of U_n. The proof of the analytic side of the main conjecture is an application of deep results proven by P. Deligne and K. A. Ribet, cf. [**12**]. In particular [**12**] implies the existence of F_n-adic Hilbert Eisenstein series E_n such that the constant term of $2^{r(n)}E_n$, $r(n) = [F_n : \mathbb{Q}]$, is the p-adic zeta function ξ_n (see [**37**]). Let g_n be the restriction of E_n to the Hilbert modular variety of F. Let $\varphi(g_{n-1})$ be the restriction of the image of E_{n-1} under a map induced by the transfer homomorphism $U_{n-1}/V_{n-1} \to U_n/V_n$. We show that all non-constant coefficients of $g_n - \varphi(g_{n-1})$ lie in I_n. Then, by the q-expansion principle [**12**], the constant term $2^{-r(n)}\xi_n - 2^{-r(n-1)}\varphi(\xi_{n-1})$ is also in I_n. We show that this is equivalent to $\xi_n\varphi(\xi_{n-1})^{-1} \in 1 + I_n$, and hence $(\xi_n)_n \in \Psi_S$.

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CHAPTER 1

Preliminaries

1. Notation

- We assume that all rings are associative and have a unit element.
- For two topological groups G_1, G_2, we write $G_1 \trianglelefteq_o G_2$ if G_1 is an open normal subgroup of G_2.
- For a ring R, let $R[[T]]$ be the ring of formal power series in one variable T over R.
- Let R be a commutative topological ring and let G be a profinite group. We define
$$R[\![G]\!] := \varprojlim_{U \trianglelefteq_o G} R[G/U].$$
- Let p be an odd prime number. For a profinite group G let
$$\Lambda(G) := \mathbb{Z}_p[\![G]\!]$$
be the Iwasawa algebra of G.
- Let R be a ring. By an R-module M, we always mean a left R-module.
- For a group G let $[G,G] := \langle ghg^{-1}h^{-1} | g,h \in G\rangle$ be the commutator subgroup of G. Let R be a ring and let A be an R-algebra. We define the commutator R-algebra of A to be
$$[A,A] := \langle ab - ba | a,b \in A\rangle_R,$$
where $\langle * \rangle_R$ is the R-module generated by $*$.

2. Noncommutative Localisation

Let R be a ring and $X \subset R$ a multiplicatively closed subset (i. e. $1 \in X$ and $x, y \in X \Rightarrow xy \in X$).

Definition 1.1. A right ring of fractions for R with respect to X is a ring RX^{-1} with a ring homomorphism $\varphi : R \to RX^{-1}$ such that

- $\varphi(x) \in (RX^{-1})^\times$ for all $x \in X$
- $RX^{-1} = \{\varphi(a)\varphi(x)^{-1} | a \in R,\ x \in X\}$
- $\ker\varphi = \{r \in R | rx = 0 \text{ for some } x \in X\}$

A left ring of fractions is defined analogously. (Notation: $X^{-1}R$)

DEFINITION 1.2. X is a right Ore set in R if for each $x \in X$, $r \in R$ there exist $x' \in X$, $r' \in R$ such that $rx' = xr'$. A left Ore set is defined analogously.

PROPOSITION 1.3 ([**18**, theorem 10.3, proposition 10.6, 10.7]). *Assume R is Noetherian and X is a right and left Ore set in R. Then the right and left ring of fractions for R with respect to X exists and $RX^{-1} = X^{-1}R$. We denote it by R_X.*

PROPOSITION 1.4 ([**18**, corollary 10.16]). *Under the above assumptions, R_X is Noetherian.*

DEFINITION 1.5. An element y of a ring R is called regular if whenever $ay = 0$ or $ya = 0$ for some $a \in R$, then $a = 0$.

Let R be a commutative ring. Let X be the set of regular elements of R. Then X is multiplicatively closed. We write $Q(R) := R_X$ for the total ring of fractions of R, the localisation of R by X.

3. p-adic Lie Groups

A map $\varphi = (\varphi_j)_j : U \to \mathbb{Z}_p^m$, where $U \subset \mathbb{Z}_p^n$ is an open subset, is called (locally) analytic if we can locally represent it by power series over $\mathbb{Q}_p$: For all $y \in U$, $j = 1, \dots, m$, there are $h \in \mathbb{N}$, $a_\nu \in \mathbb{Q}_p$ such that

$$\varphi_j(y + p^h x) = \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \quad \forall\, x \in \mathbb{Z}_p^n$$

(with $x^\nu := \prod_i x_i^{\nu_i} \in \mathbb{Z}_p$).

A *p-adic analytic manifold* of dimension n is a topological space $\mathcal{M}$, such that there is an open cover $(U_i)_{i \in I}$ of $\mathcal{M}$ with homeomorphisms φ_i of U_i onto open subsets of $\mathbb{Z}_p^n$, such that $\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i \cap U_j)}$ are analytic. We call such a family $(U_i, \varphi_i)_{i \in I}$ an atlas of $\mathcal{M}$. A global atlas is an atlas with $\#I = 1$. A morphism of p-adic analytic manifolds is a map $f : \mathcal{M} \to \mathcal{N}$ such that $\psi_j \circ f \circ \varphi_i^{-1}$ is analytic where it is defined for atlases $(U_i, \varphi_i)_{i \in I}$ of $\mathcal{M}$ and $(V_j, \psi_j)_{j \in J}$ of $\mathcal{N}$ and every $i \in I$, $j \in J$.

DEFINITION 1.6. A *p-adic Lie group* (of dimension n) is a group object in the category of (n-dimensional) p-adic analytic manifolds. (I. e. it is a p-adic analytic manifold with a group law, where multiplication and inversion are analytic.) We denote the dimension of a p-adic Lie group by $\dim(G)$.

We call a pro-p group G powerful if $[G, G] \subset \overline{G^p}$ for $p \neq 2$ (respectively $[G, G] \subset \overline{G^4}$ for $p = 2$), where G^n is the group generated (as an abstract group) by the elements g^n, $g \in G$. We define inductively the lower p-series

$$P_1(G) := G, \quad P_{i+1}(G) := \overline{P_i(G)^p[P_i(G), G]}.$$

REMARK. For topologically finitely generated powerful groups, we have $P_i(G) = G^{p^{i-1}}$ for $i \geq 1$, see [**13**, theorem 3.6].

DEFINITION 1.7. A topologically finitely generated powerful pro-p group is *uniform* if the p-power map induces isomorphisms

$$P_i(G)/P_{i+1}(G) \xrightarrow{\cong} P_{i+1}(G)/P_{i+2}(G) \quad \text{for all } i \geq 1.$$

PROPOSITION 1.8 ([**13**, theorem 8.36]). *Let $H \leq_o G$ be an open uniform pro-p subgroup of the p-adic Lie group G. Then* $\dim(G)$ *is the minimal cardinality of a topological generating set of H.*

PROPOSITION 1.9 ([**13**, corollary 8.34]). *A topological group G is a compact p-adic Lie group if and only if there is an open normal uniform pro-p subgroup U of G.*

COROLLARY 1.10. *For any compact p-adic Lie group G, the Iwasawa algebra $\Lambda(G)$ is a semi-local ring.*

PROOF. By [**33**, prop. 5.2.16], $\Lambda(G)$ is semi-local if G has an open p-Sylow subgroup. □

COROLLARY 1.11. *For any compact p-adic Lie group G, the Iwasawa algebra $\Lambda(G)$ is right and left Noetherian.*

PROOF. By proposition 1.9, there is an an open normal uniform pro-p subgroup U of G. Then $\Lambda(U)$ is right and left Noetherian (cf. [**13**, corollary 7.25]). But this implies that $\Lambda(G) \cong \bigoplus_{G/U} \Lambda(U)$ is right and left Noetherian as a module over $\Lambda(U)$ and therefore over $\Lambda(G)$. □

The following proposition is the reason for increased technical difficulties when working with groups with elements of order p.

PROPOSITION 1.12 ([**43**]). *Let G be a compact p-adic Lie group. The p-cohomological dimension $cd_p(G)$ is finite if and only if G does not contain an element of order p.*

PROPOSITION 1.13 ([**27**, V.2.5.8]). *Let G be a p-adic Lie group with cohomological dimension $cd(G) < \infty$. Then G is a Poincaré group of dimension* $\dim(G)$. *In particular, $cd(G) =$* $\dim(G)$.

PROPOSITION 1.14 ([**48**, corollary 2.8]). *Let G be a compact p-adic Lie group. There is an open subgroup U of G such that $\Lambda(U)$ is an integral domain.*

4. K-Theory

Let $\mathcal{C}$ be a full additive subcategory of an abelian category, closed under extensions. We assume that $\mathcal{C}$ has a small skeleton $\mathcal{C}_{\text{sk}}$ (i. e., $\mathcal{C}_{\text{sk}}$ is a full subcategory of $\mathcal{C}$ such that the class of objects $Ob(\mathcal{C}_{\text{sk}})$ is a set and the inclusion $\mathcal{C}_{\text{sk}} \hookrightarrow \mathcal{C}$ is an equivalence of categories). We define the following K-groups by specifying generators and relations of an abelian group. We denote the group law of K_0-groups additively and of K_1-groups multiplicatively.

$\mathbf{K_0}(\mathcal{C})$ Generators: The objects M of $\mathcal{C}_{\text{sk}}$.
Relations:

- $[M] = [N]$ if $M \cong N$,
- $[M_2] = [M_1] + [M_3]$ if there is an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$.

$\mathbf{K_1}(\mathcal{C})$ Generators: The pairs (M, f), where M is an object of $\mathcal{C}_{\text{sk}}$ and f an automorphism of M.
Relations:

- $[(M, gf)] = [(M, f)] \cdot [(M, g)]$,
- $[(M_2, f_2)] = [(M_1, f_1)] \cdot [(M_3, f_3)]$ if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & 0. \end{tikzcd}$$

Let $\mathcal{P}(R)$ be the category of finitely generated projective left R-modules. We write $K_0(R) := K_0(\mathcal{P}(R))$ and $K_1(R) = K_1(\mathcal{P}(R))$.

Remarks.

- Let $\mathcal{P}_{\text{right}}(R)$ be the category of finitely generated projective right R-modules Then
$$K_i(R) = K_i(\mathcal{P}_{\text{right}}(R))$$
for $i = 1, 2$. This follows from $K_i(R) = K_i(R^{op})$, where R^{op} is the opposite ring of R (cf. [**28**, prop. 9.10]) and from the isomorphism of categories $\mathcal{P}(R^{op}) \cong \mathcal{P}_{\text{right}}(R)$.
- For $x \in R^\times$, let
$$m_x : R \to R, \quad r \mapsto rx$$

be the left R-module homomorphism of right multiplication by x. We can define the natural homomorphism

$$[\,\cdot\,]_R : R^\times \to K_1(R), \quad x \mapsto [x]_R := [(R, m_x)].$$

Similarly, we get for every $n \geq 1$ the homomorphisms

$$[\,\cdot\,]_R : GL_n(R) \to K_1(R), \quad x \mapsto [x]_R := [(R^n, r \mapsto rx)].$$

We give an alternative definition of $K_1(R)$: For $a \in R$, let $e_{ij}(a) \in GL_n(R)$ be the matrix with 1's on the diagonal, with an a in the (i,j)-slot and 0's elsewhere. Define

$$E_n(R) := \langle e_{ij}(a) | \, a \in R, \ i,j \in \{1,\ldots,n\}, \ i \neq j \rangle$$

to be the subgroup of $GL_n(R)$ generated by such matrices. We define injections $E_n(R) \hookrightarrow E_{n+1}(R)$ and $GL_n(R) \hookrightarrow GL_{n+1}(R)$ by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ and set $E_\infty(R) := \varinjlim_n E_n(R)$, $GL_\infty(R) := \varinjlim_n GL_n(R)$, where the limits are taken with respect to the above injections.

Proposition 1.15 ([**38**, theorem 3.1.7]). *Let R be a ring. Then*

$$K_1(R) \cong GL_\infty(R)/E_\infty(R).$$

Proposition 1.16 (Whitehead Lemma, [**3**, V.1.5 and V.1.9]). *For $n \geq 3$,*

$$E_n(R) = [E_n(R), E_n(R)]$$

and for any $n \geq 1$,

$$[GL_n(R), GL_n(R)] \subset E_{2n}(R).$$

Hence

$$E_\infty(R) = [E_\infty(R), E_\infty(R)] = [GL_\infty(R), GL_\infty(R)].$$

Thus we have isomorphisms

$$K_1(R) \cong GL_\infty(R)/E_\infty(R) \cong GL_\infty(R)/[GL_\infty(R), GL_\infty(R)],$$

where the inverse of the first isomorphism is defined by $[f] \mapsto [(R^n, f)]$ for $f \in GL_n(R) \subset GL_\infty(R)$, $n \geq 1$.

Define the Jacobson radical $J(R)$ to be the intersection of all left maximal ideals of R. A ring R is called *semi-local* if the number of maximal ideals of R is finite. Equivalently, $R/J(R)$ is semisimple or artinian (see [**3**, III.2]). Therefore, a ring R is semi-local if and only if $R/J(R)$ is isomorphic to a finite product of matrix rings over division rings.

PROPOSITION 1.17 (Stable range theorem).

- *Surjective stability theorem (Bass): If R is a finitely generated module over an integral domain or if R is semi-local, then the canonical homomorphisms*
$$GL_d(R) \to K_1(R), \quad x \mapsto [x]_R$$
are surjective for $d \geq 1$.
- *Injective stability theorem (Bass, Vaserstein): For R as above, there are isomorphisms*
$$K_1(R) \cong GL_d(R)/E_d(R)$$
$$K_1(R) \cong GL_{d+1}(R)/[GL_{d+1}(R), GL_{d+1}(R)]$$
for any $d \geq 2$.
- *If R is semi-local and $R/J(R)$ is isomorphic to a product of full matrix rings over division algebras such that none of these matrix rings has order 16 and that no more than one of these matrix rings has order 2 or if R is a commutative semi-local ring or if R is a local ring, then*
$$K_1(R) \cong R^\times/[R^\times, R^\times]$$
(Vaserstein, Dieudonné, Bass).

PROOF. See [**3**, V.9.1, V.9.2], [**11**, 40.41, 40.44], [**38**, corollary 2.2.6] and [**47**]. □

COROLLARY 1.18. *For any compact p-adic Lie group G and $d \geq 2$,*
$$K_1(\Lambda(G)) \cong GL_d(\Lambda(G))/E_d(\Lambda(G))$$
$$K_1(\Lambda(G)) \cong \Lambda(G)^\times/[\Lambda(G)^\times, \Lambda(G)^\times].$$

PROOF. The the fact that the assumptions of proposition 1.17 are satisfied for $\Lambda(G)$ follows from corollary 1.10 and the fact that a $\mathbb{Z}_p$-module ($p \neq 2$) has no quotient of 2-power order. □

PROPOSITION 1.19. *Let R be a ring and assume that there is a two-sided ideal $I \subset R$ such that R/I^n is finite of order a power of p for any $n \geq 1$ and such that the canonical homorphism $R \to \varprojlim_n R/I^n$ is an isomorphism. Let $\mathcal{L}$ be a set of ideals contained in $J(R)$ such that R/L is finite for $L \in \mathcal{L}$, $\bigcap_{L\in\mathcal{L}} L = 0$ and for $L_1, L_2 \in \mathcal{L}$, there is $L_3 \in \mathcal{L}$ with $L_3 \subset L_1 \cap L_2$. Then there is an isomorphism*

$$K_1(R) \cong \varprojlim_{L\in\mathcal{L}} K_1(R/L).$$

REMARK. The above assumptions on R are equivalent to the fact that $R/J(R)^n$ is finite for any $n \geq 1$ and that the canonical homomorphism $R \to \varprojlim_n R/J(R)^n$ is an isomorphism. In this case, R is semi-local. (See [**16**, lemma 1.4.4].)

PROOF OF PROPOSITION 1.19. By [**16**, prop. 1.5.1],

$$K_1(R) \cong \varprojlim_n K_1(R/J(R)^n).$$

For $n \geq 1$, there is $L \in \mathcal{L}$ with $L \subset J(R)^n$ (since $R/J(R)^n$ is finite by the above remark) and for $L \in \mathcal{L}$, $J(R)^n \subset L$ for some $n \geq 1$ (since $\bigcap_n J(R)^n = 0$ by the above remark). Hence the proposition follows. □

LEMMA 1.20. *Let I be a directed partially ordered set and let (R_i, f_{ij}) be an inverse system of rings that satisfy the Mittag-Leffler condition (i. e. for each $k \in I$, there exists $j \in I$ such that the image of $f_{ki} : R_i \to R_k$ equals the image of $f_{kj} : R_j \to R_k$ for all $i \geq j$). Put $R := \varprojlim_I R_i$. Assume that for some $d \geq 2$,*

$$K_1(R_i) = GL_d(R_i)/E_d(R_i) \quad \textit{for all } i \in I.$$

Then the canonical homomorphism

$$K_1(R) \to \varprojlim_{i\in I} K_1(R_i)$$

is surjective.

REMARK. The assumptions of the lemma are satisfied when R is a semi-local ring and the homomorphisms $f_{ij} : R_j \to R_i$ are surjective.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} & & GL_d(R) \arrow[r] \arrow[d] & K_1(R) \arrow[d] & \\ 1 \arrow[r] & \varprojlim_{i\in I} E_d(R_i) \arrow[r] & \varprojlim_{i\in I} GL_d(R_i) \arrow[r] & \varprojlim_{i\in I} K_1(R_i) \arrow[r] & 1, \end{tikzcd}$$

The Mittag-Leffler condition is satisfied for the sets of generators

$$\{e_{kl}(a)|a \in R_i,\ k,l \in \{1,\dots,d\},\ k \neq l\}$$

of $E_d(R_i)$, $i \in I$. Hence it is satisfied for $E_d(R_i)$, $i \in I$. Using the assumption on the K_1-groups we get that the lower row is exact.

We show that the the left vertical arrow

$$GL_d(R) \to \varprojlim_{i\in I} GL_d(R_i) \tag{1}$$

in the above diagram is an isomorphism. (We only need the fact that it is surjective.) Then the right vertical arrow is surjective. Clearly, there is a ring isomorphism

$$M_d(R) \to \varprojlim_{i\in I} M_d(R_i). \tag{2}$$

This shows injectivity of the homomorphism (1).

Let $(x_i)_i \in \varprojlim_{i\in I} GL_d(R_i)$ and let $x \in M_d(R)$ be an inverse image of $(x_i)_i$ under the isomorphism (2). For $i,j \in I$ with $i \geq j$, we get

$$f_{ij}(x_j^{-1}) = f_{ij}(x_j)^{-1} = x_i^{-1} \in M_d(R_i)$$

Hence

$$y := (x_i^{-1})_{i\in I} \in \varprojlim_{i\in I} M_d(R_i).$$

Then $xy = yx = 1$ and hence $x \in GL_d(R)$. Thus the homomorphism (1) is surjective. □

Let $R \to R'$ be a ring homomorphism. We state Swan's definition of the relative K_0-group (cf. [**48**, §3]):

$\mathbf{K_0(R,R')}$ Generators: (M,N,f) where M, N are objects of $\mathcal{P}(R)_{\mathrm{sk}}$ and $f : R' \otimes_R M \xrightarrow{\cong} R' \otimes_R N$ an isomorphism in $\mathcal{P}(R')$.
Relations:

- $[(L,N,gf)] = [(L,M,f)] + [(M,N,g)]$

- $[(M_2, N_2, f_2)] = [(M_1, N_1, f_1)] + [(M_3, N_3, f_3)]$ if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R' \otimes_R M_1 \arrow[r] \arrow[d, "f_1"] & R' \otimes_R M_2 \arrow[r] \arrow[d, "f_2"] & R' \otimes_R M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\ 0 \arrow[r] & R' \otimes_R N_1 \arrow[r] & R' \otimes_R N_2 \arrow[r] & R' \otimes_R N_3 \arrow[r] & 0 \end{tikzcd}$$

$\mathbf{K_1(R, I)}$ Let R be a ring and let $I \subset R$ be a two-sided ideal. We define

$$\begin{aligned} GL_n(R, I) &:= \ker(GL_n(R) \to GL_n(R/I)) \\ E_n(R, I) &:= \langle e_{ij}(a) |\, a \in I,\ i, j \in \{1, \dots, n\},\ i \neq j \rangle \end{aligned}$$

and set $GL_\infty(R, I) := \varinjlim_n GL_n(R, I)$, $E_\infty(R, I) := \varinjlim_n E_n(R, I)$.

Proposition 1.21 (Relative Whitehead Lemma, [**3**, V.1.5 and V.1.9]). *For $n \geq 3$,*

$$E_n(R, I) = [E_n(R), E_n(R, I)]$$

and for any $n \geq 1$,

$$[GL_n(R), GL_n(R, I)] \subset E_{2n}(R, I).$$

Hence

$$E_\infty(R, I) = [E_\infty(R), E_\infty(R, I)] = [GL_\infty(R), GL_\infty(R, I)].$$

We define the relative K_1-group

$$\begin{aligned} K_1(R, I) &:= GL_\infty(R, I)/E_\infty(R, I) \\ &= GL_\infty(R, I)/[GL_\infty(R), GL_\infty(R, I)]. \end{aligned}$$

Proposition 1.22 (Stable range theorem for relative K-theory). *Let R be a ring and let $I \subset R$ be a two-sided ideal. Assume that R is a finitely generated module over an integral domain or that R is semi-local or that I is contained in the Jacobson radical of R.*

- *Surjective stability theorem: The canonical homomorphisms*

$$GL_d(R, I) \to K_1(R, I)$$

are surjective for $d \geq 1$. If R is commutative, then there is an isomorphism $1 + I \cong K_1(R, I)$.

- *Injective stability theorem: There are isomorphisms*

$$\begin{aligned} K_1(R, I) &\cong GL_d(R, I)/E_d(R, I) \\ K_1(R, I) &\cong GL_{d+1}(R, I)/[GL_{d+1}(R), GL_{d+1}(R, I)] \end{aligned}$$

for any $d \geq 2$.

Proof. See [**3**, V.9.1, V.9.2] and [**11**, 40.41, 44.17]. $\square$

Let $\varphi : R \to S$ and $\varphi' : R' \to S'$ be ring homomorphisms and assume that the diagram

$$\begin{tikzcd} R' \arrow[r, "\varphi'"] & S' \\ R \arrow[u] \arrow[r, "\varphi"] & S, \arrow[u] \end{tikzcd}$$

is commutative. We define the K-groups $K_0(R, R')$ and $K_0(S, S')$ corresponding to the vertical arrows in the above diagram. We define the following homomorphisms of K-groups:

- $\varphi_* : K_0(R) \to K_0(S), \quad [M] \mapsto [S \otimes_\varphi M]$
- $\varphi_* : K_1(R) \to K_1(S), \quad [(M, f)] \mapsto [(S \otimes_\varphi M, \mathrm{id}_S \otimes_\varphi f)]$
- $(\varphi, \varphi')_* : K_0(R, R') \to K_0(S, S')$,
  $[(M, N, f)] \mapsto [(S \otimes_\varphi M, S \otimes_\varphi N, \mathrm{id}_{S'} \otimes_{\varphi'} f)]$

It is not difficult to see that these homomorphisms are well-defined: If $M \in Ob(\mathcal{P}(R))$, then there is an R-module L such that $M \oplus L \cong R^n$. But then

$$(S \otimes_\varphi M) \oplus (S \otimes_\varphi L) \cong S \otimes_\varphi (M \oplus L) \cong S^n,$$

and hence $S \otimes_\varphi M \in Ob(\mathcal{P}(S))$. Since the tensor product commutes with direct sums, the above maps are homomorphisms.

Lemma 1.23 (Morita invariance, [**28**, 6.7, 9.11]). *The homomorphisms*

$$\begin{array}{rcl} K_0(R) & \cong & K_0(M_n(R)) \\ [M] & \mapsto & [R^n \otimes_R M] \\ [R^n \otimes_{M_n(R)} N] & \leftarrow\!\shortmid & [N] \end{array}$$

and

$$\begin{array}{rcl} K_1(R) & \cong & K_1(M_n(R)) \\ [(M, f)] & \mapsto & [(R^n \otimes_R M, \mathrm{id}_{R^n} \otimes_R f)] \\ [(R^n \otimes_{M_n(R)} N, \mathrm{id}_{R^n} \otimes_{M_n(R)} g)] & \leftarrow\!\shortmid & [(N, g)] \end{array}$$

are isomorphisms.

We now assume that $S \in Ob(\mathcal{P}(R))$ and $S' \in Ob(\mathcal{P}(R'))$. For any S-module M, let $r_R M$ be the R-module obtained from M by restriction of scalars. (This is the abelian group M with scalar multiplication $r \cdot m := \varphi(r)m$ for $r \in R$, $m \in M$.) We define:

- $\mathrm{Tr} : K_0(S) \to K_0(R), \quad [M] \mapsto [r_R M]$
- $\mathrm{N} : K_1(S) \to K_1(R), \quad [(M, f)] \mapsto [(r_R M, f)]$
- $\mathrm{Tr} : K_0(S, S') \to K_0(R, R'), \quad [(M, N, f)] \mapsto [(r_R M, r_R N, f)]$

Clearly, the composition of two norm (trace) maps is again a norm (trace) map.

Remark. Assume that S is a finitely generated free R-algebra of dimension n. For $x \in R^\times$, we have

$$\mathrm{N}([\varphi(x)]_S) = [x]_R^n \in K_1(R),$$

where φ, $[-]_S$ and $[-]_R$ are defined as above.

We give an explicit description of the above norm map when S is a semi-local ring that is finitely generated and free over R and R is commutative (e. g. if $S = \Lambda(W_1)$, $R = \Lambda(W_2)$ are Iwasawa algebras, where W_2 is a commutative open subgroup of the compact p-adic Lie group W_1).

We work with the matrix description $K_1(R) \cong GL_\infty(R)/E_\infty(R)$. Let $\{\nu_i\}_{i=1,\ldots,n}$ be a basis of the left R-module S. For $x \in S^\times$, we define the elements $y_{ij} \in R$ by the equations

$$\nu_j x = \sum_{i=1}^{n} y_{ij}\nu_i \qquad \text{for } j = 1, \ldots, n.$$

Let $y = (y_{ij})_{ij} \in GL_n(R)$ be the corresponding matrix. Then

$$\mathrm{N}([x]_S) = [y]_R = [\det y]_R = [\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} y_{i,\sigma(i)}]_R. \tag{3}$$

The first identity follows from the definition of the norm map. The second equation follows from [**50**, lemma III.1.4].

Let R, R', S, S' be rings as above. We define the homomorphisms

$$\partial : K_1(R') \to K_0(R, R'), \quad [(R' \otimes_R M, f)] \mapsto [(M, M, f)]$$
$$\lambda : K_0(R, R') \to K_0(R), \quad [(M, N, f)] \mapsto [M] - [N],$$

where M, N are finitely generated projective R-modules. (Note that by proposition 1.15,

$$\begin{aligned} K_1(R') &= \langle [((R')^n, f)] | n \in \mathbb{N},\ f \in \mathrm{End}_{R'}((R')^n) \rangle \\ &\subset \langle [(R' \otimes_R M, f)] | M \in Ob(\mathcal{P}(R)_{\mathrm{sk}}),\ f \in \mathrm{End}_{R'}(R' \otimes_R M) \rangle \\ &\subset K_1(R'), \end{aligned}$$

and hence ∂ is well-defined.)

We get the following exact sequence (cf. [**48**, §3] or [**45**, theorem 15.5]):

$$K_1(R) \to K_1(R') \xrightarrow{\partial} K_0(R,R') \xrightarrow{\lambda} K_0(R) \to K_0(R'). \tag{4}$$

Remark. Assume that $S' \cong R' \otimes_R S$ and that $S \to S'$ and $\varphi' : R' \to S'$ are the natural ring homormorphisms. Then there is a commutative diagram

$$\begin{tikzcd} K_1(S) \arrow[r] \arrow[d, "\mathrm{N}"] & K_1(S') \arrow[r, "\partial"] \arrow[d, "\mathrm{N}"] & K_0(S,S') \arrow[r, "\lambda"] \arrow[d, "\mathrm{Tr}"] & K_0(S) \arrow[r] \arrow[d, "\mathrm{Tr}"] & K_0(S') \arrow[d, "\mathrm{Tr}"] \\ K_1(R) \arrow[r] & K_1(R') \arrow[r, "\partial"] & K_0(R,R') \arrow[r, "\lambda"] & K_0(R) \arrow[r] & K_0(R'). \end{tikzcd}$$

Lemma 1.24. *Let $D^\bullet$ be a bounded exact sequence of projective R-modules. Then the kernels and images of all coboundary operators $d^i : D^i \to D^{i+1}$ are projective and*

$$D^i \cong \ker d^i \oplus \operatorname{im} d^i.$$

Proof. This is clearly true for i sufficiently large. Assume we have shown it for some fixed $i \in \mathbb{Z}$. Then $\ker d^i = \operatorname{im} d^{i-1}$ is projective. From the surjection $D^{i-1} \to \operatorname{im} d^{i-1}$, we get the decomposition $D^{i-1} \cong \ker d^{i-1} \oplus \operatorname{im} d^{i-1}$. Hence $\ker d^{i-1}$ is projective. □

We now assume that $R' = R_S$ is the localisation of a Noetherian ring R by a (right and left) Ore set $S \subset R$. An R-module M is defined to be S-torsion if $R_S \otimes_R M = 0$. Let $\mathcal{H}_S^R$ be the category whose objects are finitely generated S-torsion R-modules which have a finite resolution by finitely generated projective R-modules. Let $\mathcal{C}_S^R$ be the category of bounded cochain complexes $D^\bullet$ of finitely generated projective R-modules such that $R_S \otimes_R D^\bullet$ is acyclic (i. e. $R_S \otimes_R D^\bullet$ is an exact sequence).

Proposition 1.25. *Assume that $\partial : K_1(R_S) \to K_0(R, R_S)$ is surjective. Then there are natural isomorphisms*

$$K_0(R, R_S) \cong K_0(\mathcal{H}_S^R) \cong K_0(\mathcal{C}_S^R).$$

We first prove the following lemma:

Lemma 1.26. *Let M be a finitely generated free R-bimodule.*

(1) *Let $f : R_S \otimes_R M \cong R_S \otimes_R M$ be an isomorphism of left R_S-modules. Then there is $s \in S$ such that $f(M)s \subset M$.*
(2) *M/Ms is a left S-torsion R-module.*

Proof.

(1) By [**29**, 2.1.8], for every element $x \in R_S^n$, there is $s \in S$ such that $xs \in R^n \subset R_S^n$. Let $\{m_i\}_i$ be a basis of generators of M and let $s_i \in S$ be elements such that $f(m_i)s_i \in M$. By [**29**, 2.1.8], there are elements $s \in S$ and $r_i \in R$ such that $s = s_i r_i$. Then $f(M)s \subset M$.
(2) By tensoring the exact sequence
$$Rs \to R \to R/Rs \to 0$$
with R_S, we get the exact sequence
$$R_S \otimes_R Rs \to R_S \to R_S \otimes_R R/Rs \to 0$$
of R_S-modules. Since $s \in R_S^\times$, the left arrow is surjective. Hence R/Rs is a left S-torion R-module. There is an R-bimodule isomorphism $\bigoplus_i R \cong M$. This defines an isomorphism $\bigoplus_i R/Rs \cong M/Ms$ of left R-modules. By tensoring with R_S, we get
$$R_S \otimes_R M/Ms = 0.$$
□

Proof of proposition 1.25. We first describe generating elements of these groups. Let M be a finitely generated free R-module and let

$$f : R_S \otimes_R M \to R_S \otimes_R M$$

be an isomorphism of R_S-modules. Since ∂ is surjective, $K_0(R, R_S)$ is generated by elements of the form $[(M, M, f)]$. We endow M with the natural R-bimodule structure.

By lemma 1.26, there is $s \in S$ such that $f(M)s \subset M$. Hence we can write $f = gs^{-1}$ with $s \in S$ and $g : M \to M$ an R-homomorphism. Let $H \in Ob(\mathcal{H}_S^R)$ be an object of $\mathcal{H}_S^R$ with a resolution $0 \leftarrow H \leftarrow P_\bullet$ of finitely generated projective R-modules. Then clearly $P_\bullet \in Ob(\mathcal{C}_S)$ (via reindexing: $P^i = P_{-i}$).

Let $D^\bullet$ be an object of $\mathcal{C}_S^R$ and let

$$d_\otimes^i : R_S \otimes_R D^i \to R_S \otimes_R D^{i+1}$$

be the map induced by the coboundary operator d^i of $D^\bullet$. By lemma 1.24,

$$R_S \otimes_R D^i \cong \ker d_\otimes^i \oplus \operatorname{im} d_\otimes^i \cong \operatorname{im} d_\otimes^{i-1} \oplus \operatorname{im} d_\otimes^i.$$

We define an isomorphism φ by the following commutative diagram:

$$\begin{tikzcd}
\bigoplus_{i\in\mathbb{Z}} R_S \otimes_R D^{2i} \arrow[r, "\varphi"] \arrow[d, "\wr"] & \bigoplus_{i\in\mathbb{Z}} R_S \otimes_R D^{2i+1} \arrow[d, "\wr"] \\
\bigoplus_{i\in\mathbb{Z}} \left(\operatorname{im} d_\otimes^{2i} \oplus \operatorname{im} d_\otimes^{2i+1}\right) \arrow[r, equal] & \bigoplus_{i\in\mathbb{Z}} \left(\operatorname{im} d_\otimes^{2i+1} \oplus \operatorname{im} d_\otimes^{2i+2}\right)
\end{tikzcd}$$

The isomorphisms of the lemma are given explicitly as follows:

$$K_0(R, R_S) \to K_0(\mathcal{H}_S^R), \quad [(M, M, f)] \mapsto [M/g(M)] + [M/Ms]$$
$$K_0(\mathcal{H}_S^R) \to K_0(\mathcal{C}_S^R), \quad [H] \mapsto [P_\bullet]$$
$$K_0(\mathcal{C}_S^R) \to K_0(R, R_S), \quad [D^\bullet] \mapsto [\bigoplus_{i\in\mathbb{Z}} D^{2i}, \bigoplus_{i\in\mathbb{Z}} D^{2i+1}, \varphi]$$

Since $R_S \otimes_R g(M) \cong R_S \otimes_R M$, we get that $M/g(M)$ is S-torsion by tensoring the exact sequence

$$0 \to g(M) \to M \to M/g(M) \to 0$$

of left R-modules with R_S. By lemma 1.26 (2), M/Ms is a left S-torsion R-module. Hence it is clear that the above homomorphisms are well-defined.

We define the following homomorphisms corresponding to the above maps ∂ and λ:

$$\partial : K_1(R_S) \to K_0(\mathcal{H}_S^R), \quad [(R_S \otimes_R M, f)] \mapsto [M/g(M)] + [M/Ms],$$
$$\partial : K_1(R_S) \to K_0(\mathcal{C}_S^R),$$
$$[(R_S \otimes_R M, f)] \mapsto [M \xrightarrow{g} M] + [M \xrightarrow{-\cdot s} M],$$
$$\lambda : K_0(\mathcal{H}_S^R) \to K_0(R), \quad [H] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i [P_i],$$
$$\lambda : K_0(\mathcal{C}_S^R) \to K_0(R), \quad [D^\bullet] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i [D^i].$$

By construction, the homomorphisms of the lemma commute with the ∂'s and λ's. Hence, for $A, B \in \{K_0(R, R_S), K_0(\mathcal{H}_S^R), K_0(\mathcal{C}_S^R)\}$, the diagram

$$\begin{tikzcd}
K_1(R) \arrow[r] \arrow[d, equal] & K_1(R_S) \arrow[r, "\partial"] \arrow[d, equal] & A \arrow[r, "\lambda"] \arrow[d] & K_0(R) \arrow[r] \arrow[d, equal] & K_0(R_S) \arrow[d, equal] \\
K_1(R) \arrow[r] & K_1(R_S) \arrow[r, "\partial"] & B \arrow[r, "\lambda"] & K_0(R) \arrow[r] & K_0(R_S)
\end{tikzcd}$$

is commutative with exact rows. The five lemma implies $A \cong B$. $\square$

Let M be a left R-module and let N be a right R-module. An R-antihomomorphism is a group homomorphism $f : M \to N$ with the additional property $f(m)r = f(rm) \in N$ for $r \in R$ and $m \in M$.

Let

$$\mathrm{Tr} : M_n(R) \to R, \quad (a_{ij})_{ij} \mapsto \sum_{i=1}^{n} a_{ii}$$

be the trace map. Clearly, Tr is a left and right R-module homomorphism.

Definition 1.27. Let S be a ring and let R be a subring of S. Assume that S is a finitely generated free left R-module. Let $\mathrm{End}_R(S)$ be the right R-module of left R-endomorphisms of S. We define the trace antihomomorphism

$$\mathrm{Tr} : S \to R$$

(from the left R-module S to the right R-module R) to be the composition of the maps

$$S \to \mathrm{End}_R(S) \cong M_n(R) \xrightarrow{\mathrm{Tr}} R,$$

where the first map is the antihomomorphism that assigns to $x \in S$ the homomorphism of right multiplication by x, the central isomorphism is the natural isomorphism of right R-modules and the map on the right hand side is the right R-module homomorphism defined above.

Remarks.

- Let $\{\nu_i\}_{i=1,\dots,n}$ be a left R-basis of S. For $x \in S$, we define the elements $y_{ij} \in R$ by the equations

$$\nu_j x = \sum_{i=1}^{n} y_{ij} \nu_i \qquad \text{for } j = 1, \dots n.$$

Let $y = (y_{ij})_{ij} \in M_n(R)$ be the corresponding matrix. Then

$$\mathrm{Tr}(x) = \mathrm{Tr}(y) = \sum_{i=1}^{n} y_{ii} \in R.$$

(Compare this with the explicit description of the norm map.)

- Since $\mathrm{Tr}(xy) \equiv \mathrm{Tr}(yx) \bmod [R,R]$ for $x, y \in M_n(R)$ and since

$$\lambda r = r\lambda \qquad \text{for all } \lambda \in R, \quad r \in R/[R,R],$$

the trace map induces the left R-module homomorphism

$$\mathrm{Tr} : S/[S,S] \to R/[R,R].$$

If R and S are topological R-modules, we get the continuous homomorphism

$$\mathrm{Tr} : S/\overline{[S,S]} \to R/\overline{[R,R]}.$$

Now assume $\nu_i \in S^\times$ for $i = 1, \ldots, n$. Let

$$\pi_i : S \cong \bigoplus_{j=1}^{n} R\nu_j \to R\nu_i \cong R$$

be the projection map. Then $y_{ij} = \pi_i(\nu_j x) = \pi_0(\nu_j x \nu_i^{-1})$ and hence

$$\mathrm{Tr}(x) = \sum_{i=1}^{n} y_{ii} = \sum_{i=1}^{n} \pi_0(\nu_i x \nu_i^{-1}). \tag{5}$$

5. Homological Algebra

In this section, we only fix notation. For definitions and proofs, we refer to the literature ([**49**] and [**33**]).

Let $\mathcal{A}$ be an abelian category. We write $K(\mathcal{A})$ for the homotopy category of chain complexes in $\mathcal{A}$ and $D(\mathcal{A})$ for the derived category of $\mathcal{A}$. (We obtain $D(\mathcal{A})$ from $K(\mathcal{A})$ by localising quasi-isomorphisms.) We denote by $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ the full subcategories of $D(\mathcal{A})$ that arise from the category of cochain complexes that are bounded below or above, respectively. (For the definition of these objects, see [**49**, §10.1 -10.4].)

Let $\mathcal{A}$, $\mathcal{B}$ be two abelian categories and let $F : K^+(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. Let

$$RF : D^+(\mathcal{A}) \to D(\mathcal{B})$$

be the *(total) right derived functor* of F (cf. [**49**, def. 10.5.1]). We dually write LF for the *(total) left derived functors* of a right exact functor F.

Assume that $\mathcal{A}$ has enough injectives (respectively projectives). Let $F : \mathcal{A} \to \mathcal{B}$ be an additive left (respectively right) exact functor. We denote the induced morphism of triangulated categories $K^+(\mathcal{A}) \to K(\mathcal{B})$ also by F. Let X be an object of $\mathcal{A}$ and $X^\bullet$ be the corresponding complex concentrated in 0. When R^n (respectively L^n) are the classical right (respectively left) derived functors, we get

$$H^n(RF(X^\bullet)) = R^nF(X)$$
$$(\text{respectively } H^n(LF(X^\bullet)) = L^nF(X))$$

(see [**49**, Corollary 10.5.7]).

We assume that $\mathcal{A}$ has enough injectives. Let $A^\bullet$ be a cochain complex in $\mathcal{A}$ and consider the right derived functor

$$R\mathrm{Hom}^\bullet_{\mathcal{A}}(A^\bullet, -) : D^+(\mathcal{A}) \to D(\mathbf{Ab}),$$

where $\mathbf{Ab}$ is the category of abelian groups (cf. [**49**, def. 10.7.2]). If G is a p-adic Lie group and A, B are left $\Lambda(G)$-modules, then $\mathrm{Hom}_{\mathbb{Z}_p}(A, B)$ is a left $\Lambda(G)$-module, where the $\Lambda(G)$-module structure is given by $(gf)(a) := gf(g^{-1}a)$ for $f \in \mathrm{Hom}_{\mathbb{Z}_p}(A, B)$, $a \in A$ and $g \in G$. Hence, for a cochain complex $A^\bullet$ of left $\Lambda(G)$-modules, we may define the right derived functor

$$R\mathrm{Hom}^\bullet_{\mathbb{Z}_p}(A^\bullet, -) : D^+(\mathcal{B}) \to D(\mathcal{B}),$$

where $\mathcal{B}$ is a suitable category of left $\Lambda(G)$-modules.

For a cochain complex $B^\bullet$ in $\mathcal{A}$, we define the shifted complex $B[t]^\bullet$, $t \in \mathbb{Z}$, by $B[t]^n := B^{t+n}$ with differentials $d^n_{B[t]} := (-1)^t d^{t+n}_B$. By [**49**, §10.7], we get

$$H^n(R\mathrm{Hom}^\bullet_{\mathcal{A}}(A^\bullet, B^\bullet)) = \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B[-n]^\bullet) \tag{6}$$

and there is a similar equation for $R\mathrm{Hom}^\bullet_{\mathbb{Z}_p}$.

For two cochain complexes $A^\bullet$, $B^\bullet$ of R-modules, we define the complex

$$\mathrm{Tot}^\oplus(A^\bullet \otimes_R B^\bullet)^n := \bigoplus_{p+q=n} A^p \otimes_R B^q.$$

This defines the functor

$$\mathrm{Tot}^\oplus(A^\bullet \otimes_R -)^\bullet : K(R\text{-mod}) \to K(\mathbf{Ab}),$$

where R-mod is the category of all R-modules. Since R-mod has enough projectives, the left derived functor

$$L\mathrm{Tot}^\oplus(A^\bullet \otimes_R -)^\bullet : D^-(R\text{-mod}) \to D(\mathbf{Ab})$$

exists.

Definition 1.28. The *total tensor product* of $A^\bullet$ and $B^\bullet$ is

$$A^\bullet \otimes_R^{\mathbb{L}} B^\bullet := L^-\mathrm{Tot}^{\oplus}(A^\bullet \otimes_R -)(B^\bullet) \in Ob(D(\mathbf{Ab}))$$

(cf. [**49**, definition 10.6.1]).

Let G be a profinite group and let $\mathcal{D}(G)$ be the category of discrete $\Lambda(G)$-modules. Let $\mathcal{C}(G)$ be the category of compact $\Lambda(G)$-modules. By [**33**, lemma 2.2.5] $\mathcal{D}(G)$ has enough injectives. By Pontryagin duality, $\mathcal{C}(G)$ is dual to $\mathcal{D}(G)$ and therefore, it has enough projectives. The fixed module functor $-^G$ is a left exact functor from $\mathcal{D}(G)$ to $\mathcal{D}(1) = \mathbf{Ab}$. The cofixed module functor $-_G$ (with $M_G := M/\langle gm - m | g \in G, m \in M\rangle_{\mathbb{Z}_p}$) is a right exact functor from $\mathcal{C}(G)$ to $\mathcal{C}(1)$.

Definition 1.29. We define

$$R\Gamma(G, M) := R(-^G)(M) \in Ob(D(\mathcal{D}(1)))$$

for an object M of $\mathcal{D}(G)$.

Remarks.

- $R(-^G)(M^\vee) = L(-_G)(M)^\vee$ (cf. [**33**, 2.6.9]),
- $R(-^G)(M) = R\mathrm{Hom}_{\Lambda(G)}(\mathbb{Z}_p, M)$,
- $L(-_G)(M) = \mathbb{Z}_p \otimes_{\Lambda(G)}^{\mathbb{L}} M$ (cf. [**49**, lemma 6.1.1]).

CHAPTER 2

Noncommutative Iwasawa Theory for Totally Real Fields

We fix a prime number $p \neq 2$. Our aim is to study the main conjecture for field extensions $F_\infty|F$ with the following properties:

Assumption 2.1.

- *$F|\mathbb{Q}$ is a finite field extension.*
- *$F_\infty|F$ is an infinite Galois extension.*
- *$G := G(F_\infty|F)$ is a compact p-adic Lie group.*
- *Only finitely many primes of F ramify in F_∞.*
- *F_∞ contains $F(\mu_{p^\infty})^+$, the maximal real subfield of $F(\mu_{p^\infty})$.*
- *F_∞ is totally real.*
- *$\mu(F_\infty|F) = 0$ (cf. definition 2.27)*

Let Σ be a finite set of primes of F containing all primes ramified in $F_\infty|F$. Let $X = X_\Sigma(F_\infty|F)$ be the Galois group of the maximal abelian pro-p extension of F_∞, unramified outside Σ, considered as a $\Lambda(G)$-module. X is a fundamental arithmetic object. Following [**48**], we define an Ore set $S \subset \Lambda(G)$ and conjecturally define the zeta function $\xi = \xi_\Sigma(F_\infty|F) \in K_1(\Lambda(G)_S)$. The main conjecture states that essential arithmetic information of X can be calculated from the zeta function ξ.

Remark. The main conjecture for elliptic curves is studied in [**48**] and [**8**]. In this situation, the fundamental arithmetic object X is defined to be the dual of the Selmer group.

1. Algebraic Part

Let P be a compact p-adic Lie group with a distinguished surjective homomorphism $P \twoheadrightarrow \mathbb{Z}_p$. In this subsection, we recall the definition of the subset $S(P) \subset \Lambda(P)$ of [**48**] (in the form given in [**8**]) and show that this set is multiplicatively closed and satisfies the Ore condition. Hence, the localisation $\Lambda(P)_{S(P)}$ exists. The ring $\Lambda(P)_{S(P)}$ is semi-local with stable range two. The norm map $\mathrm{N} : \Lambda(P)_{S(P)} \to \Lambda(U)_{S(U)}$ for open subgroups U of P exists. For $f \in K_1(\Lambda(P)_{S(P)})$ and an

Artin representation $\rho : G \to GL_n(\overline{\mathbb{Q}})$, we recall the evaluation of f at ρ given in [**48**] and [**8**]. We need this later for the definition of the p-adic zeta function. We show that the evaluation map behaves naturally with respect to induction, inflation and direct sum of the Artin representation. Finally, we prove that

$$\partial : K_1(\Lambda(P)_{S(P)}) \to K_0(\Lambda(P), \Lambda(P)_{S(P)})$$

(the connecting homomorphism of K-theory) is always surjective.

Definition 2.2. We define

$$\begin{aligned} S(P) &:= S(P, \omega) \\ &:= \{f \in \Lambda(P) | \Lambda(P)/\Lambda(P)f \text{ is finitely generated as a} \\ &\qquad \Lambda(\ker \omega)\text{-module}\}. \end{aligned}$$

Lemma 2.3. *Let P be a pro-p p-adic Lie group with a surjective homomorphism $\omega : P \twoheadrightarrow \mathbb{Z}_p$. Define*

$$\psi_P : \Lambda(P) \xrightarrow{\omega} \Lambda(\mathbb{Z}_p) \twoheadrightarrow \mathbb{F}_p[\![\mathbb{Z}_p]\!] \cong \mathbb{F}_p[[T]]$$

to be the continuous ring homomorphism induced by ω. Then

$$S(P) = \Lambda(P) \setminus \ker \psi_P.$$

Proof. For $f \in \Lambda(P)$, we have the equivalences

$$\begin{aligned} & f \in S(P) \\ \Leftrightarrow\ & \Lambda(P)/\Lambda(P)f \text{ is a finitely generated } \Lambda(\ker \omega)\text{-module} \\ \Leftrightarrow\ & \mathbb{F}_p[[T]]/\mathbb{F}_p[[T]]\psi_P(f) \text{ is finite} \end{aligned}$$

(see [**8**, lemma 2.1]). The last assertion clearly implies $\psi_P(f) \neq 0$. Assume $f \in \Lambda(P) \setminus \ker \psi_P$. Then, by the division lemma (cf. [**4**, ch. VII, §3, no. 8]), $\mathbb{F}_p[[T]]/\mathbb{F}_p[[T]]\psi_P(f)$ is a finitely generated $\mathbb{F}_p$-module and hence $f \in S(P)$. □

Remark. The above set (for pro-p groups, in the form $\Lambda(P) \setminus \ker \psi_P$) was first defined in [**48**]. The description of definition 2.2 was first given in [**8**].

Lemma 2.4. *In case P is one-dimensional (i. e. $\# \ker \omega < \infty$), $S(P)$ is the set of elements $f \in \Lambda(P)$, whose image in $\Lambda(P)/p\Lambda(P)$ is regular. For $P \cong \mathbb{Z}_p$, we have $S(P, \mathrm{id}_P) = \Lambda(P) \setminus p\Lambda(P)$.*

Proof. We only prove the first assertion since the second one is obvious. Let $\psi : \Lambda(P) \to \mathbb{F}_p[\![P]\!]$ be the canonical projection. For a ring R, define the prime radical $\mathcal{N}(R)$ to be the intersection of all left prime ideals of R. From [**1**, §4.1], we get that $\mathbb{F}_p[\![P]\!]$ is semiprime, i. e. $\mathcal{N}(\mathbb{F}_p[\![P]\!]) = 0$ (this result is due to Lazard). Then obviously $\psi^{-1}(\mathcal{N}(\mathbb{F}_p[\![P]\!]) = \ker \psi = p\Lambda(P)$. We use [**8**, proposition 2.6] to complete the proof. □

Let $\mathcal{S}_{\mathbb{Z}_p}$ be the category whose objects are compact p-adic Lie groups P with a distinguished surjective homomorphism $\omega : P \twoheadrightarrow \mathbb{Z}_p$ and where the set of morphisms from (P_1, ω_1) to (P_2, ω_2) is the set of commutative diagrams

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, "\omega_1"] & P_2 \arrow[d, "\omega_2"] \\ \mathbb{Z}_p \arrow[r, hook] & \mathbb{Z}_p, \end{tikzcd}$$

where the lower homomorphism $\mathbb{Z}_p \to \mathbb{Z}_p$ is injective.

If $(P, \omega) \in Ob(\mathcal{S}_{\mathbb{Z}_p})$ and U is an open subgroup of P and V is a closed subgroup of $\ker \omega$, then $\operatorname{im} \omega|_U = p^n \mathbb{Z}_p$ and we can define the homomorphisms

$$\begin{aligned} \omega_U &: U \xrightarrow{\omega|_U} p^n \mathbb{Z}_p \cong \mathbb{Z}_p \\ \omega_{P/V} &: P/V \to \mathbb{Z}_p, \quad gV \mapsto \omega(g). \end{aligned}$$

Then (U, ω_U) is a subobject of (P, ω) and $(P/V, \omega_{P/V})$ is a quotient object of (P, ω). Without further reference, we will always assume that subgroups and quotients of P of the above type are equipped with the above homomorphisms. In particular, we write $S(U) := S(U, \omega_U)$ and $S(P/V) := S(P/V, \omega_{P/V})$.

THEOREM 2.5. *$S(P)$ satisfies the left and right Ore condition, i. e. the localisation $\Lambda(P)_{S(P)}$ exists. $\Lambda(P)$ is $S(P)$-torsion free, and hence the canonical homomorphism $\Lambda(P) \hookrightarrow \Lambda(P)_{S(P)}$ is injective.*

PROOF. [**8**, theorem 2.4] $\square$

COROLLARY 2.6. *Let* **Rng** *be the category of rings and let $\mathcal{S}^{(p)}_{\mathbb{Z}_p}$ be the category of pro-p groups in $\mathcal{S}_{\mathbb{Z}_p}$. There is a functor*

$$\Lambda_S : \mathcal{S}^{(p)}_{\mathbb{Z}_p} \to \mathbf{Rng}$$

that sends an object P of $\mathcal{S}^{(p)}_{\mathbb{Z}_p}$ to $\Lambda(P)_{S(P)} \in Ob(\mathbf{Rng})$.

PROOF. The fact that Λ_S sends objects of $\mathcal{S}^{(p)}_{\mathbb{Z}_p}$ to objects of **Rng** follows from the theorem.

Let $\varphi : (P_1, \omega_1) \to (P_2, \omega_2)$ be a morphism in $\mathcal{S}^{(p)}_{\mathbb{Z}_p}$. Put $S_1 := S(P_1, \omega_1)$ and $S_2 := S(P_2, \omega_2)$. For $i = 1, 2$, let

$$\psi_i := \psi_{P_i} : \Lambda(P) \to \mathbb{F}_p[[T]]$$

be the homomorphism defined in lemma 2.3. Then there is a commutative diagram

$$\begin{tikzcd} \Lambda(P_1) \arrow[r, "f"] \arrow[d, "\psi_1"] & \Lambda(P_2) \arrow[d, "\psi_2"] \\ \mathbb{F}_p[[T]] \arrow[r, hook] & \mathbb{F}_p[[T]]. \end{tikzcd}$$

For $x \in \Lambda(P_1)$, $x \in \ker \psi_1$ implies $f(x) \in \ker \psi_2$. Hence

$$f(S_1) \subset S_2.$$

Now the universal property of the localisation implies the existence of a homomorphism

$$\Lambda_S(f) : \Lambda(P_1)_{S_1} \to \Lambda(P_2)_{S_2}. \qquad \square$$

Proposition 2.7. *The ring $\Lambda(P)_{S(P)}$ is noetherian.*

Proof. This follows from corollary 1.11 and proposition 1.4. $\square$

Proposition 2.8 ([**8**, proposition 4.2]). *For $(P, \omega) \in Ob(\mathcal{S}_{\mathbb{Z}_p})$, the ring $\Lambda(P)_{S(P)}$ is semi-local. In particular, we get for $d \geq 2$*

$$K_1(\Lambda(P)_{S(P)}) \cong GL_d(\Lambda(P)_{S(P)})/E_d(\Lambda(P)_{S(P)})$$
$$K_1(\Lambda(P)_{S(P)}) \cong \Lambda(P)^\times_{S(P)}/[\Lambda(P)^\times_{S(P)}, \Lambda(P)^\times_{S(P)}].$$

Proposition 2.9 ([**8**, proposition 2.3]). *For a finitely generated left or right $\Lambda(P)$-module M, we get: M is an $S(P)$-torsion module if and only if M is finitely generated as a $\Lambda(\ker \omega)$-module.*

Proposition 2.10. *Let U be an open subgroup of P. Then*

$$\Lambda(P)_{S(P)} = \Lambda(P)_{S(U)}$$

and this ring is a finitely generated free $\Lambda(U)_{S(U)}$-module of dimension $(P : U)$.

Proof. We first show that $S(U)$ is also an Ore set in $\Lambda(P)$. The proof of this fact is essentially the same as the one given for $S(U) \subset \Lambda(U)$ in [**8**, theorem 2.4]. We put $H := \ker \omega_P$ and $H' := \ker \omega_U = H \cap U$. Note that $(H : H') < \infty$. For $f \in S(U)$, the left $\Lambda(H')$-module $\Lambda(P)/\Lambda(P)f$ and the right $\Lambda(H')$-module $\Lambda(P)/f\Lambda(P)$ are finitely generated. By proposition 2.9, for every $x \in \Lambda(P)$, there are elements $s, s' \in S(U)$ with $sx \in \Lambda(P)f$ and $xs' \in f\Lambda(P)$, i. e. the Ore condition is satisfied.

For any multiplicatively closed Ore set $S \subset \Lambda(P)$, define

$$S_{sat} := \{x \in \Lambda(P) | \exists\, y \in \Lambda(P) \text{ such that } yx \in S\}.$$

Then $\Lambda(P)_S = \Lambda(P)_{S_{sat}}$. Since every p-adic Lie group contains an open pro-p subgroup (see proposition 1.9), we have proven the first part of the proposition when we can show that $S(U)_{sat} = S(P)$ for pro-p open subgroups U of P. The following argument is due to R. Sujatha.

"$\subset$" Let x be an element of $S(U)_{sat}$. Then there is $y \in \Lambda(P)$ such that $yx \in S(U)$. We get the natural surjection

$$\Lambda(P)/\Lambda(P)\cdot yx \twoheadrightarrow \Lambda(P)/\Lambda(P)\cdot x$$

of $\Lambda(H)$-modules. Since $\Lambda(P)/\Lambda(P)\cdot yx$ is finitely generated over $\Lambda(H')$ and hence over $\Lambda(H)$, this is also true for $\Lambda(P)/\Lambda(P)\cdot x$. Hence $x \in S(P)$.

"$\supset$" Let $x \in S(P)$. Since $\Lambda(P)/\Lambda(P)\cdot x$ is finite over $\Lambda(U)/\Lambda(U)\cap\Lambda(P)\cdot x$, this implies that $\Lambda(U)/\Lambda(U)\cap\Lambda(P)\cdot x$ is a finitely generated $\Lambda(H)$-module.

Let $\psi_U : \Lambda(U) \to \mathbb{F}_p[[T]]$ be the homomorphism defined in lemma 2.3. Then $S(U) = \Lambda(U)\setminus\ker\psi_U$. Assume $x \notin S(U)_{sat}$. Then $S(U)\cap\Lambda(P)\cdot x = \emptyset$, or equivalently $\Lambda(U)\cap\Lambda(P)\cdot x \subset \ker\psi_U$. Hence there is a natural surjection

$$\Lambda(U)/\Big(\Lambda(U)\cap\Lambda(P)\cdot x\Big) \twoheadrightarrow \Lambda(U)/\ker\psi_U.$$

But $\Lambda(U)/\ker\psi_U \cong \mathbb{F}_p[[T]]$ is not finitely generated over $\Lambda(H')$, and this yields a contradiction.

The fact that for any open subgroup U of P, $\Lambda(P)_{S(U)}$ is a finitely generated free $\Lambda(U)_{S(U)}$-module of dimension $(P:U)$ is obvious. $\square$

We will always write $\Lambda(P)_S$ for $\Lambda(P)_{S(P)}$, since, by the above lemma, there is little chance of confusion.

Corollary 2.11. *Let P, U be as in the preceding lemma. There is a commutative diagram*

$$\begin{tikzcd} K_1(\Lambda(P)_S) \arrow[r, "\mathrm{N}"] & K_1(\Lambda(U)_S) \\ K_1(\Lambda(P)) \arrow[u] \arrow[r, "\mathrm{N}"] & K_1(\Lambda(U)). \arrow[u] \end{tikzcd}$$

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. We endow $\overline{\mathbb{Q}}$ with the discrete topology. In the following, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Definition 2.12. An Artin representation of a compact p-adic Lie group P is a continuous representation $P \to GL_n(\overline{\mathbb{Q}})$, $n \in \mathbb{N}$.

Remark. An Artin representation $\rho : P \to GL_n(\overline{\mathbb{Q}})$ factors through $P/U \to GL_n(\overline{\mathbb{Q}})$, where U is an open subgroup of P. Hence, the image of ρ in $GL_n(\overline{\mathbb{Q}})$ is finite. Let

$$L := \mathbb{Q}_p(\{\rho(g)_{ij} | g \in P,\ i, j = 1, \ldots, n\})$$

be the field obtained by adjoining all entries of elements of the image of ρ. Then L is a p-adic number field (i. e. a finite extension of $\mathbb{Q}_p$) and we can realize ρ over L. Hence ρ is isomorphic to a representation

$$\rho : P \to GL_n(\mathcal{O}),$$

where $\mathcal{O}$ is the ring of integers of L.

Let P be a compact p-adic Lie group with a surjective homomorphism $\omega : P \twoheadrightarrow Z := \mathbb{Z}_p$. Put $S := S(P, \omega)$. We will now define the *evaluation of an element* $f \in K(\Lambda(P)_S)$ on certain representations of P (cf. [**8**, section 3] or [**48**, section 5.2]). Let $\mathcal{O}$ be the ring of integers in some p-adic number field L and let $\rho : P \to GL_n(\mathcal{O})$ be a continuous representation. Then ρ extends to a continuous ring homomorphism $\rho : \Lambda(P) \to M_n(\mathcal{O})$. Set

$$\Lambda_{\mathcal{O}}(P) := \mathcal{O}[\![P]\!] = \varprojlim_{U \leq_o P} \mathcal{O}[P/U] = \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(P).$$

Let $Q_{\mathcal{O}}(Z) := Q(\Lambda_{\mathcal{O}}(Z))$ be the quotient field of $\Lambda_{\mathcal{O}}(Z)$. We extend the group homomorphism

$$P \to \big(M_n(\mathcal{O}) \otimes_{\mathbb{Z}_p} \Lambda(Z)\big)^{\times} \cong GL_n(\Lambda_{\mathcal{O}}(Z)), \quad \sigma \mapsto \rho(\sigma) \otimes \omega(\sigma).$$

to the continuous ring homomorphism

$$\Phi_\rho : \Lambda(P) \to M_n(\mathcal{O}) \otimes_{\mathbb{Z}_p} \Lambda(Z) \cong M_n(\Lambda_{\mathcal{O}}(Z)).$$

By [**8**, lemma 3.3], this extends to a ring homomorphism

$$\Phi_\rho : \Lambda(P)_S \to M_n(Q_{\mathcal{O}}(Z)).$$

Let $\varepsilon : \Lambda_{\mathcal{O}}(Z) \to \mathcal{O}$ be the augmentation map and define

$$\mathfrak{p} := \ker \varepsilon, \qquad \Lambda_{\mathcal{O}}(Z)_{\mathfrak{p}} := (\Lambda_{\mathcal{O}}(Z) \setminus \mathfrak{p})^{-1} \Lambda_{\mathcal{O}}(Z).$$

Then ε extends to a map

$$\varepsilon : \Lambda_{\mathcal{O}}(Z)_{\mathfrak{p}} \to L.$$

We can now define a map as the composition

$$K_1(\Lambda(P)_S) \to K_1(M_n(Q_{\mathcal{O}}(Z))) \cong Q_{\mathcal{O}}(Z)^{\times} \to L \cup \{\infty\}. \tag{7}$$

Here, the first map is the homomorphism induced by Φ_ρ, the isomorphism in the middle is given by Morita invariance and the third map is $x \mapsto \varepsilon(x)$ for $x \in \Lambda(Z)_{\mathfrak{p}}$ and $x \mapsto \infty$ for $x \notin \Lambda(Z)_{\mathfrak{p}}$.

Definition 2.13. The evaluation of an element $f \in K_1(\Lambda(P)_S)$ at the continuous representation ρ is the image $f(\rho) \in L \cup \{\infty\}$ of f under the homomorphism (7).

REMARK. We get the following basic property of the evaluation map: If g is an inverse image of f under the natural map

$$\Lambda(P)_S^\times \cap \Lambda(P) \to K_1(\Lambda(P)_S),$$

where $\Lambda(P)_S^\times := (\Lambda(P)_S)^\times$, then $f(\rho) = \det(\rho(g))$.

LEMMA 2.14. *Let f be an element of $K_1(\Lambda(P)_S)$ and let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$. We get the following properties of the evaluation homomorphism:*

(1) *Let $U \subset P$ be an open normal subgroup and let χ be a one dimensional representation of U. Let $\mathrm{N} : K_1(\Lambda(P)_S) \to K_1(\Lambda(U)_S)$ be the norm map. Then*

$$\mathrm{N}(f)(\chi) = f(\mathrm{ind}_P^U(\chi)).$$

(2) *Let U be a normal subgroup of P contained in the kernel of ω_P and let $p_* : K_1(\Lambda(P)_S) \to K_1(\Lambda(P/U)_S)$ be the projection map. Let $\rho : P/U \to GL_n(\mathcal{O})$ be a continuous representation. Let $\mathrm{inf}_P^{P/U}(\rho) : P \to GL_n(\mathcal{O})$ be the composition of the natural surjection $P \to P/U$ with ρ. Then*

$$f(\mathrm{inf}_P^{P/U}(\rho)) = p_*(f)(\rho).$$

(3) *Let ρ, ρ' be two continuous representations of P. Then*

$$f(\rho \oplus \rho') = f(\rho)f(\rho').$$

PROOF. (1) Recall that the groups P and U are equipped with the surjective homomorphisms $\omega_P : P \to Z$ and $\omega_U : U \to Z'$, where $Z' \subset_o Z \cong \mathbb{Z}_p$. Since $\omega_P|U = \omega_U$, we may denote both maps by ω. Put $\rho := \mathrm{ind}_P^U(\chi)$ and $n := (P : U)$. By definition of the evaluation map, it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
K_1(\Lambda(P)_S) \arrow[r, "\Phi_\rho"] \arrow[d, "\mathrm{N}"] & K_1(Q_{\mathcal{O}}(Z)) \\
K_1(\Lambda(U)_S) \arrow[r, "\Phi_\chi"] & K_1(Q_{\mathcal{O}}(Z')) \arrow[u, hook]
\end{tikzcd}$$

Let $(\nu_i)_{i=1,\dots n}$ be a system of representatives of P/U in P. Then it is a $\Lambda(U)$-basis of $\Lambda(P)$ and a $\Lambda(U)_S$-basis of $\Lambda(P)_S$. It determines the isomorphisms

$$\Lambda(P) \cong \Lambda(U)^n \qquad \Lambda(P)_S \cong \Lambda(U)_S^n$$

of $\Lambda(U)$-modules and $\Lambda(U)_S$-modules, respectively. We consider $\mathcal{O}$ as a $\Lambda(U)$-module via χ. For any right R-module M, let $\mathrm{End}_R(M)$ denote the left R-module of right R-homomorphisms on M. Then the above isomorphisms induce the isomorphisms

$$\begin{aligned}\alpha &: \mathrm{End}_{\Lambda(U)_S}(\Lambda(P)_S) \xrightarrow{\sim} M_n(\Lambda(U)_S)\\ \beta &: \mathrm{Aut}_{\mathcal{O}}(\Lambda(P) \otimes_{\Lambda(U)} \mathcal{O}) \xrightarrow{\sim} GL_n(\mathcal{O})\end{aligned}$$

of $\Lambda(U)_S$-modules and groups, respectively. Let $\tilde{\mathrm{N}}$ be the composition

$$\begin{aligned}\Lambda(P)_S \cong \mathrm{End}_{\Lambda(P)_S}(\Lambda(P)_S) &\longrightarrow \mathrm{End}_{\Lambda(U)_S}(\Lambda(P)_S)\\ &\xrightarrow{\alpha} M_n(\Lambda(U)_S)\end{aligned}$$

and let ρ be the homomorphism

$$\rho : P \to \mathrm{Aut}_{\mathcal{O}}(\Lambda(P) \otimes_{\Lambda(U)} \mathcal{O}) \xrightarrow{\beta} GL_n(\mathcal{O}).$$

Put

$$A := \begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_n \end{pmatrix} \in M_n(Q_{\mathcal{O}}(Z)).$$

We show that there is a commutative diagram

$$\begin{array}{ccccc} \Lambda(P)_S & & \xrightarrow{\Phi_\rho} & & M_n(\mathcal{O}) \otimes_{\mathbb{Z}_p} Q(Z) \\ \downarrow \tilde{\mathrm{N}} & & & & \downarrow X \mapsto AXA^{-1} \\ M_n(\Lambda(U)_S) & \xrightarrow{M_n(\Phi_\chi)} & M_n(\mathcal{O} \otimes_{\mathbb{Z}_p} Q(Z')) & \hookrightarrow & M_n(\mathcal{O} \otimes_{\mathbb{Z}_p} Q(Z)). \end{array}$$

Since all maps in the above diagram are continuous ring homomrophisms, it suffices to prove commutativity for elements $\sigma \in P$. For every $i \in \{1, \ldots, n\}$, there is exactly one $j \in \{1, \ldots, n\}$ such that

$$\nu_i \sigma = \tau_{ij} \nu_j \tag{8}$$

for some $\tau_{ij} \in U$. For all other j, we put $\tau_{ij} := 0 \in \Lambda(U)_S$. Then $\tilde{\mathrm{N}}(\sigma) = (\tau_{ij})_{ij} \in M_n(\Lambda(U)_S)$ and hence

$$\begin{aligned}\Phi_\rho(\sigma) &= \left(\chi(\tau_{ij})\omega(\sigma)\right)_{ij}\\ \left(M_n(\Phi_\chi) \circ \tilde{\mathrm{N}}\right)(\sigma) &= \left(\chi(\tau_{ij})\omega(\tau_{ij})\right)_{ij}.\end{aligned}$$

Thus we get

$$\begin{aligned} A\Phi_\rho(\sigma)A^{-1} &= \Big(\omega(\nu_i)\Big)_{ij}\Big(\chi(\tau_{ij})\omega(\sigma)\Big)_{ij}\Big(\omega(\nu_j)^{-1}\Big)_{ij} \\ &= \Big(\chi(\tau_{ij})\omega(\sigma)\omega(\nu_i\nu_j^{-1})\Big)_{ij} \\ &\overset{(*)}{=} \Big(\chi(\tau_{ij})\omega(\tau_{ij})\Big)_{ij} \\ &= \Big(M_n(\Phi_\chi)\circ\tilde{\mathrm{N}}\Big)(\sigma), \end{aligned}$$

where equation $(*)$ follows from the definition of τ_{ij} (cf. equation (8)). This completes the proof of (1).

(2) We write $\rho : P/U \to GL(V)$ and $\inf_P^U(\rho) : P \to GL(W)$. The diagram

$$\begin{tikzcd} \Lambda(P)_S \arrow[r, "\Phi_{\inf_U^P(\rho)}"] \arrow[d] & M_n(Q_{\mathcal{O}}(Z)) \arrow[d, equal] \\ \Lambda(P/U)_S \arrow[r, "\Phi_\rho"] & M_n(Q_{\mathcal{O}}(Z)). \end{tikzcd}$$

is clearly commutative. By passing to the corresponding K-groups, we get the commutativity of the diagram

$$\begin{tikzcd} K_1(\Lambda(P)_S) \arrow[r, "\Phi_{\inf_P^U(\rho)}"] \arrow[d, "p_*"] & K_1(M_n(Q_{\mathcal{O}}(Z))) \arrow[d, equal] \\ K_1(\Lambda(P/U)_S) \arrow[r, "\Phi_\rho"] & K_1(M_n(Q_{\mathcal{O}}(Z))). \end{tikzcd}$$

(3) This follows directly from the definition of the evaluation homomorphism. $\square$

Remark. Let $\chi : U \to \mathcal{O}$ be a continuous character and put $V := \Lambda(P)\otimes_{\Lambda(U)}\mathcal{O}$, where χ induces the $\Lambda(U)$-module structure on $\mathcal{O}$. Let

$$\rho : P \to \mathrm{End}_{\mathcal{O}}(V), \quad g \mapsto (x\otimes y \mapsto (gx)\otimes y)$$

be the representation induced by χ. Then there is a commutative diagram

$$\begin{tikzcd} K_1(\Lambda(P)) \arrow[r, "\rho"] \arrow[d, "\mathrm{N}"] & K_1(\mathrm{End}_{\mathcal{O}}(V)) \arrow[d, equal] \\ K_1(\mathrm{End}_{\Lambda(U)}\Lambda(P)) \arrow[r] \arrow[d, "\cong"] & K_1(\mathrm{End}_{\mathcal{O}}(\Lambda(P)\otimes_{\Lambda(U)}\mathcal{O})) \arrow[d, "\cong"] \\ K_1(\Lambda(U)) \arrow[r, "\chi"] & K_1(\mathcal{O}), \end{tikzcd}$$

where the two lower isomorphisms are given by Morita invariance. The above lemma generalises this fact.

LEMMA 2.15. *Let P be a compact p-adic Lie group with a surjective homomorphism $\omega : P \twoheadrightarrow Z := \mathbb{Z}_p$. Then the connecting homomorphism of K-theory*

$$\partial : K_1(\Lambda(P)_S) \to K_0(\Lambda(P), \Lambda(P)_S)$$

is surjective. In particular, proposition 1.25 holds for $R = \Lambda(P)$ and $R_S = \Lambda(P)_S$.

PROOF. We use a generalisation of [**8**, proposition 3.4], given in [**24**, lemma 1.5].

We define a homomorphism η on $K_0(\Lambda(P))$ and show that η is injective and $\eta \circ \lambda = 0$. Then $\lambda = 0$ and we get the exact sequence (cf. (4))

$$K_1(\Lambda(P)) \to K_1(\Lambda(P)_S) \xrightarrow{\partial} K_0(\Lambda(P), \Lambda(P)_S) \to 0. \tag{9}$$

Let W be a pro-p open normal subgroup of G and set $\Delta = P/W$. Let $\mathcal{V}$ be the set of irreducible representations of Δ over $\overline{\mathbb{Q}}_p$ and let L be a fixed finite extension of $\mathbb{Q}_p$ such that all representations in $\mathcal{V}$ can be realised over L. We define η to be the composition of natural maps

$$K_0(\Lambda(P)) \xrightarrow{\eta_1} K_0(\mathbb{Z}_p[\Delta]) \xrightarrow{\eta_2} K_0(\mathbb{Q}_p[\Delta]) \xrightarrow{\eta_3} K_0(L[\Delta]) \xrightarrow{\eta_4} \prod_{\rho \in \mathcal{V}} K_0(L).$$

Here, η_4 is the isomorphism $K_0(L[\Delta]) \cong \prod_\rho K_0(M_{n_\rho}(L)) \cong \prod_\rho K_0(L)$ (n_ρ is the dimension of ρ), where the first map is induced by the Wedderburn decomposition of $L[\Delta]$ and the second map is Morita invariance.

We will prove injectivity of η_1 in a short lemma below. Injectivity of η_2 and η_3 is well known (see [**42**, chapter 16, theorem 34, corollary 2] and loc. cit. §14.6, respectively).

In order to show $\eta \circ \lambda = 0$, we give an alternative description of

$$\eta = (\eta_\rho)_\rho : K_0(\Lambda(P)) \to \prod_{\rho \in \mathcal{V}} K_0(L)$$

by writing η_ρ as the composition

$$K_0(\Lambda(P)) \xrightarrow{\mathrm{tw}_\rho} K_0(\Lambda_{\mathcal{O}}(P)) \xrightarrow{\varepsilon_Z} K_0(\Lambda_{\mathcal{O}}(Z)) \xrightarrow{\varepsilon_1} K_0(\mathcal{O}) \xrightarrow{j} K_0(L), \tag{10}$$

where the composing maps are defined as follows: The homomorphisms ε_Z, ε_1 and j are induced by the natural surjections $\Lambda_{\mathcal{O}}(P) \twoheadrightarrow \Lambda_{\mathcal{O}}(Z)$, $\Lambda_{\mathcal{O}}(Z) \twoheadrightarrow \mathcal{O}$ and the injection $\mathcal{O} \hookrightarrow L$, respectively. For a representation $\rho : P \to GL_n(\mathcal{O})$ and a finitely generated projective $\Lambda(P)$-module M, we set

$$\mathrm{tw}_\rho(M) = M \otimes_{\mathbb{Z}_p} \mathcal{O}^n,$$

and endow $\mathrm{tw}_\rho(M)$ with the diagonal action, i. e. $\sigma(m \otimes z) = (\sigma m) \otimes (\rho(\sigma)z)$ for $\sigma \in P$, $m \otimes z \in \mathrm{tw}_\rho(M)$. Obviously, this induces a homomorphism

$$\mathrm{tw}_\rho : K_0(\Lambda(P)) \to K_0(\Lambda_{\mathcal{O}}(P)).$$

It is easily verified that the composition (10) indeed equals η_ρ.

Let U be an object of $\mathcal{H}_S^{\Lambda(P)}$ and choose a finite projective resolution $Q_\bullet$ of U. Then

$$\begin{aligned}\varepsilon_Z \circ \mathrm{tw}_\rho \circ \lambda([U]) &= \sum_{i\in\mathbb{Z}}(-1)^i[\mathrm{tw}_\rho(Q_i)_{\ker\omega}] \overset{(*)}{=} \sum_{i\in\mathbb{Z}}(-1)^i[H_i(\mathrm{tw}_\rho(Q_\bullet)_{\ker\omega})]\\ &= \sum_{i\in\mathbb{Z}}(-1)^i[H_i(\ker\omega, \mathrm{tw}_\rho(U))]\end{aligned}$$

where the identity $(*)$ follows from [**50**, chapter II, proposition 6.6] and the last identity follows from the definition of homology groups.

By [**8**, lemmata 3.1 and 3.2], $H_i(\ker\omega, \mathrm{tw}_\rho(U))$ are finitely generated torsion $\Lambda(Z)$-modules for all $i \geq 0$. Since $K_0(\Lambda_{\mathcal{O}}(Z)) \cong \mathbb{Z}$ via the rank map,

$$[H_i(\ker\omega, \mathrm{tw}_\rho(U))] = 0 \in K_0(\Lambda_{\mathcal{O}}(Z))$$

for all $i \geq 0$. Hence $\eta \circ \lambda([U]) = 0$. □

Lemma 2.16 ([**8**, lemma 3.5]). *Let W be a pro-p open normal subgroup of P and put $\Delta = P/W$. Then the canonical map*

$$K_0(\Lambda(P)) \to K_0(\mathbb{Z}_p[\Delta])$$

is injective.

Proof. By [**28**, proposition 4.3], we can write every element of $K_0(\Lambda(P))$ in the form $[M] - [\Lambda(P)^n]$, where M is a finitely generated projective $\Lambda(P)$-module and $n \in \mathbb{N}$. Assume that $[M] - [\Lambda(P)^n]$ is an element of the kernel of this map. Then there is an isomorphism

$$\alpha : M_W \oplus \mathbb{Z}_p[\Delta]^r \xrightarrow{\sim} \mathbb{Z}_p[\Delta]^{r+n},$$

where $M_W = M/\overline{\langle\sigma - 1|\sigma \in W\rangle}M$ is the module of W-coinvariants. Since M is projective, α lifts to a homomorphism

$$\beta : M \oplus \Lambda(P)^r \to \Lambda(P)^{r+n}.$$

It suffices to show that β is an isomorphism. Since $(\cdot)_W$ is right exact, $(\operatorname{coker}\beta)_W = 0$. Since W is pro-p, the augmentation ideal $I(W)$ is contained in the Jacobson radical

$$J(\Lambda(W)) = p\Lambda(W) + I(W)$$

of $\Lambda(W)$ (cf. [**33**, proposition 5.2.16]). Hence, by the topological Nakayama lemma, $\operatorname{coker}\beta = 0$. We take W-homology of the short exact sequence

$$0 \to \ker\beta \to M \oplus \Lambda(P)^r \to \Lambda(P)^{r+n} \to 0.$$

Since $\Lambda(P)$ is a free $\Lambda(W)$-module of finite rank, we get

$$H_1(W, \Lambda(P)^{r+n}) = 0.$$

This yields the short exact sequence

$$0 \to (\ker\beta)_W \to M_W \oplus \mathbb{Z}_p[\Delta]^r \xrightarrow{\sim} \mathbb{Z}_p[\Delta]^{r+n} \to 0,$$

and we get (using again Nakayama's lemma) $\ker\beta = 0$. □

2. Arithmetic Part

For a number field K, we denote by K^{cyc} the cyclotomic $\mathbb{Z}_p$-extension of K. (K^{cyc} is the fixed field of the torsion part of $G(K(\mu_{p^\infty})|K)$ in $K(\mu_{p^\infty})$.) Let $F_\infty|F$ be an extension of number fields that satisfies assumption 2.1. Put $G := G(F_\infty|F)$, $\Gamma := G(F^{cyc}|F)$ and $S := S(G, G \twoheadrightarrow \Gamma)$. If G has elements of order p, then $X = X_\Sigma(F_\infty|F)$ may not have a finite resolution by finitely generated projective $\Lambda(G)$-modules. This prevents us from mapping X in the K-group

$$K_0(\Lambda(G), \Lambda(G)_S).$$

In our situation, where G may have elements of order p, we define a cochain complex which is closely related to X and plays a role similar to that of X in case G has no element of order p.

Let Σ be a fixed finite set of primes of F, such that all primes which ramify in $F_\infty|F$ are contained in Σ. For any field F', $F \subset F' \subset F_\infty$, we denote the set of primes of F' lying over primes of Σ also by Σ.

Lemma 2.17. *Σ contains all primes of F which divide p.*

Proof. Let $\mathfrak{P}$ be a prime of F_∞ which divides p, $\mathfrak{P}' := \mathfrak{P} \cap F^{cyc}$ and $\mathfrak{p} := \mathfrak{P} \cap F$. By [**33**, proposition 11.1.1 (ii)], $\mathfrak{P}'$ ramifies in $F^{cyc}|F$, and hence $\mathfrak{P}$ is ramified in $F_\infty|F$. That is, $\mathfrak{p} \in \Sigma$. □

Let $\mathbb{Q}_p/\mathbb{Z}_p$ be the $\Lambda(G)$-module with trivial action of G. For any $\Lambda(G)$-module M, we define the associated complex $M^\bullet$ (concentrated in 0) by $M^0 := M$ and $M^i := 0$ for $i \neq 0$.

Definition 2.18. Let

$$\begin{aligned} C^\bullet &:= C_\Sigma(F_\infty|F)^\bullet \\ &:= R\mathrm{Hom}^\bullet_{\mathbb{Z}_p}\Big(R\Gamma^\bullet_{\acute{e}t}(\operatorname{Spec}\mathcal{O}_{F_\infty}[1/\Sigma], \mathbb{Q}_p/\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^\bullet\Big) \end{aligned}$$

be the object of the derived category $D(\mathcal{P}(\Lambda(G)))$, where $\mathbb{Q}_p/\mathbb{Z}_p$ is the locally constant sheaf on the étale site of $\operatorname{Spec}\mathcal{O}_{F_\infty}[1/\Sigma]$.

For an intermediate field F' of $F_\infty|F$, let $(F')_\Sigma(p)$ be the maximal pro-p extension of F' unramified outside Σ. We put $G_\Sigma := G((F_\infty)_\Sigma(p)|F_\infty)$. Let M_Σ be the maximal abelian pro-p extension of F_∞ unramified outside Σ. We set

$$X := X_\Sigma(F_\infty|F) := G(M_\Sigma|F_\infty) = G_\Sigma/\overline{[G_\Sigma, G_\Sigma]}.$$

For $\sigma \in G(F_\infty|F)$, let $\hat{\sigma}$ be an inverse image of σ under the natural map $G(M_\Sigma|F) \twoheadrightarrow G$. By setting $\sigma \cdot x := \hat{\sigma}x\hat{\sigma}^{-1}$ for $\sigma \in G$, $x \in X$, we give X a $\Lambda(G)$-module structure.

Lemma 2.19. *The cohomology groups of $C^\bullet$ are given as follows:*

$$\begin{aligned} H^0(C^\bullet) &= \mathbb{Z}_p \\ H^{-1}(C^\bullet) &= X \\ H^i(C^\bullet) &= 0 \text{ for } i \notin \{0, -1\} \end{aligned}$$

Proof. For any $\Lambda(G)$-module M, we set $M^\vee := \operatorname{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is a direct limit of finite abelian groups of p-power order, we get

$$R\Gamma_{ét}(\operatorname{Spec}\mathcal{O}_{F_\infty}[1/\Sigma], \mathbb{Q}_p/\mathbb{Z}_p) = R\Gamma(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p). \tag{11}$$

Hence

$$H^i(C^\bullet) = H^{-i}(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p)^\vee.$$

Let U be a pro-p open subgroup of G and let $F_U \subset F_\infty$ be the fixed field of U. Then $F_\infty|F_U$ is a pro-p extension unramified outside Σ and hence

$$(F_\infty)_\Sigma(p) = (F_U)_\Sigma(p).$$

By [**33**, cor. 10.4.9(iii)],

$$scd_p\, G((F_\infty)_\Sigma(p)|(F_U)^{cyc}) = scd_p\, G((F_U)_\Sigma(p)|(F_U)^{cyc}) \le 2.$$

Since $G_\Sigma = G((F_\infty)_\Sigma(p)|F_\infty)$ is a closed subgroup of

$$G((F_\infty)_\Sigma(p)|(F_U)^{cyc})$$

we get (using [**33**, proposition 3.3.5])

$$scd_p G_\Sigma \le 2.$$

This implies (since $H^j(G_\Sigma, \mathbb{Q}_p) = 0$ for $j \ge 1$ by [**33**, 1.6.2c])

$$H^i(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = H^{i+1}(G_\Sigma, \mathbb{Z}_p) = 0 \text{ for all } i \ne 0, 1.$$

We get

$$\begin{aligned} H^0(C^\bullet) &= (\mathbb{Q}_p/\mathbb{Z}_p)^\vee = (\varinjlim_n \frac{1}{p^n}\mathbb{Z}_p/\mathbb{Z}_p)^\vee \\ &= \varprojlim_n (\frac{1}{p^n}\mathbb{Z}_p/\mathbb{Z}_p)^\vee = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p. \end{aligned}$$

We have $H^{-1}(C^\bullet) = (G_\Sigma^\vee)^\vee = X$. □

Let $U \subset G$ be an open subgroup and let $V \subset U$ be a normal subgroup. Let $F_V \subset F_\infty$ be the fixed field of V.

Definition 2.20.

$$\begin{aligned} C^\bullet_{U,V} &:= R\mathrm{Hom}^\bullet_{\mathbb{Z}_p}(R\Gamma^\bullet_{\acute{e}t}(\mathrm{Spec}\,\mathcal{O}_{F_V}[1/\Sigma], \mathbb{Q}_p/\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^\bullet) \\ &\in Ob(D(\mathcal{P}(\Lambda(U/V)))). \end{aligned}$$

Lemma 2.21.

$$\Lambda(U/V)^\bullet \otimes^{\mathbb{L}}_{\Lambda(U)} C^\bullet = C^\bullet_{U,V}$$

Proof. Recall that

$$R\Gamma(W,A)^\vee = R(-^W)(A)^\vee = L(-_W)(A^\vee) = \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(W)} A^\vee$$

for a group W and a $\Lambda(W)$-module A. Hence, due to (11),

$$\begin{aligned} C^\bullet &= \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G_\Sigma)} \mathbb{Z}_p \\ C^\bullet_{U,V} &= \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G_\Sigma(F_V))} \mathbb{Z}_p, \end{aligned}$$

where $G_\Sigma(F_V) := G((F_V)_\Sigma(p)|F_V)$. Since $\Lambda(U/V) = \mathbb{Z}_p \otimes_{\Lambda(V)} \Lambda(U)$, we have $\Lambda(U/V)^\bullet = \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(V)} \Lambda(U)$. Since $F_\infty|F_V$ is pro-p and unramified outside Σ, we get the exact sequence

$$1 \to G_\Sigma \to G_\Sigma(F_V) \to V \to 1.$$

By the Hochschild-Serre spectral sequence (with respect to the above exact sequence), we get (cf. [**49**, exercise 10.8.5])

$$\begin{aligned} C^\bullet_{U,V} &= \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G_\Sigma(F_V))} \mathbb{Z}_p \\ &= \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(V)} \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G_\Sigma)} \mathbb{Z}_p \\ &= \big(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(V)} \Lambda(U)\big) \otimes^{\mathbb{L}}_{\Lambda(U)} \big(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G_\Sigma)} \mathbb{Z}_p\big) \\ &= \Lambda(U/V)^\bullet \otimes^{\mathbb{L}}_{\Lambda(U)} C^\bullet. \end{aligned}$$

□

We fix the following objects assigned to our extension $F_\infty|F$:

- $\Gamma := G(F^{cyc}|F) \cong \mathbb{Z}_p$
- $H := G(F_\infty|F^{cyc}) = \ker(G \twoheadrightarrow \Gamma)$

- $S := S(G, G \twoheadrightarrow \Gamma) = \{f \in \Lambda(G) | \Lambda(G)/\Lambda(G)f$ is finitely generated as a $\Lambda(H)$-module$\} \subset \Lambda(G)$

Let $K_\infty|K$ be a $\mathbb{Z}_p$-extension over a totally real number field and let Σ be a finite set of primes of K that contains all primes lying over p. Let $K_n = K_\infty^{p^n\mathbb{Z}_p}$ be the fixed field of $p^n\mathbb{Z}_p$ and let Σ_n be the set of primes of K_n lying over a prime of Σ. For any prime $\mathfrak{p}$ of K_n, let $U_n^{\mathfrak{p}} := \mathcal{O}_{K_{n,\mathfrak{p}}}^\times$ be the group of units of $K_{n,\mathfrak{p}}$. Put $U_n := \prod_{\mathfrak{p} \in \Sigma_n} U_n^{\mathfrak{p}}$. Let E_n be the image of $\mathcal{O}_{K_n}^\times$, the group of units of K_n, under the diagonal embedding in U_n. Let $\bar{E}_n$ be the topological closure of E_n.

Definition 2.22. The Leopoldt defect of K_n is defined to be the integer δ_n such that the $\mathbb{Z}_p$-rank of the pro-p part of $\bar{E}_n$ is $[K_n : \mathbb{Q}] - 1 - \delta_n$.

Conjecture 2.23 (Leopoldt). *We say that the weak Leopoldt conjecture holds for $K_\infty|K$ when δ_n is bounded independent of n. We say that the Leopoldt conjecture holds for K when δ_0 is zero.*

For $n \in \mathbb{N} \cup \{\infty\}$, let L_n be the maximal abelian unramified extension of K_n and let M_n be the maximal abelian extension of K_n unramified outside Σ_n.

The following two lemmata are applications of [**51**, §13.1].

Lemma 2.24. *For $n \in \mathbb{N}$, $G(M_n|L_n) \cong U_n/\bar{E}_n$.*

Proof. By class field theory,
$$G(M_n|L_n) \cong U_n/\left(U_n \cap \overline{K^\times U_n'}\right)$$
where $U_n' := \prod_{\mathfrak{p} \notin \Sigma_n} U_n^{\mathfrak{p}}$. Hence it suffices to show that
$$U_n \cap \overline{K^\times U_n'} = \bar{E}_n.$$

The inclusion $\supset$ is plain. We prove the other inclusion. Since
$$\overline{K^\times U'} = \bigcap_m K^\times U_n' U_n^m$$
and
$$\bar{E}_n = \bigcap_m E_n U_n^m,$$
it suffices to show that
$$U_n \cap K^\times U_n' U_n^m \subset \bar{E}_n U_n^m$$
for all $m \geq 0$. Let $x \in K^\times$, $u' \in U_n'$ and $u \in U_n^m$ be such that $xu'u \in U_n$. Then $xu' \in U_n$. Since the $\mathfrak{p}$-component of elements of U_n at $\mathfrak{p} \notin \Sigma$ is 1, this implies $x \in E_n$. Hence $xu' \in E_n$. Thus we get
$$xu'u \in E_n U_n^m.$$
□

Lemma 2.25. *Assume that the weak Leopoldt conjecture holds and that* K_∞ *containes* $\mathbb{Q}^{cyc}$. *Then the pro-*p *part of* $G(M_\infty|L_\infty)$ *is a finitely generated* $\mathbb{Z}_p$*-module.*

Proof. For $m \in \mathbb{N}$ and $\mathfrak{p} \in \Sigma_n$, let $U_n^{m,\mathfrak{p}} := \{x \in U_n^{\mathfrak{p}} | x \equiv 1 \bmod \mathfrak{p}^m\}$. Then for $\mathfrak{p} \in \Sigma_n$, $\mathfrak{p}|p$, there is $m \geq 0$ such that the logarithm homomorphism induces the isomorphism

$$U_n^{m,\mathfrak{p}} \cong \mathfrak{p}^m \cong \mathcal{O}_{K_{n,\mathfrak{p}}} \cong \mathbb{Z}_p^{e_\mathfrak{p} f_\mathfrak{p}},$$

where $e_\mathfrak{p}$ is the ramification index and $f_\mathfrak{p}$ is the inertia degree of $\mathfrak{p}$ over $\mathbb{Q}$. Hence there are integers $a_{m,\mathfrak{p}} \in \mathbb{N}$ such that

$$U_n^{\mathfrak{p}} \cong \mu(K_{n,\mathfrak{p}}) \times U_n^{m,\mathfrak{p}} \cong \mu_{q-1} \times \mu_{p^{a_{m,\mathfrak{p}}}} \times \mathbb{Z}_p^{e_\mathfrak{p} f_\mathfrak{p}},$$

where $\mu(*)$ is the group of roots of unity of $*$ and μ_l is the group of l-th roots of unity. Since K_n is totally real, we have $\mu(K_n) = \{\pm 1\}$. Hence

$$\begin{aligned} U_n &\cong \left(\prod_{\mathfrak{p} \in \Sigma_n} \mu(K_{n,\mathfrak{p}})(p) \right) \times \mathbb{Z}_p^{[K_n:\mathbb{Q}]} \times Q_{1,n} \\ \bar{E}_n &\cong \mathbb{Z}_p^{[K_n:\mathbb{Q}]-1-\delta_n} \times Q_{2,n} \\ U_n/\bar{E}_n &\cong \left(\prod_{\mathfrak{p} \in \Sigma_n} \mu_{p^{a_{n,\mathfrak{p}}}} \right) \times \mathbb{Z}_p^{1+\delta_n} \times (Q_{1,n}/Q_{2,n}), \end{aligned}$$

where $Q_{1,n}$ and $Q_{2,n}$ are profinite groups with trivial pro-p part. Since $G(M_\infty|L_\infty) \cong \varprojlim_n G(M_n|L_n)$, this implies

$$G(M_\infty|L_\infty) \cong \left(\varprojlim_n \prod_{\mathfrak{p} \in \Sigma_n} \mu_{p^{a_{n,\mathfrak{p}}}} \right) \times \left(\varprojlim_n \mathbb{Z}_p \right)^{1+\delta_\infty} \times \left(\varprojlim_n Q_{1,n}/Q_{2,n} \right),$$

where $\delta_\infty = \lim_{n\to\infty} \delta_n < \infty$. Clearly, $\varprojlim_n \mathbb{Z}_p$ is always a subgroup of $\mathbb{Z}_p$ and hence has finite $\mathbb{Z}_p$-rank. Let $\mathfrak{p} \in \Sigma$ be a prime. By the following lemma 2.26, $\mathfrak{p}$ is finitely decomposed and hence $\varprojlim_n \prod_{\mathfrak{p}_n|\mathfrak{p}} \mu_{p^{a_{n,\mathfrak{p}_n}}}$ (where the product is over all primes $\mathfrak{p}_n \in \Sigma_n$ lying over $\mathfrak{p}$) is a quotient of $\mathbb{Z}_p^r$, $r < \infty$. □

Lemma 2.26. *There is no prime* $q \in \mathbb{Q}$ *that is completely decomposed in* $\mathbb{Q}^{cyc}$.

Proof. Let $q \in \mathbb{Q}$ be a prime number. Since $q \not\equiv 1 \bmod p^n$ for some $n \geq 1$, q is not completely decomposed in $\mathbb{Q}(\mu_{p^\infty})$ (cf. [**51**, theorem 2.13]). Hence $G(\mathbb{Q}(\mu_{p^\infty})_\mathfrak{q}|\mathbb{Q}_q) \neq 1$ and $G_\mathfrak{q}$ is the pro-p part of this group.

Assume that q is completely decomposed in F_H. Then every prime $\mathfrak{q}|q$ of $\mathbb{Q}(\mu_p)$ is completely decomposed in $\mathbb{Q}(\mu_{p^n})$, i. e. splits into p^{n-1} distinct primes. Let f_n be the minimal positive integer such that $q^{f_n} \equiv$

1 mod p^n. By [**51**, theorem 2.13], q splits into $\frac{\varphi(p^n)}{f_n}$ primes in $\mathbb{Q}(\mu_{p^n})$. Therefore,

$$\frac{\varphi(p^n)}{f_n} = p^{n-1}\frac{\varphi(p)}{f_1}$$

or equivalently $f_n = f_1$ for all $n \geq 1$. That is, $q^{f_1} \equiv 1 \bmod p^n$ for all $n \geq 1$ and hence $q \in \{\pm 1\}$. Contradiction. □

For any field K, let L_K be the maximal abelian unramified pro-p extension of K.

DEFINITION 2.27. The Iwasawa μ invariant of $F_\infty|F$ is defined to be zero if and only if there is a pro-p open subgroup H' of H such that $G(L_{F_{H'}}|F_{H'})$ is a finitely generated $\mathbb{Z}_p$-module. In this case, we write $\mu(F_\infty|F) = 0$.

CONJECTURE 2.28 (Iwasawa). *For any number field K, the μ-invariant $\mu(K^{cyc}|K)$ is zero.*

Ferrero and Washington proved this in case $K|\mathbb{Q}$ is abelian (cf. [**15**] or [**51**, theorem 7.15]).

REMARK. There are non-cyclotomic $\mathbb{Z}_p$-extensions such that $\mu > 0$ (cf. [**23**]).

PROPOSITION 2.29. *$X = X_\Sigma(F_\infty|F)$ is an S-torsion module if and only if $\mu(F_\infty|F) = 0$.*

PROOF. We follow the proof given in [**24**, lemma 1.7].

By proposition 2.9, X is S-torsion if and only if X is a finitely generated $\Lambda(H)$-module. By [**33**, proposition 5.2.16], $\Lambda(H')$ is a local ring and hence, by the topological Nakayama lemma (cf. [**33**, lemma 5.2.18]), X is a finitely generated $\Lambda(H)$-module if and only if $X_{H'}$ is a finitely generated $\mathbb{Z}_p$-module.

Let $F_\Sigma := (F_\infty)_\Sigma(p)$ be the maximal pro-p extension of F_∞ unramified outside Σ. There is the five term exact sequence

$$0 \to H^1(H') \to H^1(G(F_\Sigma|F_{H'})) \to H^1(G(F_\Sigma|F_\infty))^{H'} \to H^2(H')$$

of cohomology groups with coefficients in $\mathbb{Q}_p/\mathbb{Z}_p$. We dualise this sequence and get the exact sequence

$$H_2(H', \mathbb{Z}_p) \to X_{H'} \to G(M_\Sigma|F_{H'}) \to H_1(H', \mathbb{Z}_p) \to 0,$$

where M_Σ is the maximal abelian pro-p extension of $F_{H'}$ (equivalently of F_∞) unramified outside Σ. Since

$$H_i(H', \mathbb{Z}_p) = \mathrm{Tor}_i^{\Lambda(H')}(\mathbb{Z}_p, \mathbb{Z}_p)$$

(cf. [**33**, proposition 5.2.6]), and since $\mathbb{Z}_p$ has a resolution by finitely generated projective $\Lambda(H')$-modules (cf. [**46**, theorem 5.1.2]), the $\mathbb{Z}_p$-modules $H_i(H', \mathbb{Z}_p)$ are finitely generated for all $i \geq 0$. Hence $X_{H'}$ is finitely generated over $\mathbb{Z}_p$ if and only if $G(M_\Sigma|F_{H'})$ is finitely generated over $\mathbb{Z}_p$. Let K be an intermediate field of $F_{H'}|F$ such that $G(F_{H'}|K) \cong \mathbb{Z}_p$. ($K$ exists since there is a surjection $G/H' \twoheadrightarrow \mathbb{Z}_p$.) By [**33**, theorem 10.3.25], the weak Leopoldt conjecture is true for $F_{H'}|K$. Hence $G(M_\Sigma|L_{F_{H'}})$ is a finitely generated $\mathbb{Z}_p$-module (see lemma 2.25). Since $G(L_{F_{H'}}|F_{H'})$ is a finitely generated $\mathbb{Z}_p$-module if and only if $\mu(F_\infty|F) = 0$, this proves the theorem. □

Remark. If Iwasawa's conjecture is true, then X is always an S-torsion module.

Corollary 2.30. *If $\mu(F_\infty|F) = 0$, then $C^\bullet \in Ob(\mathcal{C}_S^{\Lambda(G)})$ and hence we can write*

$$[C^\bullet] \in K_0(\mathcal{C}_S^{\Lambda(G)}) = K_0(\Lambda(G), \Lambda(G)_S).$$

Remark. Let $\Lambda(G)\text{-mod}_{S\text{-tors}}$ be the category of finitely generated S-torsion $\Lambda(G)$-modules. For any ring R, let $\mathrm{gl.dim}\, R \in \mathbb{N} \cup \{\infty\}$ be the global dimension of R, i. e. the supremum of the set of projective dimensions of all R-modules. (The projective dimension of an R-module M is the minimal length of a finite projective resolution of M, if such a resolution exists, and ∞ otherwise.) Let $cd_p(G)$ be the p-cohomological dimension of G. By [**6**, theorem 4.1],

$$\mathrm{gl.dim}\, \Lambda(G) = \mathrm{gl.dim}\, \mathbb{Z}_p + cd_p(G).$$

Assume that G is a compact p-adic Lie group which contains no element of order p. By proposition 1.12, this implies $cd_p(G) < \infty$. Since $\mathbb{Z}_p$ is a principal ideal domain, we have $\mathrm{gl.dim}\, \mathbb{Z}_p = 1$. Hence $\mathrm{gl.dim}\, \Lambda(G) < \infty$. That means

$$\mathcal{H}_S^{\Lambda(G)} = \Lambda(G)\text{-mod}_{S\text{-tors}}.$$

By corollary 1.18, every element in $K_1(\Lambda(G)_S)$ may be represented by a 1×1-matrix. We then get

$$\partial([(\Lambda(G)_S, f)]) = [\mathrm{coker}\, f] = [\Lambda(G)/\Lambda(G)f] \in K_0(\Lambda(G)\text{-mod}_{S\text{-tors}})$$

if $f \in \Lambda(G)_S^\times \cap \Lambda(G)$ (cf. [**48**, §3]).

Lemma 2.31. *Assume that G has no element of order p. Let $D^\bullet \in Ob(\mathcal{C}_S^{\Lambda(G)})$ be a cochain complex. Then the image of $[D^\bullet]$ in $K_0(\mathcal{H}_S^{\Lambda(G)})$ is $\sum_{i\in\mathbb{Z}}(-1)^i[H^i(D^\bullet)]$.*

Proof. Let H be an object of $\mathcal{H}_S^{\Lambda(G)}$ and let $0 \leftarrow H \leftarrow P_\bullet$ be a projective resolution of H. By proposition 1.25, the map

$$\varphi : K_0(\mathcal{H}_S^{\Lambda(G)}) \to K_0(\mathcal{C}_S^{\Lambda(G)}), \quad [H] \mapsto [P_\bullet]$$

is an isomorphism. Put

$$\psi : K_0(\mathcal{C}_S^{\Lambda(G)}) \to K_0(\mathcal{H}_S^{\Lambda(G)}), \quad [D^\bullet] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i [H^i(D^\bullet)].$$

We have seen above that $\mathcal{H}_S^{\Lambda(G)} = \Lambda(G)\text{-mod}_{S\text{-tors}}$. Hence $H^i(D^\bullet) \in Ob(\mathcal{H}_S^{\Lambda(G)})$ for all $i \in \mathbb{Z}$. Obviously, ψ does not depend on the choice of representatives. Hence ψ is a homomorphism. Since

$$\psi \circ \varphi([H]) = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(P_\bullet)] = [H^0(P_\bullet)] = [H],$$

$\psi \circ \varphi$ is the identity map on $K_0(\mathcal{H}_S^{\Lambda(G)})$. Since φ is an isomorphism, $\psi = \varphi^{-1}$. $\square$

Using lemma 2.19, we get the

Corollary 2.32. *If G has no element of order p, then*

$$-[C^\bullet] = [X] - [\mathbb{Z}_p] \in K_0(\Lambda(G)\text{-}mod_{S\text{-}tors}).$$

3. Analytic Part

In this subsection, we give the definition of the p-adic zeta function ξ_Σ for $F_\infty|F$ with respect to Σ (if it exists). It is defined to be an element of $K_1(\Lambda(G)_S)$ that interpolates the Artin L-function of $F_\infty|F$, with the Euler factors at Σ removed, for all Artin characters and all odd negative integers.

Let $\mathfrak{P} \notin \Sigma$ be a prime ideal of F_∞ and put $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_F$. Since $\mathfrak{P}|\mathfrak{p}$ is unramified, the decomposition group $G_\mathfrak{P}$ of $\mathfrak{P}$ over F is generated by the Frobenius element $\sigma_\mathfrak{P} \in G(F_\infty|F)$. ($\sigma_\mathfrak{P}$ is defined by $\sigma_\mathfrak{P} x \equiv x^{\#\mathcal{O}_F/\mathfrak{p}} \bmod \mathfrak{P}$ for all $x \in \mathcal{O}_{F_\infty}$.) Let $\mathbf{1}_n \in GL_n(\overline{\mathbb{Q}})$ be the unit matrix.

Definition 2.33. For a representation

$$\rho : G \to GL_n(\overline{\mathbb{Q}}) \subset GL_n(\mathbb{C})$$

and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the complex Artin L-function with respect to Σ is defined by

$$L_\Sigma(s, \rho) = \prod_{\mathfrak{p}\notin\Sigma} \det(\mathbf{1}_n - \rho(\sigma_\mathfrak{P})\mathfrak{N}(\mathfrak{p})^{-s})^{-1} \in \mathbb{C},$$

where the product is over all prime ideals $\mathfrak{p}$ of F with $\mathfrak{p} \notin \Sigma$ and where for each $\mathfrak{p}$, $\mathfrak{P} \subset \mathcal{O}_{F_\infty}$ is a prime ideal with $\mathfrak{P}|\mathfrak{p}\mathcal{O}_{F_\infty}$. (This is the ordinary Artin L-function, with the Euler factors at Σ removed, cf. [**31**, ch. VII, def. 10.1].)

For $\mathrm{Re}(s) \geq 1 + \delta$ with $\delta > 0$, the Artin L-series converges absolutely and uniformly (cf. [**31**]). The characteristic polynomial $\det(\mathbf{1}_n - \rho(\sigma_{\mathfrak{P}})t) \in \mathbb{C}[t]$ depends only on $\mathfrak{p}$, not on $\mathfrak{P}$. Hence $L_\Sigma(s,\rho)$ is well-defined.

PROPOSITION 2.34. (1) *Let $\rho : G \to GL_n(\overline{\mathbb{Q}}), \rho' : G \to GL_m(\overline{\mathbb{Q}})$ be two Artin representations of G. Then*

$$L_\Sigma(s, \rho \oplus \rho') = L_\Sigma(s,\rho) L_\Sigma(s,\rho').$$

(2) *Let $\tilde{F}|F_\infty$ be a Galois extension and set $\tilde{G} := G(\tilde{F}|F)$. Assume that $\tilde{G}$ is a compact p-adic Lie group. Let $\inf_{\tilde{G}}^{G}(\rho) : \tilde{G} \to GL_n(\overline{\mathbb{Q}})$ be the Artin representation that factors through $\rho : G \to GL_n(\overline{\mathbb{Q}})$. Then*

$$L_\Sigma(s,\rho) = L_\Sigma(s, \inf_{\tilde{G}}^{G}(\rho)).$$

(3) *Let U be an open normal subgroup of G and let $\rho : U \to GL_n(\overline{\mathbb{Q}})$ be an Artin representation. Then*

$$L_\Sigma(s,\rho) = L_\Sigma(s, \mathrm{ind}_G^U(\rho)).$$

PROOF. We use the proof given in [**31**, 10.4], which we can obviously apply to our situation.

(1) This follows from

$$\det(\mathbf{1}_{n+m} - (\rho \oplus \rho')(\sigma_{\mathfrak{P}})t) = \det(\mathbf{1}_n - \rho(\sigma_{\mathfrak{P}})t) \det(\mathbf{1}_m - \rho'(\sigma_{\mathfrak{P}})t).$$

(2) Let $\mathfrak{P}'|\mathfrak{P}|\mathfrak{p}$ be prime ideals of $\tilde{F}|F_\infty|F$, lying one above the other, with $\mathfrak{p} \notin \Sigma$. The natural projection $\tilde{G} \to G$ induces the homomorphism $G_{\mathfrak{P}'} \to G_{\mathfrak{P}}$, which maps $\sigma_{\mathfrak{P}'}$ to $\sigma_{\mathfrak{P}}$. Hence

$$\det(\mathbf{1}_n - \inf_{\tilde{G}}^{G}(\rho)(\sigma_{\mathfrak{P}'})t) = \det(\mathbf{1}_n - \rho(\sigma_{\mathfrak{P}})t).$$

(3) We write $\rho : U \to GL(W)$, $\mathrm{ind}_G^U(\rho) : G \to GL(V)$, where $V = \mathrm{ind}_G^U(W)$. Let $Z := F_\infty^U$ be the fixed field of U. Set

$$f := (G : U) = [Z : F] < \infty.$$

Let $\mathfrak{p} \notin \Sigma$ be a prime ideal of F. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the prime ideals of Z lying over $\mathfrak{p}$. For $i = 1, \ldots, r$, let $\mathfrak{P}_i$ be a prime ideal of F_∞ lying over $\mathfrak{q}_i$. Let $G_i := G_{\mathfrak{P}_i}$ be the decomposition group of $\mathfrak{P}_i$ over $\mathfrak{p}$. Then $U_i := G_i \cap U$ is the decomposition group of $\mathfrak{P}_i$ over $\mathfrak{q}_i$. Put $f_i := (G_i : U_i)$. Then

$$\mathfrak{N}(\mathfrak{q}_i) = \mathfrak{N}(\mathfrak{p})^{f_i},$$

where $\mathfrak{N}(\cdot)$ denotes the norm of an ideal. Let $\tau_i \in G$ be elements such that $\tau_i(\mathfrak{P}_i) = \mathfrak{P}_1$. Then $G_i = \tau_i^{-1} G_1 \tau_i$, $\sigma_{\mathfrak{P}_i} =$

$\tau_i^{-1}\sigma_{\mathfrak{P}_1}\tau_i \in G_i$ and $\sigma_{\mathfrak{P}_i}^{f_i} \in U_i$ is the Frobenius homomorphism of $\mathfrak{P}_i$ over Z. Set $\sigma := \sigma_{\mathfrak{P}_1}$. We need to show

$$\det(\mathbf{1}_{nf} - \mathrm{ind}_G^U(\rho)(\sigma)t; V) = \prod_{i=1}^r \det(\mathbf{1}_n - \rho(\sigma_{\mathfrak{P}_i}^{f_i})t^{f_i}; W).$$

We reduce this to the case $G_1 = G$. For each $i \in \{1, \dots, r\}$, let $\{\alpha_{ij}\}_{j=1,\dots,f_i}$ be a system of representatives of $G_1/(G_1 \cap \tau_i U \tau_i^{-1})$ in G_1. Then $\{\alpha_{ij}\tau_i\}_{ij}$ is a set of representatives of G/U in G. Thus $V = \bigoplus_{ij} \alpha_{ij}\tau_i W$. We put

$$V_i := \bigoplus_{j=1}^{f_i} \alpha_{ij}\tau_i W$$

and get the decomposition $V = \bigoplus_{i=1}^r V_i$ of the G_1-module V. Hence

$$\det(\mathbf{1}_{nf} - \mathrm{ind}_G^U(\rho)(\sigma)t; V) = \prod_{i=1}^r \det(\mathbf{1}_{nf_i} - \mathrm{ind}_{G_1}^U(\rho)(\sigma)t; V_i),$$

and since $\det(\mathbf{1}_n - \rho(\sigma_{\mathfrak{P}_i}^{f_i})t^{f_i}; W) = \det(\mathbf{1}_n - \rho(\sigma^{f_i})t^{f_i}; \tau_i W)$, it suffices to prove

$$\det(\mathbf{1}_{nf_i} - \mathrm{ind}_{G_1}^U(\rho)(\sigma)t; V_i) = \det(\mathbf{1}_n - \rho(\sigma^{f_i})t^{f_i}; \tau_i W).$$

We may assume $G = G_1$, $V = V_1$ and $f = f_1$. Then $G = \langle \sigma \rangle$ and hence

$$V = \bigoplus_{i=0}^{f-1} \mathrm{ind}_G^U(\sigma^i)W.$$

Let A be the matrix of $\rho(\sigma^f)$ with respect to a basis $w_1, \dots, w_n$ of W. Then

$$\begin{pmatrix} 0 & \mathbf{1}_n & & \\ & \ddots & \ddots & \\ & & \ddots & \mathbf{1}_n \\ A & & & 0 \end{pmatrix}$$

is the matrix of $\mathrm{ind}_G^U(\rho)(\sigma)$ with respect to the basis

$$\{\mathrm{ind}_G^U(\rho)(\sigma^i)w_j\}_{\substack{i=f-1,\dots,0 \\ j=1,\dots,n}}.$$

Hence

$$\det(\mathbf{1}_{nf} - \mathrm{ind}_G^U(\rho)(\sigma)t) = \det \begin{pmatrix} \mathbf{1}_n & -t\mathbf{1}_n & & \\ & \ddots & \ddots & \\ & & \ddots & -t\mathbf{1}_n \\ -tA & & & \mathbf{1}_n \end{pmatrix}$$
$$= \det(\mathbf{1}_n - \rho(\sigma^f)t^f).$$

(We get the last equation by adding the t-fold of the first column to the second column, etc.) $\square$

For $\sigma \in G$, let $\tilde{\sigma} \in G(F(\mu_{p^\infty})|F)$ be an element, whose image under $G(F(\mu_{p^\infty})|F) \to G(F(\mu_{p^\infty})^+|F)$ coincides with the image of σ under $G \to G(F(\mu_{p^\infty})^+|F)$. Let $\rho : G \to GL_n(\overline{\mathbb{Q}})$ be an Artin representation and let $\kappa : G(F(\mu_{p^\infty})|F) \to \mathbb{Z}_p^\times$ be the cyclotomic character. For even integers r, we define the continuous representation

$$\rho\kappa^r : G \to GL_n(\overline{\mathbb{Q}}_p), \quad \sigma \mapsto \rho(\sigma)\kappa(\tilde{\sigma})^r.$$

We fix an isomorphism $\mathbb{C}_p \cong \mathbb{C}$ and hence we may define $f(\rho\kappa^r) \in \mathbb{C}$ for $f \in K_1(\Lambda(G)_S)$.

Conjecture 2.35. *There is a unique element $\xi \in K_1(\Lambda(G)_S)$, such that*

$$\xi(\rho\kappa^r) = L_\Sigma(1-r, \rho) \in \mathbb{C}$$

for any Artin representation ρ of G and any even integer $r \geq 2$.

Definition 2.36. If the element $\xi = \xi_\Sigma(F_\infty|F)$ in conjecture 2.35 exists, it is called the *p-adic zeta function for $F_\infty|F$ with respect to Σ.*

4. The Main Conjecture and Burns' Theorem

The mysterious connection between the p-adic zeta function and the complex $[C^\bullet]$ is conjectured as follows (cf. [**25**]):

Conjecture 2.37 (Main Conjecture). *Assume the p-adic zeta function for $F_\infty|F$ with respect to Σ of conjecture 2.35, $\xi \in K_1(\Lambda(G)_S)$, exists. Then*

$$\partial(\xi) = -[C^\bullet] \in K_0(\Lambda(G), \Lambda(G)_S).$$

Theorem 2.38 (Main Conjecture of Commutative Iwasawa Theory). *Assume that G is an abelian group. Then the main conjecture for G is true.*

Proof. Assume that G is one dimensional as a p-adic Lie group. Then this well-known theorem follows from deep results of Kubota and Leopoldt, Iwasawa, Deligne and Ribet, Mazur and Wiles among others. The case $\dim(G) > 1$ can be reduced to the case $\dim(G) = 1$, cf. [**24**, theorem 1.15]. $\square$

Remark. K. Iwasawa first constructed a p-adic zeta function and formulated a main conjecture for $\mathbb{Q}^{cyc}|\mathbb{Q}$, following work of T. Kubota and H. W. Leopoldt. J. Coates formulated certain hypotheses under which the p-adic zeta function for the cyclotomic extension of arbitrary totally real number fields exists. Deligne and Ribet and also

P. Cassou-Noguès proved these hypotheses. The main conjecture in this situation was formulated by Coates and R. Greenberg. The main conjecture for $\mathbb{Q}^{cyc}|\mathbb{Q}$ was proven by B. Mazur and A. Wiles, after deep results in this direction by Iwasawa. K. Rubin gave another proof using V. Kolyvagin's Euler systems. For the cyclotomic extension of totally real number fields, the main conjecture was proven by Wiles.

We try to deduce the noncommutative Main Conjecture from the Main Conjectures for all abelian subquotients of G. In the following, we specify an upper bound for the set of subquotients that we are going to consider.

Let P be a compact p-adic Lie group with a surjection $\omega : P \twoheadrightarrow \mathbb{Z}_p$. Let $\mathcal{I} = \mathcal{I}(P)$ be a set of pairs (U, V), where U is an open normal subgroup of P and V is a closed subgroup of $\ker\omega$, such that V is a normal subgroup of U and U/V is commutative.

Let $\mathcal{I}$ be such a set for G with the surjection $G \twoheadrightarrow \Gamma$. For all $(U, V) \in \mathcal{I}$, we define

$$\theta_{U,V} : K_1(\Lambda(G)) \to \Lambda(U/V)^\times$$

to be the composition of homomorphisms

$$\begin{aligned} \mathrm{N} &: K_1(\Lambda(G)) \to K_1(\Lambda(U)) \quad \text{and} \\ p_* &: K_1(\Lambda(U)) \to K_1(\Lambda(U/V)) = \Lambda(U/V)^\times. \end{aligned}$$

(The latter identity follows from proposition 1.17 and the commutativity of U/V.) This induces the homomorphism

$$\theta : K_1(\Lambda(G)) \to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times, \quad x \mapsto (\theta_{U,V}(x))_{(U,V)\in\mathcal{I}}.$$

We define

$$\theta_{S,U,V} : K_1(\Lambda(G)_S) \to \Lambda(U/V)_S^\times$$

to be the composition of homomorphisms

$$\begin{aligned} \mathrm{N} &: K_1(\Lambda(G)_S) \to K_1(\Lambda(U)_S) \quad \text{and} \\ p_* &: K_1(\Lambda(U)_S) \to K_1(\Lambda(U/V)_S) = \Lambda(U/V)_S^\times. \end{aligned}$$

(The latter identity follows from proposition 2.8.)

Then we get the following homomorphism:

$$\theta_S : K_1(\Lambda(G)_S) \to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)_S^\times, \quad x \mapsto (\theta_{S,U,V}(x))_{(U,V)\in\mathcal{I}}$$

Remark. Let $f \in K_1(\Lambda(G)_S)$ and let $\chi : U/V \to \overline{\mathbb{Q}}^\times$ be a one-dimensional Artin representation. By lemma 2.14,

$$\theta_{S,U,V}(f)(\chi) = f(\mathrm{ind}_G^U(\mathrm{inf}_U^{U/V}(\chi))).$$

By corollary 2.11, there is a commutative diagram

$$\begin{tikzcd} K_1(\Lambda(G)_S) \arrow[r, "\theta_{S,U,V}"] & \Lambda(U/V)^\times_S \\ K_1(\Lambda(G)) \arrow[u] \arrow[r, "\theta_{U,V}"] & \Lambda(U/V)^\times . \arrow[u, hook] \end{tikzcd}$$

Property 2.39. *Let*

$$\Psi_S \leq \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times_S$$

and

$$\Psi \leq \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times$$

be subgroups. We assume that the following holds:

(1) $\Psi = \Psi_S \cap \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times$
(2) $\mathrm{im}\,(\theta_S) \subset \Psi_S$
(3) $\theta : K_1(\Lambda(G)) \to \Psi$ *is an isomorphism.*
(4) *Every Artin representation ρ of G is – interpreted as a virtual representation – a $\mathbb{Z}$-linear combination of induced representations $\mathrm{ind}_{U_i}^G \circ \inf_{U_i}^{U_i/V_i}(\chi_i)$ with $(U_i, V_i) \in \mathcal{I}$ and with $\chi_i : U_i/V_i \to \overline{\mathbb{Q}}^\times$ a character of finite order.*

For $(U, V) \in \mathcal{I}$ let

$$\xi_{U,V} := \xi_\Sigma(F_V|F_U) \in K_1(\Lambda(U/V)_S) = \Lambda(U/V)^\times_S \subset Q(U/V)$$

be the p-adic zeta function for $F_V|F_U$ with respect to Σ. The following theorem is due to D. Burns and K. Kato (cf. [**25**]).

Theorem 2.40. *Let $\Psi \leq \prod_{\mathcal{I}} \Lambda(U/V)$ and $\Psi_S \leq \prod_{\mathcal{I}} \Lambda(U/V)_S$ be subgroups for which property 2.39 holds and such that $(\xi_{U,V})_{(U,V)\in\mathcal{I}} \in \Psi_S$. Then the p-adic zeta function $\xi = \xi_\Sigma(F_\infty|F)$ for $F_\infty|F$ (with respect to Σ) exists uniquely and the main conjecture $\partial(\xi) = -[C]$ is true.*

Remark. In chapter 3, we define groups Ψ and Ψ_S for a certain class of Galois groups G and show that they satisfy property 2.39. In chapter 4, we deal with certain congruences of the zeta functions $\xi_{U,V}$ in order to show $(\xi_{U,V})_{\mathcal{I}} \in \Psi_S$.

Proof. Surjectivity of $\partial : K_1(\Lambda(G)_S) \to K_0(\Lambda(G), \Lambda(G)_S)$ (cf. lemma 2.15) implies that there is $f \in K_1(\Lambda(G)_S)$ such that $\partial(f) = -[C]$. For $(U, V) \in \mathcal{I}$, we define

$$\begin{aligned} f_{U,V} &:= \theta_{S,U,V}(f) \in \Lambda(U/V)^\times_S \quad \text{and} \\ u_{U,V} &:= \xi_{U,V} f_{U,V}^{-1} \in \Lambda(U/V)^\times_S . \end{aligned}$$

Consider the commutative diagram

$$\begin{tikzcd}
K_1(\Lambda(G)_S) \arrow[r, "\mathrm{N}"] \arrow[d, "\partial"] & K_1(\Lambda(U)_S) \arrow[r, "p_*"] \arrow[d, "\partial"] & \Lambda(U/V)_S^\times \arrow[d, two heads, "\partial"] \\
K_0(\Lambda(G), \Lambda(G)_S) \arrow[r, "\mathrm{Tr}"] & K_0(\Lambda(U), \Lambda(U)_S) \arrow[r, "p_*"] & \Lambda(U/V)_S^\times/\Lambda(U/V)^\times.
\end{tikzcd}$$

By lemma 2.21, the image of $-[C]$ under $p_* \circ \mathrm{Tr}$ is $-[C_{U,V}]$. Hence $\partial(f_{U,V}) = -[C_{U,V}]$. Theorem 2.38 (the main conjecture of commutative Iwasawa theory) implies

$$\partial(\xi_{U,V}) = -[C_{U,V}].$$

Therefore $u_{U,V} \in \ker \partial = \Lambda(U/V)^\times$. Property 2.39 (2) implies

$$(f_{U,V})_{(U,V)\in\mathcal{I}} \in \Psi_S.$$

Using the assumption $(\xi_{U,V})_{(U,V)\in\mathcal{I}} \in \Psi_S$, we get that

$$(u_{U,V})_{(U,V)\in\mathcal{I}} \in \Psi_S \cap \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times = \Psi.$$

By property 2.39 (3), there is a unique element $u \in K_1(\Lambda(G))$, such that

$$\theta_{U,V}(u) = u_{U,V} \quad \text{for all } (U,V) \in \mathcal{I}.$$

By property 2.39 (3) and the commutativity of the diagram

$$\begin{tikzcd}
K_1(\Lambda(G)_S) \arrow[r, "\theta_S"] & \prod_{\mathcal{I}} \Lambda(U/V)_S^\times \\
K_1(\Lambda(G)) \arrow[r, "\theta"] \arrow[u] & \prod_{\mathcal{I}} \Lambda(U/V)^\times, \arrow[u, hook]
\end{tikzcd}$$

the natural map $K_1(\Lambda(G)) \hookrightarrow K_1(\Lambda(G)_S)$ is injective. We identify $K_1(\Lambda(G))$ with its image in $K_1(\Lambda(G)_S)$. Now, we can define

$$\xi := uf \in K_1(\Lambda(G)_S).$$

Then

$$\partial(\xi) = \partial(uf) = \partial(f) = -[C]$$

(since $u \in K_1(\Lambda(G)) = \ker \partial$ by the exact sequence (4)) and

$$\theta_{S,U,V}(\xi) = \theta_{S,U,V}(u)\theta_{S,U,V}(f) = u_{U,V} f_{U,V} = \xi_{U,V}$$

for all $(U,V) \in \mathcal{I}$ ($\theta_{S,U,V}(u) = \theta_{U,V}(u)$ by corollary 2.11).

Now, we will show that $\xi(\rho\kappa^r) = L_\Sigma(1-r,\rho)$ for $r \geq 2$ and all Artin representations ρ of G. By property 2.39 (4), we can write $\rho = \sum_{i=0}^m n_i \mathrm{ind}_{U_i}^G(\chi_i)$. By proposition 2.34,

$$L_\Sigma(1-r,\rho) = \prod_{i=0}^m L_\Sigma(1-r, \mathrm{ind}_G^{U_i}(\chi_i))^{n_i} = \prod_{i=0}^m L_\Sigma(1-r,\chi_i)^{n_i}.$$

For two characters χ, χ', we have $\xi(\chi+\chi')=\xi(\chi)\xi(\chi')$ by lemma 2.14. Therefore

$$\xi\left(\rho\kappa^r\right)=\prod_{i=0}^{m}\xi\left(\operatorname{ind}_G^{U_i}(\chi_i)\kappa^r\right)^{n_i}.$$

Let $\kappa_U : G(F_U(\mu_{p^\infty})|F_U)\to\mathbb{Z}_p^\times$ be the cyclotomic character of the field F_U. By the remark after the definition of θ_S (see also lemma 2.14), we have

$$\xi\left(\operatorname{ind}_G^{U_i}(\chi_i)\kappa^r\right)=\theta_{S,U,V}(\xi)(\chi_i\kappa_U^r)=\xi_{U,V}(\chi_i\kappa_U^r).$$

The interpolation property of $\xi_{U,V}$ in the commutative case implies

$$\xi_{U,V}(\chi_i\kappa_U^r)=L_\Sigma(1-r,\chi_i).$$

This proves the existence of the p-adic zeta function. We will now show its uniqueness.

Let $\tilde{\xi}$ be another element that satisfies the conditions of the main conjecture. Then $\partial(\xi\tilde{\xi}^{-1})=0$ and hence $\xi\tilde{\xi}^{-1}\in K_1(\Lambda(G))$. By the uniqueness of the p-adic zeta function in the commutative case, we get that $\theta_{S,U,V}(\tilde{\xi})=\xi_{U,V}$. Hence $\theta(\xi\tilde{\xi}^{-1})=1$, and thus $\xi=\tilde{\xi}$. $\square$

CHAPTER 3

K_1 of Certain Noncommutative Iwasawa Algebras

Let P be a pro-p p-adic Lie group with a surjection $P \twoheadrightarrow \mathbb{Z}_p$ that is a quotient of the product of the p-adic Heisenberg group and a commutative p-adic Lie group. In this chapter, we show the existence of groups Ψ and Ψ_S that satisfy property 2.39.

We consider the following more general situation: Let R be a topological ring and let $\underline{\underline{c}}$ be an index set. For $n \in \underline{\underline{c}}$, let $R_n \subset R$ be a subring such that R is a free finite dimensional R_n-module and let $J_n \subset R_n$ be a two sided ideal. Define

$$\begin{aligned}\theta_n &: K_1(R) \xrightarrow{\mathrm{N}} K_1(R_n) \xrightarrow{p_*} K_1(R_n/J_n)\\ \theta = (\theta_n)_n &: K_1(R) \to \prod_{n \in \underline{\underline{c}}} K_1(R_n/J_n).\end{aligned}$$

We will define a trace homomorphism

$$\mathrm{Tr} : R/\overline{[R,R]} \to R_n/\overline{[R_n,R_n]}$$

and put

$$\begin{aligned}\tau_n : R/\overline{[R,R]} &\xrightarrow{\mathrm{Tr}} R_n/\overline{[R_n,R_n]}\\ &\xrightarrow{p_*} (R_n/J_n)/\overline{[(R_n/J_n),(R_n/J_n)]}\\ \tau = (\tau_n)_n : R/\overline{[R,R]} &\to \prod_{n \in \underline{\underline{c}}} (R_n/J_n)/\overline{[(R_n/J_n),(R_n/J_n)]}.\end{aligned}$$

We set $\Psi := \operatorname{im}\theta$ and $\Omega := \operatorname{im}\tau$. Assume that the following holds:

(1) τ is injective.
(2) There is a commutative diagram

$$\begin{tikzcd} K_1(R) \arrow[r, "\mathscr{L}"] \arrow[d, "\theta"] & R/\overline{[R,R]} \arrow[d, "\tau"] \\ \Psi \arrow[r, "\widetilde{\mathscr{L}}"] & \Omega \end{tikzcd}$$

of abelian groups.
(3) $\ker \mathscr{L} \cap \ker\theta = \{0\}$

Then, by the snake lemma, there is a commutative diagram of abelian groups

$$
\begin{tikzcd}
1 \arrow[r] & \ker \mathscr{L} \arrow[r] \arrow[d, "\cong"] & K_1(R) \arrow[r, "\mathscr{L}"] \arrow[d, "\theta"] & R/\overline{[R,R]} \arrow[r] \arrow[d, "\cong"', "\tau"] & \operatorname{coker} \mathscr{L} \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & \theta(\ker \mathscr{L}) \arrow[r] & \Psi \arrow[r, "\widetilde{\mathscr{L}}"] & \Omega \arrow[r] & \tau(\operatorname{coker} \mathscr{L}) \arrow[r] & 1
\end{tikzcd}
\tag{12}
$$

and the five lemma implies that θ is an isomorphism.

For $n = \underline{\underline{c}}$, we will define certain open normal subgroups U_n of P. Put $V_n := [U_n, U_n]$. We will prove the above assumptions for

$$R = \Lambda(P) \qquad R_n = \Lambda(U_n) \qquad J_n = I(V_n)\Lambda(U_n).$$

It is not difficult to show assumption (1). In this setting, the homomorphism $\mathscr{L}$ of (2) is called the *integral logarithm*. For its definition, we need the fact that P is pro-p. We will use an explicit description of Ψ and Ω in terms of certain generators of U_n to define $\widetilde{\mathscr{L}}$. For the proof of assumption (3), we use the fact that $\ker \mathscr{L} = \mu_{p-1} \times P^{ab}$, which follows from a theorem of R. Oliver.

Now assume that

$$R = \Lambda(P)_S \qquad R_n = \Lambda(U_n)_S \qquad J_n = I(V_n)\Lambda(U_n)_S,$$

where R is endowed with the discrete topology. In general, R is not p-adically complete, and hence the above logarithm may not exist. We will define completions $\left(R/\overline{[R,R]}\right)^{\wedge}$ and $\widehat{\Omega_S}$ containing the corresponding p-adic completions (cf. definition 3.10 and corollary 3.13). We will show that there is a commutative diagram

$$
\begin{tikzcd}
K_1(R) \arrow[r, "\mathscr{L}"] \arrow[d, "\theta"] & \left(R/\overline{[R,R]}\right)^{\wedge} \arrow[d, "\tau"] \\
\Psi_S \arrow[r, "\widetilde{\mathscr{L}}"] & \widehat{\Omega_S},
\end{tikzcd}
$$

where Ψ_S and Ω_S are groups defined in analogy to the explicit description of Ψ and Ω. Then property 2.39 (2) is plain and property 2.39 (1) follows from the definition of Ψ_S.

Property 2.39 (4) (for groups P as above) is a consequence of a well-known fact of the representation theory of finite groups.

1. The Heisenberg Group

Let p be an odd prime. Let $\mathcal{H}$ be the p-adic Heisenberg group, i. e.

$$\mathcal{H} = \begin{pmatrix} 1 & \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 1 & \mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix}.$$

Define

$$\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}.$$

We first observe some group theoretic properties of $\mathcal{H}$ and P. By direct calculations, we get the following useful identities (with $i, j, k, l \in \mathbb{Z}_p$):

(1) $\alpha^i \beta^j \gamma^k = \begin{pmatrix} 1 & i & ij+k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix}$
(2) $\gamma = \alpha\beta\alpha^{-1}\beta^{-1} = \alpha^{-1}\beta^{-1}\alpha\beta$
(3) $\alpha\gamma = \gamma\alpha, \quad \beta\gamma = \gamma\beta$
(4) $[\alpha^i\beta^j, \alpha^k\beta^l] = \gamma^{il-jk}$
(5) $\alpha^k(\alpha^i\beta^j)\alpha^{-k} = (\alpha^i\beta^j)\gamma^{jk}, \quad \beta^k(\alpha^i\beta^j)\beta^{-k} = (\alpha^i\beta^j)\gamma^{-ik}$

Remark. $\mathcal{H}$ is a p-adic Lie group. There is a global atlas of $\mathcal{H}$:

$$\mathcal{H} \to \mathbb{Z}_p^3, \quad \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \mapsto (i, j, k)$$

Let P be a compact pro-p p-adic Lie group with surjective homomorphism $\omega : P \twoheadrightarrow \mathbb{Z}_p$.

Assumption 3.1. *We assume that there is a commutative p-adic Lie group N and a surjective homomorphism of p-adic Lie groups*

$$s : \mathcal{H} \times N \longrightarrow P.$$

Remark. We will prove the main conjecture for extensions $F_\infty|F$ satisfying assumption 2.1 such that $G = G(F_\infty|F)$ with the surjection $G \twoheadrightarrow \Gamma$ satisfies assumption 3.1.

Henceforth, we denote the elements $s(\alpha, 1), s(\beta, 1), s(\gamma, 1) \in P$ by α, β, γ. To simplify notation, we define the following constant $c = c(\gamma) \in \mathbb{N} \cup \{\infty\}$: If γ is of finite order we require

$$\mathrm{ord}(\gamma) = p^c.$$

If γ is of infinite order, we set $c = \infty$. For $n \in \mathbb{N}$, we define $\underline{\underline{n}} := \{0, \dots, n\} \subset \mathbb{N}$ and put $\underline{\underline{\infty}} := \mathbb{N}$.

For a group W, we define the centre of W by

$$Z(W) := \{g \in W | gh = hg \text{ for all } h \in W\}.$$

For $n \in \mathbb{N}$, we define the following subgroups of P, where $\langle \cdot \rangle$ denotes the closed subgroup of P, generated by " $\cdot$ ":

$$U_n := \langle \alpha, \beta^{p^n}, Z(P) \rangle$$
$$V_n := [U_n, U_n] = \langle \gamma^{p^n} \rangle$$

The last identity follows from $V_n = \langle [g, h] | g, h \in \{\alpha, \beta^{p^n}\} \cup Z(P) \rangle = \langle [\alpha, \beta^{p^n}] \rangle$.

We calculate the centre $Z(P)$. Any element $x \in P$ can be written in the form $x = \alpha^i \beta^j \gamma^k z$ with $z \in s(1 \times N)$. The element

$$[\alpha^i \beta^j \gamma^k z, \alpha^{\tilde{\imath}} \beta^{\tilde{\jmath}} \gamma^{\tilde{k}} \tilde{z}] = \gamma^{\tilde{\jmath} i - j \tilde{\imath}}$$

is trivial for all $\tilde{\imath}, \tilde{\jmath}, \tilde{k} \in \mathbb{Z}_p$ and all $\tilde{z} \in s(1 \times N)$ if and only if $i, j \in p^c \mathbb{Z}_p$ for $c < \infty$ and $i = j = 0$ for $c = \infty$. Hence

$$Z(P) = \langle \alpha^{p^c}, \beta^{p^c}, \gamma, s(1 \times N) \rangle \tag{13}$$

if we put $g^{p^c} := 1$ for $c = \infty$ and $g \in P$. We will later use the facts that $[P, P] \subset Z(P)$, and that $(P : Z(P)) < \infty$ if and only if $c < \infty$.

Now, we get

$$U_n = \langle \alpha, \beta^{p^{\min\{n,c\}}}, \gamma, s(1 \times N) \rangle.$$

We set $U_\infty := \bigcap_n U_n = \langle \alpha, \beta^{p^c}, \gamma, s(1 \times N) \rangle$.

LEMMA 3.2. *The group U_n is an open normal subgroup of P, V_n is a closed subgroup of* $\ker \omega$ *and a normal subgroup of U_n and U_n/V_n is commutative. Hence the definition $\mathcal{I} := \mathcal{I}(P) := \{(U_n, V_n)\}_{n \in \underline{\underline{c}}}$ is consistent with the notation in the previous chapter.*

PROOF. U_m is an open normal subgroup of U_n for $n \leq m$, since $gU_m g^{-1} \subset U_m$ for every generator g of U_m. For $m, n \in \underline{\underline{c}}$, we get that

$$U_n/U_m \cong (U_n/U_\infty)/(U_m/U_\infty) \cong \langle \beta^{p^n} \rangle / \langle \beta^{p^m} \rangle$$

is a finite cyclic group of order p^{n-m} (since $\operatorname{ord}(\beta) \geq \operatorname{ord}(\gamma)$), generated by the image of β^{p^n}. In particular, U_n is an open subgroup of $U_0 = P$, and we have the identity

$$G/U_n \cong \langle \beta \rangle / \langle \beta^{p^n} \rangle \cong \mathbb{Z}/p^n\mathbb{Z}.$$

Since $\mathbb{Z}_p$ is commutative, we get $V_n = [U_n, U_n] \subset \ker \omega$. The fact that V_n is a normal subgroup of U_n and that U_n/V_n is commutative follow directly from the definition of V_n. $\square$

REMARK. For $n \in \underline{\underline{c}}$,

$$U_n/V_n = \langle \alpha, \beta^{p^n}, \gamma, s(1 \times N) \rangle / \langle \gamma^{p^n} \rangle.$$

Hence there is a surjection

$$\mathbb{Z}_p^2 \times (\mathbb{Z}/p^n\mathbb{Z}) \times s(1 \times N) \twoheadrightarrow U_n/V_n.$$

The kernel of this map depends on s.

LEMMA 3.3. *Assume that Leopoldt's conjecture is true for F and that $G = G(F_\infty|F)$ with the surjection $G \twoheadrightarrow \Gamma$ satisfies assumption 3.1. Then G is of dimension 1 as a p-adic Lie group.*

PROOF. Since F is totally real, Leopoldt's conjecture implies

$$\dim G^{ab} = 1. \tag{14}$$

Since $G^{ab} = G/\langle \gamma \rangle$, it suffices to show that γ is of finite order. By equation (14), the homomorphism

$$\mathbb{Z}_p^2 \to G^{ab}, \quad (m, n) \mapsto \bar{\alpha}^m \bar{\beta}^n$$

is not injective. Let (m, n) be a non-trivial element of the kernel. Without loss of generality, we may assume that $m \neq 0$. Since $[G, G] \subset Z(G)$, the commutator map

$$[-, -] : G \times G \to G$$

factors through $G^{ab} \times G^{ab}$ and hence

$$1 = [\alpha^m \beta^n, \beta] = \gamma^m \in G. \qquad \square$$

2. The Additive Homomorphism τ

We start with some heuristics that lead us to the definition of the additive version τ of θ. Let W be a finite p-group and assume that $\mathscr{L}_W$ is a homomorphism defined on

$$K_1(\Lambda(W)) = \Lambda(W)^\times / [\Lambda(W)^\times, \Lambda(W)^\times],$$

with values in a quotient of $\Lambda(W)$, that generalises the usual logarithm on $\mathbb{Z}_p^\times$. It seems natural that its codomain should be

$$\Lambda(W)/[\Lambda(W), \Lambda(W)].$$

If W is a p-adic Lie group, we demand that $\mathscr{L}_W$ commutes with the inverse limit functor. Let $\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]$ be the $\mathbb{Z}_p$-module topologically generated by the conjugacy classes of W. There are the isomorphisms

$$K_1(\Lambda(W)) \cong \varprojlim_U \Lambda(W/U)^\times/[\Lambda(W/U)^\times, \Lambda(W/U)^\times]$$
$$\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \cong \varprojlim_U \Lambda(W/U)/[\Lambda(W/U), \Lambda(W/U)],$$

where the limit is over all open normal subgroups U of W. Hence the integral logarithm is of the following form:

$$\mathscr{L}_W : K_1(\Lambda(W)) \to \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!].$$

Let P be a group that satisfies assumption 3.1. We define the homomorphism

$$\tau = (\tau_n)_n : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \Omega \subset \prod_n \Lambda(U_n/V_n)$$

in analogy to the homomorphism θ. More precisely, we define it to be the composition of a trace map

$$\mathrm{Tr}_{P|U_n} : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!]$$

and a projection homomorphism

$$\pi : \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n/V_n)]\!] = \Lambda(U_n/V_n).$$

We use the following construction to define a "localised version" of $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ (note that $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ is not a ring): If the centre $Z(P)$ of P is an open subgroup of P, then $\Lambda(P)_S = \Lambda(P)_{S(Z(P))}$. In this case, it is natural to define

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S := \Lambda(Z(P))_S \otimes_{\Lambda(Z(P))} \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!].$$

We show that every object P of $\mathcal{S}_{\mathbb{Z}_p}$ can be written as an inverse limit

$$P = \varprojlim_{W \in \mathfrak{W}_P} P/W,$$

where $\mathfrak{W}_P$ is a set of normal subgroups of P such that P/W is a quotient object of P with open centre for all $W \in \mathfrak{W}_P$. Hence we can define

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S^\wedge := \varprojlim_{W \in \mathfrak{W}_P} \left(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]_S\right)^{\langle p \rangle} \quad \text{and}$$
$$(\Lambda(P)_S)^\wedge := \varprojlim_{W \in \mathfrak{W}_P} (\Lambda(P/W)_S)^{\langle p \rangle}$$

for all $P \in Ob(\mathcal{S}_{\mathbb{Z}_p})$, where $-^{\langle p \rangle}$ denotes the p-adic completion. Note that the structure of $(\Lambda(P)_S)^\wedge$ was studied in [**41**]. We define

$$\tau_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S^\wedge \to \widehat{\Omega_S} \subset \prod_n (\Lambda(U_n/V_n)_S)^\wedge$$

to be the homomorphism induced by the composition of a trace map and a projection homomorphism.

Let R be a commutative ring. We start with the definition of the R-module $R[\mathrm{Conj}(W)]$ for a group W. For $\sigma \in W$, we set $\mathrm{class}(\sigma) := \mathrm{class}_W(\sigma) := \{\nu\sigma\nu^{-1} | \nu \in W\}$ and $\mathrm{Conj}(W) = \{\mathrm{class}(\sigma) | \sigma \in W\}$. We define

$$R[\mathrm{Conj}(W)] := \bigoplus_{S \in \mathrm{Conj}(W)} RS.$$

In general, the multiplication on W does not transfer to $\mathrm{Conj}(W)$. (Assume $\sigma, \tau \in W$, $\sigma \neq \tau$, $\mathrm{class}(\sigma) = \mathrm{class}(\tau)$. Then $\mathrm{class}(\sigma\tau^{-1}) \neq \mathrm{class}(\tau\tau^{-1}) = \{1\}$.) However, the map

$$\mathrm{Conj}(W) \to \mathrm{Conj}(W),\ \mathrm{class}(\sigma) \mapsto \mathrm{class}(\sigma^k)$$

is well-defined for any $k \geq 0$: For $\sigma, \tau, \nu \in W$ with $\sigma = \nu\tau\nu^{-1}$, we have $\sigma^k = (\nu\tau\nu^{-1})^k = \nu\tau^k\nu^{-1}$ and hence $\mathrm{class}(\sigma^k) = \mathrm{class}(\tau^k)$. For $k = p$, we define the induced R-linear map

$$\varphi : R[\mathrm{Conj}(W)] \to R[\mathrm{Conj}(W)], \quad \mathrm{class}(\sigma) \mapsto \mathrm{class}(\sigma^p).$$

Let W_1 be a group and let W_2 be a normal subgroup and W_3 a quotient group of W_1. Let $\{\nu_i\}_{i \in I}$ be a set of representatives of W_1/W_2 in W_1. We define the R-module homomorphism

$$\begin{aligned} \mathrm{Tr}_{W_1|W_2} : R[\mathrm{Conj}(W_1)] &\to R[\mathrm{Conj}(W_2)], \\ \mathrm{class}_{W_1}(\sigma) &\mapsto \begin{cases} \sum_{i \in I} \mathrm{class}_{W_2}(\nu_i\sigma\nu_i^{-1}) & \text{if } \sigma \in W_2 \\ 0 & \text{if } \sigma \notin W_2. \end{cases} \end{aligned}$$

This map is well-defined since for any two elements σ, $\tau \in W_2$ that are conjugate in W_1, there is $i \in I$ such that we get the identity

$$\mathrm{class}_{W_2}(\sigma) = \mathrm{class}_{W_2}(\nu_i\tau\nu_i^{-1}).$$

The following R-module homomorphisms are clearly well-defined:

$$\begin{aligned} \iota : R[\mathrm{Conj}(W_2)] \to R[\mathrm{Conj}(W_1)], &\quad \mathrm{class}_{W_2}(\sigma) \mapsto \mathrm{class}_{W_1}(\sigma) \\ \pi : R[\mathrm{Conj}(W_1)] \to R[\mathrm{Conj}(W_3)], &\quad \mathrm{class}_{W_1}(\sigma) \mapsto \mathrm{class}_{W_3}(\bar{\sigma}) \\ p_{conj} : R[W_1] \twoheadrightarrow R[\mathrm{Conj}(W_1)], &\quad \sigma \mapsto \mathrm{class}_{W_1}(\sigma) \end{aligned}$$

LEMMA 3.4. *Let W be a group. Then p_{conj} induces the isomorphism*

$$R[W]_W = R[W]/[R[W], R[W]] \cong R[\mathrm{Conj}(W)]$$

of R-modules, where $[-,-]$ is the commutator R-algebra. For a monoid W, we have

$$\langle gh - hg | g, h \in W \rangle_R = [R[W], R[W]].$$

PROOF. In a first step, we show that

$$T := \langle g - \nu g \nu^{-1} | g, \nu \in W \rangle_R = \ker p_{conj}$$

for a group W. Then inclusion "$\subset$" is obvious. Let

$$x = \sum_{i=1}^{n} x_i g_i \in \ker p_{conj}$$

with $x_i \in R$, $g_i \in \text{class}(g)$ for all i and some fixed $g \in W$. For $n = 1$, trivially $x = 0 \in T$. If $n > 1$, we assume that we have already proven it for $n - 1$. Then

$$x - x_n(g_n - g_{n-1}) = \sum_{i=1}^{n-2} x_i g_i + (x_{n-1} - x_n) g_{n-1} \in T,$$

by hypothesis. Hence $x \in T$.

In a second step, we show that

$$\langle gh - hg | g, h \in W \rangle_R = [R[W], R[W]].$$

for a monoid W. The inclusion "$\subset$" is obvious. For a generating element of the module on the right hand side, we have

$$\left(\sum_{g \in W} a_g g\right)\left(\sum_{h \in W} b_h h\right) - \left(\sum_{h \in W} b_h h\right)\left(\sum_{g \in W} a_g g\right) = \sum_{g,h \in W} a_g b_h (gh - hg),$$

where $a_g, b_h \in R$, and $a_g = 0$, $b_h = 0$ for almost all $g, h \in W$. This is an element of the module on the left hand side.

Since $gh - hg = gh - h(gh)h^{-1}$, we have shown that $\ker p_{conj} = [R[W], R[W]]$ for a group W and this proves the lemma. $\square$

Let W be a profinite group and let R be a commutative ring. Let $\{W_\lambda\}_\lambda$ be the set of open normal subgroups of W. For $W_{\lambda_1} \subset W_{\lambda_2}$ we have a natural map

$$R[\text{Conj}(W/W_{\lambda_1})] \to R[\text{Conj}(W/W_{\lambda_2})],$$
$$\text{class}_{W/W_{\lambda_1}}(\sigma) \mapsto \text{class}_{W/W_{\lambda_2}}(\sigma).$$

With respect to these maps, we can define the R-module

$$R[\![\text{Conj}(W)]\!] := \varprojlim_{\lambda} R[\text{Conj}(W/W_\lambda)].$$

We set $\text{class}_W(\sigma) := \big(\text{class}_{W/W_\lambda}(\sigma)\big)_\lambda \in R[\![\text{Conj}(W)]\!]$.

Let W_1 be a profinite group and let W_2 be an open normal subgroup and W_3 a quotient group of W_1. Let $\{\nu_i\}_i$ be a set of representatives of

W_1/W_2 in W_1. The following homomorphisms of R-modules are clearly well-defined:

$$\begin{aligned}
\varphi : R[\![\mathrm{Conj}(W_1)]\!] &\to R[\![\mathrm{Conj}(W_1)]\!], \quad \mathrm{class}_{W_1}(\sigma) \mapsto \mathrm{class}_{W_1}(\sigma^p) \\
\mathrm{Tr}_{W_1|W_2} : R[\![\mathrm{Conj}(W_1)]\!] &\to R[\![\mathrm{Conj}(W_2)]\!], \\
\mathrm{class}_{W_1}(\sigma) &\mapsto \begin{cases} \sum_i \mathrm{class}_{W_2}(\nu_i \sigma \nu_i^{-1}) & \text{if } \sigma \in W_2 \\ 0 & \text{if } \sigma \notin W_2 \end{cases} \\
\iota : R[\![\mathrm{Conj}(W_2)]\!] &\to R[\![\mathrm{Conj}(W_1)]\!], \quad \mathrm{class}_{W_2}(\sigma) \mapsto \mathrm{class}_{W_1}(\sigma) \\
\pi : R[\![\mathrm{Conj}(W_1)]\!] &\to R[\![\mathrm{Conj}(W_3)]\!], \quad \mathrm{class}_{W_1}(\sigma) \mapsto \mathrm{class}_{W_3}(\bar{\sigma})
\end{aligned}$$

We get the following homomorphism from the map p_{conj} defined above by passing to the inverse limit:

$$p_{conj} : \Lambda(W) \twoheadrightarrow \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]$$

Lemma 3.5. *Let W be a profinite group. Then p_{conj} induces the isomorphism*

$$\Lambda(W)/\overline{[\Lambda(W), \Lambda(W)]} \cong \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]$$

of $\mathbb{Z}_p$-modules, where $[-,-]$ is the commutator $\mathbb{Z}_p$-algebra.

Proof. Let $\{W_\lambda\}_\lambda$ be the set of all open normal subgroups of W. Since the inverse limit functor is left exact, we get

$$\begin{aligned}
&\ker\Big(\Lambda(W) \to \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]\Big) \\
&= \varprojlim_\lambda \ker\Big(\mathbb{Z}_p[W/W_\lambda] \to \mathbb{Z}_p[\mathrm{Conj}(W/W_\lambda)]\Big).
\end{aligned}$$

Now lemma 3.4 implies

$$\ker\Big(\mathbb{Z}_p[W/W_\lambda] \to \mathbb{Z}_p[\mathrm{Conj}(W/W_\lambda)]\Big) \cong [\mathbb{Z}_p[W/W_\lambda], \mathbb{Z}_p[W/W_\lambda]]$$

Let $\pi_\lambda : \Lambda(W) \to \mathbb{Z}_p[W/W_\lambda]$ be the natural projection. Then

$$[\mathbb{Z}_p[W/W_\lambda], \mathbb{Z}_p[W/W_\lambda]] = \pi_\lambda\Big([\Lambda(W), \Lambda(W)]\Big).$$

By [**36**, corollary 1.1.8],

$$\varprojlim_\lambda \pi_\lambda\Big([\Lambda(W), \Lambda(W)]\Big) = \overline{[\Lambda(W), \Lambda(W)]}.$$

Hence

$$\ker\Big(\Lambda(W) \to \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]\Big) \cong \overline{[\Lambda(W), \Lambda(W)]}. \qquad \square$$

Lemma 3.6. $[\Lambda(W), \Lambda(W)] \subset \Lambda(W)$ *is a closed subgroup if* $(W : Z(W)) < \infty$. *(In particular, this is the case for groups P that satisfy assumption 3.1 and for which* $\mathrm{ord}(\gamma) < \infty$*.)*

Proof. Put $n := (W : Z(W))$ and let $\{w_i\}_{i=1,\dots,n}$ be a set of representatives of $W/Z(W)$ in W. We define

$$\psi : \Lambda(Z(W))^{n^2} \to \Lambda(W),\ (\lambda_{i,j})_{i,j=1,\dots,n} \mapsto \sum_{1\le i,j\le n} \lambda_{i,j}[w_i, w_j].$$

Then $\operatorname{im}\psi = [\Lambda(W), \Lambda(W)]$. Since $\Lambda(Z(W))^{n^2}$ is compact and since ψ is continuous, $\operatorname{im}\psi$ is compact and hence closed as a subgroup of $\Lambda(W)$. $\square$

Let W be a profinite group. We endow $\Lambda(W)$ with the W-module structure defined by $g \cdot x := gxg^{-1}$ for $g \in W$ and $x \in \Lambda(W)$. We define the continuous homology groups by

$$H_n(W, \Lambda(W)) := \varprojlim_{U \le_o W} H_n(W/U, \Lambda(W/U))$$

(cf. [**33**, prop. 1.2.5, thm. 2.6.9, ch. II §7]). Then

$$H_0(W, \Lambda(W)) = \varprojlim_{U \le_o W} \Lambda(W/U)/[\Lambda(W/U), \Lambda(W/U)] = \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!].$$

The following lemma will elucidate the $\Lambda(Z(P))$-module structure of $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$.

Lemma 3.7. *Let W be a p-adic Lie group. Then $\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]$ is a $\Lambda(Z(W))$-module. There is a surjective homomorphism*

$$\Lambda(Z(W))[\![W/Z(W)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]$$

of $\Lambda(Z(W))$-modules.

Let P be a group that satisfies assumption 3.1. For $g \in P$, let

$$\Lambda(Z(P))\mathrm{class}_P(g) \subset \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$$

be the $\Lambda(Z(P))$-module generated by $\mathrm{class}_P(g)$. If $P/Z(P)$ is finite, then

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \cong \bigoplus_{\bar{g} \in P/Z(P)} \Lambda(Z(P))\mathrm{class}_P(g). \tag{15}$$

Proof. The $\Lambda(Z(P))$-module is defined by

$$z \cdot \mathrm{class}_W(g) = \mathrm{class}_W(zg) \qquad \text{for } z \in Z(W) \text{ and } g \in W.$$

(This is clearly unambiguous.) Let

$$s : W/Z(W) \to W$$

be a continuous section of the natural projection $W \twoheadrightarrow W/Z(W)$ of topological spaces. We can define

$$\begin{aligned} f : \Lambda(Z(W))[\![W/Z(W)]\!] &\to \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!], \\ z \cdot g &\mapsto \mathrm{class}_W(z \cdot s(g)), \end{aligned}$$

where $z \in Z(W)$ and $g \in W/Z(W)$. This is obviously a surjective $\Lambda(Z(W))$-homomorphism.

Now, we prove the isomorphism (15). For $g \in P$, define

$$Z(P)\mathrm{class}_P(g) := \{\mathrm{class}_P(zg) | z \in Z(P)\} \subset \mathrm{Conj}(P).$$

Since $\overline{[P,P]} \subset Z(P)$, we can write $\mathrm{Conj}(P)$ as a disjoint union

$$\mathrm{Conj}(P) = \dot{\bigcup_{\bar{g} \in P/Z(P)}} Z(P)\mathrm{class}_P(g)$$

of compact subsets. Since $P/Z(P)$ is assumed to be finite, this implies (15). $\square$

LEMMA 3.8. *Let P be a compact p-adic Lie group with a surjection $\omega : P \twoheadrightarrow \mathbb{Z}_p$. Let $\mathfrak{W}_P$ be the set of all closed normal subgroups W of P that are open subgroups of $H := \ker \omega$. Then $(P/W)_{W \in \mathfrak{W}_P}$ is a directed system and*

$$P = \varprojlim_{W \in \mathfrak{W}_P} P/W.$$

In other words, every object of $\mathcal{S}_{\mathbb{Z}_p}$ is a projective limit of quotient objects with open centre.

PROOF. Since $H \in \mathfrak{W}_P$, we get $\mathfrak{W}_P \neq \emptyset$. For two groups $W_1, W_2 \in \mathfrak{W}_P$, we get $W_1 \cap W_2 \in \mathfrak{W}_P$. Hence it suffices to show that

$$\bigcap_{W \in \mathfrak{W}_P} W = \{1\} \quad \text{and} \tag{16}$$

For $n \in \mathbb{N}$, put $N_n := \bigcap_{[H:U]=n} U$ if there is a subgroup U of H with $[H : U] = n$ and set $N_n := H$ otherwise. Then N_n is a normal subgroup of P and a closed subgroup of H. We show that $N_n \subset H$ is an open subgroup. Since H contains a pro-p open subgroup, we may assume that H is a pro-p group and n is a power of p. Since subgroups U of H with $[H : U] = p$ are maximal, the Frattini group $\Phi(H)$ is contained in N_p and hence (since $\Phi(H)$ is an open subgroup of H by [**13**, proposition 1.14])

$$[H : N_p] \leq [H : \Phi(H)] < \infty.$$

(The Frattini subgroup of a group is defined to be the intersection of its maximal proper subgroups.) Now assume that N_{p^n} is an open subgroup of H for some $n \geq 1$. Every subgroup U of H with $[H : U] = p^{n+1}$ is contained in a subgroup U' of H with $[H : U'] = p^n$. Then $\Phi(U') \subset U$. Hence we get

$$N_{p^{n+1}} = \bigcap_{[H:U]=p^{n+1}} U \supset \bigcap_{[H:U']=p^n} \Phi(U').$$

Since, by hypothesis, H/N_{p^n} is finite, there are only finitely many subgroups $U' \subset H$ of index p^n. Since $\Phi(U')$ is an open subgroup of H

for every open subgroup U' of H, the intersection $\bigcap_{[H:U']=p^n} \Phi(U')$ is a finite intersection of open subgroups of H. Hence $N_{p^{n+1}} \subset H$ is open.

Since

$$\bigcap_{W \in \mathfrak{W}_P} W \subset \bigcap_{n \in \mathbb{N}} N_n = \bigcap_{U \leq_o H} U = \{1\},$$

we have proven equation (16). □

DEFINITION 3.9. Let L be a $\mathbb{Z}_p$-module. We define the p-adic completion

$$L^{\langle p \rangle} := \varprojlim_n L/p^n L.$$

Let $\mathfrak{V}$ be a cofinite subset of $\mathfrak{W}_P$. Let C be a map that assigns to every group $V \in \mathfrak{V}$ an open subgroup $C(V) \subset Z(P/V)$. Assume that for $V_1 \subset V_2$, the image of $C(V_1)$ in P/V_2 is a subgroup of $C(V_2)$. Let C_{V_1,V_2} be the group of all elements of P whose image in P/V_i lies in $C(V_i)$ for $i = 1, 2$ and let $C_{\mathfrak{V}}$ be the set of all elements of P whose image in P/V lies in $C(V)$ for all $V \in \mathfrak{V}$. Let $\mathcal{M}_{C,\mathfrak{V}}$ be the category whose objects are the projective systems $(M_V)_{V \in \mathfrak{V}}$ of abelian groups such that M_V carries a $\Lambda(C(V))$-module structure and for $V_1 \subset V_2$, the transition map $M_{V_1} \to M_{V_2}$ is $\Lambda(C_{V_1,V_2})$-linear. The set of morphisms from $(M_V)_V$ to $(N_V)_V$ is defined to be the set of tuples

$$(f_V : M_V \to N_V)_{V \in \mathfrak{V}},$$

where f_V is a $\Lambda(C(V))$-linear map for all $V \in \mathfrak{V}$ and the diagram

$$\begin{tikzcd} M_{V_1} \arrow[r, "f_{V_1}"] \arrow[d] & N_{V_1} \arrow[d] \\ M_{V_2} \arrow[r, "f_{V_2}"] & N_{V_2} \end{tikzcd}$$

is a commutative diagram of $\Lambda(C_{V_1,V_2})$-modules for all $V_1, V_2 \in \mathfrak{V}$ with $V_1 \subset V_2$.

DEFINITION 3.10. Let $P \in \mathcal{S}_{\mathbb{Z}_p}$ be a group whose centre is an open subgroup of P. Let M be a $\Lambda(Z(P))$-module. We define

$$M_S := \Lambda(Z(P))_S \otimes_{\Lambda(Z(P))} M.$$

For general $P \in \mathcal{S}_{\mathbb{Z}_p}$ and $\mathfrak{V} \subset \mathfrak{W}_P$, C as above, we define the functors

$$(-)^\wedge : \mathcal{M}_{C,\mathfrak{V}} \to \Lambda(C_{\mathfrak{V}})\text{-mod} \qquad \text{and}$$
$$(-)^\wedge_S : \mathcal{M}_{C,\mathfrak{V}} \to \Lambda(C_{\mathfrak{V}})\text{-mod}$$

as follows: Let $M = (M_V)_{V\in\mathfrak{V}}$ be an object of $\mathcal{M}_{C,\mathfrak{V}}$. We put

$$(M)^\wedge := \varprojlim_{V\in\mathfrak{V}} (M_V)^{\langle p\rangle}$$

$$(M)^\wedge_S := \varprojlim_{V\in\mathfrak{V}} (M_V)^{\langle p\rangle}_S = \varprojlim_{V\in\mathfrak{V}} \big(\Lambda(C(V))_S \otimes_{\Lambda(C(V))} M_V\big)^{\langle p\rangle}$$

Let $f : M \to N$ be a morphism of the category $\mathcal{M}_{C,\mathfrak{V}}$. The homomorphisms

$$f_V : M_V \to N_V, \quad V \in \mathfrak{V}$$

induce the homomorphisms

$$(M_V)^{\langle p\rangle} \to (N_V)^{\langle p\rangle} \quad \text{and} \quad (M_V)^{\langle p\rangle}_S \to (N_V)^{\langle p\rangle}_S.$$

Since $(-)^\wedge$ ($(-)^\wedge_S$, respectively) is the composition of $\varprojlim_{V\in\mathfrak{V}}$ and $(-)^{\langle p\rangle}$ ($\varprojlim_{V\in\mathfrak{V}}$ and $(-)^{\langle p\rangle}$, respectively), f naturally induces the homomorphisms

$$(f)^\wedge : M^\wedge \to N^\wedge \quad \text{and} \quad (f)^\wedge_S : M^\wedge_S \to N^\wedge_S.$$

Thus we have defined the functors $(-)^\wedge$ and $(-)^\wedge_S$.

Remarks.
- The group $C_{\mathfrak{V}}$ and the $\Lambda(C_{\mathfrak{V}})$-modules $(M)^\wedge$ and $(M)^\wedge_S$ do not depend on the cofinal subset $\mathfrak{V}$ of $\mathfrak{W}_P$.
- We will apply the above definition to the objects

$$\Lambda(P) := \Big(\Lambda(P/W)\Big)_{W\in\mathfrak{W}_P},$$

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] := \Big(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]\Big)_{W\in\mathfrak{W}_P},$$

$$\mathbb{Q}_p[\![\mathrm{Conj}(P)]\!] := \Big(\mathbb{Q}_p[\![\mathrm{Conj}(P/W)]\!]\Big)_{W\in\mathfrak{W}_P}$$

of $\mathcal{M}_{C,\mathfrak{W}_P}$, where $C(W) := Z(P/W)$ for $W \in \mathfrak{W}_P$. (We use the same notation for the projective system and the corresponding projective limit.) For groups $P \in \mathcal{S}_{\mathbb{Z}_p}$ with open centre, we get

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S = \Lambda(Z(P))_S \otimes_{\Lambda(Z(P))} \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$$

$$\mathbb{Q}_p[\![\mathrm{Conj}(P)]\!]_S = \Lambda(Z(P))_S \otimes_{\Lambda(Z(P))} \mathbb{Q}_p[\![\mathrm{Conj}(P)]\!]$$

and for general $P \in \mathcal{S}_{\mathbb{Z}_p}$,

$$\widehat{\Lambda(P)_S} = \varprojlim_{W\in\mathfrak{W}_P} (\Lambda(P/W)_S)^{\langle p\rangle}$$

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^\wedge_S = \varprojlim_{W\in\mathfrak{W}_P} \big(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]_S\big)^{\langle p\rangle}.$$

Since the p-adic completion of a $\mathbb{Q}_p$-module is always trivial, we need the following

DEFINITION 3.11. Let $P \in \mathcal{S}_{\mathbb{Z}_p}$ be a p-adic Lie group. We define

$$\mathbb{Q}'_p[\![\mathrm{Conj}(P)]\!]^{\wedge}_S := \varprojlim_{W \in \mathfrak{W}_P} \left(\left(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]_S\right)^{\langle p \rangle} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \right)$$
$$\mathbb{Q}'_p[\![P]\!]^{\wedge}_S := \varprojlim_{W \in \mathfrak{W}_P} \left(\left(\mathbb{Z}_p[\![P/W]\!]_S\right)^{\langle p \rangle} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \right)$$

LEMMA 3.12 ([**41**, lemma 3.4]). *There is an isomorphism*

$$\widehat{\Lambda(P)_S} \cong \varprojlim_n \Lambda(P)_S / J(\Lambda(P)_S)^n.$$

COROLLARY 3.13. *The natural homomorphisms*

$$\Lambda(P)_S \hookrightarrow (\Lambda(P)_S)^{\langle p \rangle} \quad \text{and} \quad \Lambda(P)_S \hookrightarrow \widehat{\Lambda(P)_S}$$

are injective.

PROOF. This follows from the fact that $\Lambda(P)_S$ is Noetherian by proposition 2.7 and hence

$$\bigcap_n p^n \Lambda(P)_S \subset \bigcap_n J(\Lambda(P)_S)^n = \{0\}. \qquad \square$$

REMARK. In general, $\Lambda(P)_S \subsetneqq (\Lambda(P)_S)^{\langle p \rangle} \subsetneqq \widehat{\Lambda(P)_S}$. For example, when $P \cong \mathbb{Z}_p^2$ and $\omega : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ is the projection to the second factor, we may identify $\Lambda(P) = \mathbb{Z}_p[[T_1, T_2]]$ and get

$$S = \mathbb{Z}_p[[T_1, T_2]] \setminus (p, T_1)$$

(cf. lemma 2.3). Hence

$$\sum_{i \geq 0} \left(\frac{T_1}{T_2}\right)^i \in \widehat{\Lambda(P)_S} \setminus (\Lambda(P)_S)^{\langle p \rangle},$$
$$\sum_{i \geq 0} \left(\frac{p}{T_2}\right)^i \in (\Lambda(P)_S)^{\langle p \rangle} \setminus \Lambda(P)_S.$$

When P is one-dimensional, it suffices to work with the p-adic completion (cf. [**24**], [**21**]).

LEMMA 3.14. *The ring $\widehat{\Lambda(P)_S}$ is semi-local. In particular, when P is one-dimensional,*

$$(\Lambda(P)_S)^{\langle p \rangle} = \widehat{\Lambda(P)_S}$$

is semi-local.

PROOF. By [**40**] and [**41**, thm. 3.7, prop. 2.26, lem. 1.11],

$$J(\widehat{\Lambda(P)_S}) = J(\Lambda(P)_S)\widehat{\Lambda(P)_S}.$$

Hence, using lemma 3.12, we get

$$\begin{aligned}\Lambda(P)_S/J(\Lambda(P)_S) &= \Big(\Lambda(P)_S/J(\Lambda(P)_S)\Big) \otimes_{\Lambda(P)_S} \widehat{\Lambda(P)_S} \\ &= \widehat{\Lambda(P)_S}/\Big(J(\Lambda(P)_S)\widehat{\Lambda(P)_S}\Big) \\ &= \widehat{\Lambda(P)_S}/J(\widehat{\Lambda(P)_S}).\end{aligned}$$

Since $\Lambda(P)_S$ is semi-local, this implies that $\widehat{\Lambda(P)_S}$ is semi-local. □

LEMMA 3.15.

$$\widehat{\Lambda(P)_S} = \varprojlim_{W\in\mathfrak{W}_P} \left(\Lambda(P/W)_{S(Z(P/W))}\right)^{\langle p\rangle}$$

PROOF. By definition,

$$\widehat{\Lambda(P)_S} = \varprojlim_{W\in\mathfrak{W}_P} \left(\Lambda(P/W)_{S(P/W)}\right)^{\langle p\rangle}.$$

By proposition 2.10,

$$\Lambda(P/W)_{S(P/W)} = \Lambda(P/W)_{S(Z(P/W))}$$

for $W \in \mathfrak{W}_P$. □

LEMMA 3.16. *We will now apply definition 3.10 to the object*

$$M := \Big(\Lambda(P/W)_S/[\Lambda(P/W)_S, \Lambda(P/W)_S]\Big)_{W\in\mathfrak{W}_P}.$$

of $\mathcal{M}_{C,\mathfrak{W}_P}$, where $C(W) = Z(P/W)$. There is an isomorphism

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S^\wedge \cong (\Lambda(P)_S/[\Lambda(P)_S, \Lambda(P)_S])^\wedge := M^\wedge$$

of $\Lambda(Z(P))$-modules. If the centre of P is open, then we get the isomorphism

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S \cong \Lambda(P)_S/[\Lambda(P)_S, \Lambda(P)_S]$$

PROOF. Using lemma 3.5, lemma 3.6 and the fact that $\Lambda(Z(P))_S$ is a flat $\Lambda(Z(P))$-module, we see that it suffices to show

$$\Lambda(Z(P))_S \otimes_{\Lambda(Z(P))} [\Lambda(P), \Lambda(P)] \cong [\Lambda(P)_S, \Lambda(P)_S] \tag{17}$$

for groups P, where $Z(P) \subset P$ is open.

Let $s^{-1} \otimes [a, b]$, $a, b \in \Lambda(P)$ and $s \in S(Z(P))$ be a generating element of the module on left hand side. We define $[as^{-1}, b]$ to be its image in $[\Lambda(P)_{S(P)}, \Lambda(P)_{S(P)}]$. By proposition 2.10, the module on the right hand side is generated by elements of the form $[as^{-1}, bt^{-1}]$ with $a, b \in \Lambda(P)$, $s, t \in S(Z(P))$. We define $(st)^{-1} \otimes [a, b]$ to be its image in the module on the left hand side. We defined two maps that are the inverse of each other. Hence, they are isomorphisms. □

Let $P \in \mathcal{S}_{\mathbb{Z}_p}$ be a p-adic Lie group, let $U \subset P$ be an open normal subgroup of P and let V be a normal subgroup of $\ker(P \to \mathbb{Z}_p)$. The trace and projection homomorphisms

$$\begin{aligned} \mathrm{Tr}_{P|U} : \mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!] &\to \mathbb{Z}_p[\![\mathrm{Conj}(U/W)]\!] \quad \text{and} \\ \pi : \mathbb{Z}_p[\![\mathrm{Conj}(U/W)]\!] &\to \mathbb{Z}_p[\![\mathrm{Conj}(U/VW)]\!], \end{aligned}$$

where $W \in \mathfrak{W}_P \cap \mathfrak{W}_U$, induce the morphisms

$$\begin{aligned} \mathrm{Tr}_{P|U} : \Big(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]\Big)_{W \in \mathfrak{W}_P \cap \mathfrak{W}_U} &\to \Big(\mathbb{Z}_p[\![\mathrm{Conj}(U/W)]\!]\Big)_{W \in \mathfrak{W}_P \cap \mathfrak{W}_U} \\ \pi : \Big(\mathbb{Z}_p[\![\mathrm{Conj}(U/W)]\!]\Big)_{W \in \mathfrak{W}_U} &\to \Big(\mathbb{Z}_p[\![\mathrm{Conj}(U/VW)]\!]\Big)_{W \in \mathfrak{W}_U} \end{aligned}$$

in $\mathcal{M}_{C_1, \mathfrak{W}_P \cap \mathfrak{W}_U}$ and $\mathcal{M}_{C_2, \mathfrak{W}_U}$, respectively, where $C_1(W) = Z(P/W) \cap Z(U/W)$ and $C_2(W) = Z(P/W)$ for $W \in \mathfrak{W}_P \cap \mathfrak{W}_U$. By definition 3.10, they induce the homomorphisms

$$\begin{aligned} \big(\mathrm{Tr}_{P|U}\big)^\wedge_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^\wedge_S &\to \mathbb{Z}_p[\![\mathrm{Conj}(U)]\!]^\wedge_S \quad \text{and} \\ (\pi)^\wedge_S : \mathbb{Z}_p[\![\mathrm{Conj}(U)]\!]^\wedge_S &\to \mathbb{Z}_p[\![\mathrm{Conj}(U/V)]\!]^\wedge_S \end{aligned}$$

of $\Lambda\Big(Z(P) \cap Z(U)\Big)$-modules and $\Lambda(Z(U))$-modules, respectively. A similar statement for $(-)^\wedge$ also holds. The fact that these maps are well-defined follows from the following lemma:

Lemma 3.17. *Let U be an open normal subgroup of P. Then $\mathfrak{W}_P \cap \mathfrak{W}_U$ is cofinal in $\mathfrak{W}_P$ and in $\mathfrak{W}_U$. In particular,*

$$(M(P))^\wedge_S = (N(P))^\wedge_S \quad \textit{and} \quad (M(U))^\wedge_S = (N(U))^\wedge_S, \tag{18}$$

where

$$\begin{aligned} M(U') &:= \Big(\mathbb{Z}_p[\![\mathrm{Conj}(U'/W)]\!]\Big)_{W \in \mathfrak{W}_{U'}} \in Ob\left(\mathcal{M}_{C', \mathfrak{W}_{U'}}\right) \\ N(U') &:= \Big(\mathbb{Z}_p[\![\mathrm{Conj}(U'/W)]\!]\Big)_{W \in \mathfrak{W}_P \cap \mathfrak{W}_U} \in Ob\left(\mathcal{M}_{C_1, \mathfrak{W}_P \cap \mathfrak{W}_U}\right) \end{aligned}$$

for $U' = U$ or $U' = P$, $C'(W) = Z(U'/W)$ and $C_1(W) = Z(P/W) \cap Z(U/W)$.

Proof. Let W be an element of $\mathfrak{W}_P$. Then $W \cap U$ is a normal subgroup of P and of U and an open subgroup of $\ker(P \twoheadrightarrow \mathbb{Z}_p)$ and of $\ker(U \twoheadrightarrow \mathbb{Z}_p)$. Hence $W \cap U \in \mathfrak{W}_P \cap \mathfrak{W}_U$. Thus $\mathfrak{W}_P \cap \mathfrak{W}_U$ is cofinal in $\mathfrak{W}_P$.

Let $W \in \mathfrak{W}_U$ and set $V := \bigcap_{\sigma \in P} W^\sigma$, where $W^\sigma := \{\sigma g \sigma^{-1} | g \in W\}$. Since W is normal in U, V is a finite intersection of open subgroups of $\ker(U \twoheadrightarrow \mathbb{Z}_p)$. Clearly, it is a normal subgroup of U and of P. Hence $V \in \mathfrak{W}_P \cap \mathfrak{W}_U$. Thus $\mathfrak{W}_P \cap \mathfrak{W}_U$ is cofinal in $\mathfrak{W}_U$.

The equations (18) follow from proposition 2.9. $\square$

DEFINITION 3.18. Let $P \in \mathcal{S}_{\mathbb{Z}_p}$ be a group that satisfies assumption 3.1. For $n \in \underline{\underline{c}}$, the homomorphism

$$\tau_n : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \Lambda(U_n/V_n)$$

is defined to be the composition of the two maps

$$\mathrm{Tr}_{P|U_n} : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!] \quad \text{and}$$
$$\pi : \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!] \to \Lambda(U_n/V_n).$$

The homomorphism

$$\tau_{n,S} : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^{\wedge}_S \to \Lambda(U_n/V_n)^{\wedge}_S$$

is defined to be the composition of the two maps

$$\left(\mathrm{Tr}_{P|U_n}\right)^{\wedge}_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^{\wedge}_S \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!]^{\wedge}_S \quad \text{and}$$
$$(\pi)^{\wedge}_S : \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!]^{\wedge}_S \to \Lambda(U_n/V_n)^{\wedge}_S.$$

We define the homomorphisms

$$\tau : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \prod_{n \in \underline{\underline{c}}} \Lambda(U_n/V_n), \quad x \mapsto (\tau_n(x))_n,$$
$$\tau_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^{\wedge}_S \to \prod_{n \in \underline{\underline{c}}} \Lambda(U_n/V_n)^{\wedge}_S, \quad x \mapsto (\tau_{n,S}(x))_n,$$

of $\Lambda(Z(P))$-modules. (Recall that $Z(P) = \bigcap_{n \in \underline{\underline{c}}} Z(U_n)$ since $Z(P) \subset U_n$ for all $n \in \underline{\underline{c}}$.)

REMARKS.

- For $W \in \mathfrak{W}_P$, let $c_{P/W}$ be the natural number defined by the relation $\mathrm{ord}(\bar{\gamma}) = p^{c_{P/W}}$, where $\bar{\gamma}$ is the image of γ in P/W. Since $W \subset U_n$ for $n \in \underline{\underline{c_{P/W}}}$, the homomorphism τ induces a morphism

$$\tau : \left(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]\right)_{W \in \mathfrak{W}_P} \to \left(\prod_{n \in \underline{\underline{c_{P/W}}}} \Lambda(U_n/V_nW)\right)_{W \in \mathfrak{W}_P}$$

in $\mathcal{M}_{C,\mathfrak{W}_P}$, where $C(W) = Z(P/W)$ for $W \in \mathfrak{W}_P$. By definition 3.10, this induces the homomorphism

$$(\tau)^{\wedge}_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^{\wedge}_S \to \prod_{n \in \underline{\underline{c_P}}} \Lambda(U_n/V_n)^{\wedge}_S$$

of $\Lambda(Z(P))$-modules. This is just the homomorphism τ_S defined above.

- If $\mathrm{ord}(\gamma) < \infty$, we can define the homomorphism

$$\tau_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S \to \prod_n \Lambda(U_n/V_n)_S, \quad z \otimes x \mapsto (p^n z \otimes \tau_n(x))_n,$$

where $z \in \Lambda(Z(P))_S$ and $x \in \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ (since $Z(P) \subset U_\infty$). In this case, τ_S is $\Lambda(Z(P))_S$-linear.

As mentioned above, we are interested in an explicit description of the image of τ. We start with calculating the image of τ_n, $n \in \underline{\underline{c}}$. For this purpose, we will need the definition of the following elements: For $i \in \underline{\underline{n}}$, we set

$$h_{n,i} = \sum_{j=0}^{p^{n-i}-1} \gamma^{p^i j} \in \Lambda(U_n/V_n).$$

For $i, j \in \mathbb{Z}_p$, $n \in \underline{\underline{c}}$, we set

$$c_{ijn} = \left\{ \begin{array}{ll} 0 & \text{if } n > v_p(j) \\ \sum_{t=0}^{p^n-1} \gamma^{it} & \text{if } n \leq v_p(j) \end{array} \right\} \in \Lambda(U_n/V_n).$$

Let $i, j \in \mathbb{Z}_p$ and $s, t \in \mathbb{Z}_p^\times$. The s-power map permutes the set $\{1, \gamma, \ldots, \gamma^{p^n-1}\} \subset U_n/V_n$. Then for $n \leq v_p(j) = v_p(jt)$,

$$c_{ijn} = \sum_{r=0}^{p^n-1} \gamma^{ir} = \sum_{r=0}^{p^n-1} \gamma^{irs} = c_{is,jt,n}.$$

For $n > v_p(j)$, clearly $c_{ijn} = 0 = c_{is,jt,n}$. Hence

$$c_{ijn} = c_{is,jt,n} \tag{19}$$

for all $n \in \mathbb{N}$.

For $0 \leq k \leq n$, $n \in \underline{\underline{c}}$, put $U_{n,k} := \langle Z(P), \alpha^{p^k}, \beta^{p^n} \rangle$. Note that we have a descending chain of subgroups $U_n = U_{n,0} \supset U_{n,1} \supset \ldots \supset U_{n,n}$, where each subgroup is normal in P. By (13), $U_{n,n}/V_n = Z(P/V_n)$.

Lemma 3.19. *The image of τ_n, $n \in \underline{\underline{c}}$ is*

$$I_n := \langle p^i h_{n,i} g | i \in \underline{\underline{n}},\ g \in U_{n,i}/V_n,\ g \notin U_{n,i+1}/V_n \text{ if } i < n \rangle_{\Lambda(U_{n,n}/V_n)},$$

the $\Lambda(U_{n,n}/V_n)$-submodule of $\Lambda(U_n/V_n)$ generated by the elements $p^i h_{n,i} g$ mentioned above. The image of $\tau_{n,S}$ is

$$\begin{aligned} I_{n,S} &:= \langle p^i h_{n,i} g | i \in \underline{\underline{n}},\ g \in U_{n,i}/V_n,\ g \notin U_{n,i+1}/V_n \text{ if } i < n \rangle_{\Lambda(U_{n,n}/V_n)_S} \\ &= (I_n)_{S(U_n/V_n)} \subset \Lambda(U_n/V_n)_S. \end{aligned}$$

Proof. Clearly, is suffices to determine the image of τ_n. Since $\operatorname{im} \tau_0 = \Lambda(U_0/V_0) = I_0$, we may assume $n \geq 1$.

By lemma 3.7, the image of $P/Z(P)$ in $\mathrm{Conj}(P)$ topologically generates the $\Lambda(Z(P))$-module $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$. If we write $\bar{\alpha}$, $\bar{\beta}$ for the images of α, β in $P/Z(P)$, respectively, we get

$$P/Z(P) = \{\bar{\alpha}^i \bar{\beta}^j | i, j \in \mathbb{Z}_p\}.$$

Since $\alpha^{p^n}, \beta^{p^n} \in Z(P/V_n) = U_{n,n}/V_n$, this implies

$$\begin{aligned}\operatorname{im}\tau_n &= \langle \tau_n(\operatorname{class}(\alpha^i\beta^j)) | i,j \in \mathbb{Z}_p\rangle_{\Lambda(Z(P/V_n))} \\ &= \langle \tau_n(\operatorname{class}(\alpha^i\beta^j)) | i,j = 0,\dots,p^n-1\rangle_{\Lambda(U_{n,n}/V_n)}.\end{aligned} \tag{20}$$

Note that we have found a finite generating set of the (abstract) $\Lambda(Z(U_{n,n}/V_n))$-vector space $\operatorname{im}\tau_n$.

Using (19), we get for $0 \le i < p^n$

$$\tau_n(\operatorname{class}(\alpha^i)) = \sum_{k=0}^{p^n-1} \beta^k\alpha^i\beta^{-k} = \sum_{k=0}^{p^n-1} \alpha^i\gamma^{-ik} = c_{-i0n}\alpha^i = c_{i0n}\alpha^i.$$

For $0 \le i < p^n$ and $0 < j < p^n$ (i. e. $\beta^j \notin U_n$), we get

$$\tau_n(\operatorname{class}(\alpha^i\beta^j)) = 0 = c_{ijn}\alpha^i\beta^j.$$

Hence

$$\begin{aligned}\operatorname{im}\tau_n &= \langle c_{ijn}\alpha^i\beta^j | i,j = 0,\dots,p^n-1\rangle_{\Lambda(U_{n,n}/V_n)} \\ &= \langle p^{v_p(i)}h_{n,v_p(i)}\alpha^i | i = 0,\dots,p^n-1\rangle_{\Lambda(U_{n,n}/V_n)}.\end{aligned}$$

Since $\alpha^i \in U_{n,v_p(i)}$ and $\alpha^i \notin U_{n,v_p(i)+1}$ for $v_p(i) < n$, we get $\operatorname{im}\tau_n \subset I_n$. Since $U_{n,v_p(i)} = \langle \alpha^i, U_{n,n}\rangle$, this inclusion is an identity. $\square$

We have seen that $\operatorname{im}\tau \subset \prod_{n\in\underline{\underline{c}}} I_n$. We will use the trace homomorphism defined in definition 1.27 for the description of $\operatorname{im}\tau$ as a subset of $\prod_{n\in\underline{\underline{c}}} I_n$.

Let W_2 be an open subgroup of the profinite group W_1. Then there is a trace homomorphism

$$\operatorname{Tr} = \operatorname{Tr}_{W_1|W_2} : \Lambda(W_1) \to \Lambda(W_2).$$

Equation (5) clearly implies the following

LEMMA 3.20. *For $m \le n$, $m,n \in \underline{\underline{c}}$,*

$$\begin{tikzcd} \mathbb{Z}_p[\![\operatorname{Conj}(P)]\!] \arrow[r, "\operatorname{Tr}_{P|U_n}"] \arrow[d, "\pi"] & \mathbb{Z}_p[\![\operatorname{Conj}(U_n)]\!] \arrow[d, "\pi"] \\ \mathbb{Z}_p[\![\operatorname{Conj}(P/V_m)]\!] \arrow[r, "\operatorname{Tr}"] & \Lambda(U_n/V_m) \end{tikzcd}$$

is a commutative diagram of $\Lambda(Z(P))$-modules (i. e. $\tau_n = \operatorname{Tr}\circ\pi$). $\square$

We make the following observation:

LEMMA 3.21. *For $m,n \in \underline{\underline{c}}$, $m \le n$, let*

$$\operatorname{Tr}_{m,n} = \operatorname{Tr} : \Lambda(U_m/V_m) \to \Lambda(U_n/V_m)$$
$$\operatorname{Tr}_{m,n,S} = (\operatorname{Tr})_S : \Lambda(U_m/V_m)_S \to \Lambda(U_n/V_m)_S$$

be the trace homomorphisms and

$$p_{n,m} = p_* : \Lambda(U_n/V_n) \to \Lambda(U_n/V_m)$$
$$p_{n,m,S} = (p_*)_S : \Lambda(U_n/V_n)_S \to \Lambda(U_n/V_m)_S$$

be the projection homomorphisms.

We define the $\Lambda(Z(P))$*-modules*

$$\Omega := \Omega_P := \left\{ (x_n) \in \prod_{n \in \underline{\underline{c}}} I_n \;\middle|\; \begin{array}{c} Tr_{m,n}(x_m) = p_{n,m}(x_n) \\ \textit{for } m \leq n \end{array} \right\}$$

$$\Omega_S := \Omega_{P,S} := \left\{ (x_n) \in \prod_{n \in \underline{\underline{c}}} I_{n,S} \;\middle|\; \begin{array}{c} Tr_{m,n,S}(x_m) = p_{n,m,S}(x_n) \\ \textit{for } m \leq n \end{array} \right\}$$

Assume that γ *is of finite order* $\operatorname{ord}(\gamma) = p^c$*. Then*

$$\operatorname{im}\tau \subset \Omega \quad \textit{and} \quad \operatorname{im}\tau_S \subset \Omega_S.$$

PROOF. Clearly, it suffices to prove the inclusion $\operatorname{im}\tau \subset \Omega$.

Let $m, n \in \underline{\underline{c}}$ with $m \leq n$. The maps $\mathrm{Tr}_{m,n}$ and $p_{n,m}$ are $\Lambda(Z(P))$-linear. By (20), it suffices to prove the claim for the generating set $\{\tau(\mathrm{class}(\alpha^i\beta^j)) | i, j = 0, \ldots, p^c - 1\}$ of the $\Lambda(Z(P))$-module $\operatorname{im}\tau$.

If $n \leq v_p(j)$, we calculate

$$\begin{aligned}
\mathrm{Tr}_{m,n}(c_{ijm}\alpha^i\beta^j) &= \sum_{t=0}^{p^{n-m}-1} \beta^{p^m t} \left(\sum_{k=0}^{p^m-1} \gamma^{ik}\alpha^i\beta^j \right) \beta^{-p^m t} \\
&= \sum_{t=0}^{p^{n-m}-1} \sum_{k=0}^{p^m-1} \alpha^i\beta^j\gamma^{ik-p^m ti} \\
&= p^{n-m} c_{ijm}\alpha^i\beta^j \quad (\text{note: } \gamma^{p^m} = 1)
\end{aligned}$$

and

$$\begin{aligned}
p_{n,m}(c_{ijn}\alpha^i\beta^j) &= \sum_{k=0}^{p^n-1} \gamma^{ik}\alpha^i\beta^j = p^{n-m} \sum_{k=0}^{p^m-1} \gamma^{ik}\alpha^i\beta^j \\
&= p^{n-m} c_{ijm}\alpha^i\beta^j.
\end{aligned}$$

If $v_p(j) < n$, we have

$$\begin{aligned}
\mathrm{Tr}_{m,n}(c_{ijm}\alpha^i\beta^j) = 0 \qquad &\text{and} \\
p_{n,m}(c_{ijn}\alpha^i\beta^j) = p_{n,m}(0) = 0.&
\end{aligned}$$

Hence $\mathrm{Tr}_{m,n}(c_{ijm}\alpha^i\beta^j) = p_{n,m}(c_{ijn}\alpha^i\beta^j)$ in both cases. $\square$

LEMMA 3.22. *The elements*

$$\mathrm{class}_P(\alpha^i\beta^j) \in \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \text{ and } c_{i,j,v_p(j)} \in \Lambda(U_{v_p(j)}/V_{v_p(j)})$$

for $i, j \in \mathbb{N}$ *have identical* $\Lambda(Z(P))$*-annihilators:*

$$\mathrm{Ann}_{\Lambda(Z(P))}(\mathrm{class}_P(\alpha^i\beta^j)) = \mathrm{Ann}_{\Lambda(Z(P))}(c_{i,j,v_p(j)})$$

PROOF. Put $x := \mathrm{class}_P(\alpha^i\beta^j)$, $Z := Z(P)$ and let

$$\Sigma = \{z \in Z | zx = x\}$$

be the stabiliser of x in Z. There is an exact sequence of $\Lambda(Z)$-modules

$$0 \to \mathrm{Ann}_{\Lambda(Z)}(x) \to \Lambda(Z) \xrightarrow{f} \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!],$$

where f is given by $f(z) = zx$ for $z \in Z$. Clearly, f factors through

$$\Lambda(Z/\Sigma) \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!].$$

This map is injective since $Z/\Sigma \hookrightarrow \mathrm{Conj}(P)$, $z \mapsto zx$ is injective. Hence

$$\mathrm{Ann}_{\Lambda(Z)}(x) = \ker(\Lambda(Z) \to \Lambda(Z/\Sigma)) = I(\Sigma)\Lambda(Z).$$

Since

$$\Sigma = \{[\alpha^i\beta^j, g] | g \in P\} = V_{\min\{v_p(i), v_p(j)\}},$$

we get $\mathrm{Ann}_{\Lambda(Z)}(x) = I(V_{\min\{v_p(i),v_p(j)\}})\Lambda(Z)$.

We will now determine $\mathrm{Ann}_{\Lambda(Z)}(c_{i,j,v_p(j)})$. If $v_p(i) \geq v_p(j)$, we get $c_{i,j,v_p(j)} = p^{v_p(j)}$ and hence

$$\mathrm{Ann}_{\Lambda(Z)}(c_{i,j,v_p(j)}) = I(V_{v_p(j)})\Lambda(Z) = \mathrm{Ann}_{\Lambda(Z)}(x).$$

Thus we may assume $v_p(i) < v_p(j)$. Put

$$\tilde{c} := \frac{1}{\# V_{v_p(i)}/V_{v_p(j)}} \sum_{g \in V_{v_p(i)}/V_{v_p(j)}} g \in \Lambda(U_{v_p(j)}/V_{v_p(j)}).$$

Then $\tilde{c} = p^{-v_p(j)} c_{i,j,v_p(j)}$ is an idempotent with

$$\mathrm{Ann}_{\Lambda(Z)}(\tilde{c}) = \mathrm{Ann}_{\Lambda(Z)}(c_{i,j,v_p(j)}).$$

Since $x = \tilde{c}x + (1 - \tilde{c})x$ for $x \in \Lambda(Z)$, we get

$$\begin{aligned}\mathrm{Ann}_{\Lambda(Z)}(\tilde{c}) &= (1 - \tilde{c})\Lambda(Z) \\ &= I(V_{v_p(i)})\Lambda(Z) = \mathrm{Ann}_{\Lambda(Z)}(x).\end{aligned}$$

□

THEOREM 3.23. *The homomorphism*

$$\tau : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \Omega$$

is an isomorphism of $\Lambda(Z(P))$*-modules.*

PROOF. We first have a quick look at the heuristics that lead us to the definition of the inverse homomorphism. Let $x \in \Omega$ and assume that we can write $x = (c_{ijn}\alpha^i\beta^j z_n)_n$ with some elements $z_n \in \Lambda(Z(P))$ and $i, j < p^c$. (Note that i, j are independent of n.) We want to find $z \in \Lambda(Z(P))$ such that $\tau(\alpha^i\beta^j z) = x$. We believe that $z = z_n$ for suitable n will work. It seems reasonable to demand that $\mathrm{class}_P(\alpha^i\beta^j z) \in \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ and $c_{ijn}\alpha^i\beta^j z_n \in \Lambda(U_n/V_n)$ have the same annihilators. Since $\alpha^i\beta^j \in \Lambda(U_n/V_n)^\times$ and hence

$$\mathrm{Ann}_{\Lambda(Z(P))}(c_{ijn}\alpha^i\beta^j) = \mathrm{Ann}_{\Lambda(Z(P))}(c_{ijn}),$$

this leads us to the following definition of a possible inverse map of τ:

$$\tau' : \Omega \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!], \quad (x_n)_{n\in\underline{c}} \mapsto \sum_{n\in\underline{c}} \tau'_n(x_n),$$

where τ'_n is the $\Lambda(Z(P))$-linear map

$$\tau'_n : I_n \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!], \quad c_{ijn}\alpha^i\beta^j \mapsto \begin{cases} \mathrm{class}_P(\alpha^i\beta^j) & \text{if } n = v_p(j) \\ 0 & \text{if } n \neq v_p(j) \end{cases}$$

with $i, j \in \{0, \dots, p^c - 1\}$.

We first assume that γ is of finite order $\mathrm{ord}(\gamma) = p^c$. Then the $\Lambda(Z(P))$-module $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ is generated by

$$\{\alpha^i\beta^j | i, j = 0, \dots, p^c - 1\}$$

(cf. lemma 3.7). Clearly, $\tau' \circ \tau = \mathrm{id}_{\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]}$ is the identity and hence τ is injective. It suffices to show that τ' is injective.

Let $x = (x_n)_n \in \Omega$ be an element of the kernel of τ'. We can write

$$x_n = \sum_{i,j=0}^{p^c-1} c_{ijn}\alpha^i\beta^j z_{ijn},$$

with $z_{ijn} \in \Lambda(Z(P))$.

For $0 \le i, j \le p^c$, we have $0 \le v_p(j) \le c$ and hence

$$\begin{aligned} 0 = \tau'(x) = \sum_{n=0}^{c} \tau'_n(x_n) &= \sum_{n=0}^{c} \sum_{i,j=0}^{p^c-1} \tau'_n(c_{ijn}\alpha^i\beta^j) z_{ijn} \\ &= \sum_{i,j=0}^{p^c-1} \mathrm{class}(\alpha^i\beta^j) z_{i,j,v_p(j)} \end{aligned}$$

By the direct sum decomposition of $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$ (cf. (15) in lemma 3.7), we get $\mathrm{class}(\alpha^i\beta^j) z_{i,j,v_p(j)} = 0$ for all $i, j \in \{0, \dots, p^c - 1\}$. By lemma 3.22, this is equivalent to $c_{i,j,v_p(j)} z_{i,j,v_p(j)} = 0$.

We need to show $c_{ijn}z_{ijn} = 0$ for all $i, j \in \{0, \dots, p^c - 1\}$ and $n \in \underline{\underline{c}}$. For $n > v_p(j)$, clearly $c_{ijn} = 0$ and hence $c_{ijn}z_{ijn} = 0$.

For $n \leq k$, we get from the calculation of $\mathrm{Tr}_{n,k}(x_n)$ and $p_{k,n}(x_k)$ in lemma 3.21

$$\sum_{\substack{i,j=0\\ k\leq v_p(j)}}^{p^c-1} p^{k-n} c_{ijn}\alpha^i\beta^j z_{ijn} = \sum_{\substack{i,j=0\\ k\leq v_p(j)}}^{p^c-1} p^{k-n} c_{ijn}\alpha^i\beta^j z_{ijk}$$

and hence

$$c_{ijn}z_{ijn} = c_{ijn}z_{ijk} \in \Lambda(U_m/V_m)$$

for all $i, j \in \{0, \dots, p^c - 1\}$ with $k \leq v_p(j)$.

That means we have for $n \leq v_p(j)$

$$c_{ijn}z_{ijn} = c_{ijn}z_{i,j,v_p(j)} = 0.$$

Thus we have shown that $\tau'(x) = 0$ implies $x = 0$.

Now assume that γ is of infinite order. Since $\bigcap_{n\in\underline{\underline{c}}} V_n = \{1\}$,

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \cong \varprojlim_{U\trianglelefteq_o P} \mathbb{Z}_p[\![\mathrm{Conj}(P/U)]\!] \cong \varprojlim_n \mathbb{Z}_p[\![\mathrm{Conj}(P/V_n)]\!].$$

In the next lemma, we will see that $\Omega_P \cong \varprojlim_n \Omega_{P/V_n}$. We have seen above that

$$\mathbb{Z}_p[\![\mathrm{Conj}(P/V_n)]\!] \cong \Omega_{P/V_n}$$

for any $n \in \mathbb{N}$. Hence we get an isomorphism

$$\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \cong \Omega_P.$$

Since the other two isomorphisms are the natural ones, this isomorphism is the homomorphism τ. □

Lemma 3.24. *The canonical map defines the isomorphism*

$$\Omega_P \cong \varprojlim_{n\in\underline{\underline{c}}} \Omega_{P/V_n}$$

of $\Lambda(Z(P))$-modules.

Proof. For any group W that satisfies assumption 3.1, let $c_W = c \in \mathbb{N} \cup \infty$ and $I_{i,W} = \tau_i\Big(\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]\Big)$ be the corresponding objects defined above. We define $I_W := \prod_{i=0}^{c_W} I_{i,W}$. Then $I_{i,P/V_n} = I_{i,P}$ and hence $I_{P/V_n} = \prod_{i=0}^{n} I_{i,P}$ for $n \leq c$. Thus we get

$$I_P \cong \varprojlim_n I_{P/V_n}.$$

We have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Omega_P \arrow[r] \arrow[d] & I_P \arrow[r] \arrow[d, "\wr"] & \prod_{i\geq j} \Lambda(U_i/V_j) \arrow[d, "\wr"] \\
0 \arrow[r] & \varprojlim_n \Omega_{P/V_n} \arrow[r] & \varprojlim_n I_{P/V_n} \arrow[r] & \varprojlim_n \prod_{n\geq i\geq j} \Lambda(U_i/V_j),
\end{tikzcd}$$

where the map $I_P \to \prod_{i\geq j} \Lambda(U_i/V_j)$ is given by

$$(x_k)_k \mapsto (\mathrm{Tr}_{j,i}(x_j) - p_{i,j}(x_i))_{j,i}.$$

Hence $\Omega_P \cong \varprojlim_{n\in\underline{\underline{c}}} \Omega_{P/V_n}$. □

Corollary 3.25. *If we apply definition 3.10 to the objects*

$$\Omega_P := \left(\Omega_{P/W}\right)_{W\in\mathfrak{W}_P} \quad \textit{and} \quad \Omega_{P,S} := \left(\Omega_{P/W,S}\right)_{W\in\mathfrak{W}_P},$$

of $\mathcal{M}_{C,\mathfrak{W}_P}$, where $C(W) = Z(P/W)$ for $W \in \mathfrak{W}_P$, we get the identities

$$(\Omega_P)^\wedge_S = \varprojlim_{W\in\mathfrak{W}_P} \left(\Lambda(Z(P/W))_S \otimes_{\Lambda(Z(P/W))} \Omega_{P/W}\right)^{\langle p\rangle}$$

$$(\Omega_{P,S})^\wedge = \varprojlim_{W\in\mathfrak{W}_P} \left(\Omega_{P/W,S}\right)^{\langle p\rangle}.$$

Then these two groups coincide. We denote it by $\widehat{\Omega_S}$. There is an isomorphism

$$\tau_S : \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^\wedge_S \to \widehat{\Omega_S}$$

of $\Lambda(Z(P))$-modules.

Proof. If $\mathrm{ord}(\gamma) < \infty$, the identity

$$\Omega_{P,S} = \Lambda(Z(P))_S \otimes_{\Lambda(Z(P))} \Omega_P$$

follows from proposition 2.10. By passing to the p-adic completion, we get

$$(\Omega_P)^\wedge_S = (\Omega_{P,S})^\wedge.$$

This implies the first assertion. Since $\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \cong \Omega_P$ by theorem 3.23, application of $(-)^\wedge_S$ yields the second assertion. □

3. Construction of the Integral Logarithm

Let P be a pro-p p-adic Lie group. In this subsection, we will-define the integral logarithm

$$\mathscr{L}_P : K_1(\Lambda(P)) \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!].$$

We start with the investigation of convergence properties of the ordinary logarithm series log on $\Lambda(P)$. The homomorphism property will follow from a general property of formal power series. We will compose log with another homomorphism to get the integral logarithm $\mathscr{L}_P$.

Assume that there is a surjective homomorphism $\omega : P \twoheadrightarrow \mathbb{Z}_p$ and put $S := S(P, \omega)$. In this case, we will define a localised version

$$\mathscr{L}_{P,S} : K_1(\widehat{\Lambda(P)_S}) \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^{\wedge}_S.$$

of $\mathscr{L}_P$. In general, the logarithm series does not converge on $K_1(\widehat{\Lambda(P)_S})$. We will show that it suffices to define $\mathscr{L}_{P,S}$ on $\Lambda(P)^\times$ and $\Lambda(Z(P/W))^\times_S$ for all $W \in \mathfrak{W}_P$.

LEMMA 3.26. *Let W be a pro-p p-adic Lie group. Then*

$$\Lambda(W)^\times = \mu_{p-1} \times (1 + J),$$

where $J := J(\Lambda(W))$ denotes the Jacobson radical of $\Lambda(W)$ and $\mu_{p-1} \subset \mathbb{Z}_p$ is the group of $(p-1)$th roots of unity.

PROOF. Clearly, $\mu_{p-1} \subset \Lambda(W)^\times$. Let $\varepsilon : \Lambda(W) \to \mathbb{Z}_p$ be the augmentation map. By [**33**, 5.2.16], $\Lambda(W)$ is a local ring with maximal ideal

$$J = \{x \in \Lambda(W) | \varepsilon(x) \in p\mathbb{Z}_p\}.$$

(This is not true for p-adic Lie groups that are not pro-p.) Thus, $\Lambda(W)/J \cong \mathbb{Z}/p\mathbb{Z}$ and we have an exact sequence

$$1 \to 1 + J \to \Lambda(W)^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \to 1.$$

Since the image of μ_{p-1} under the projection map is $(\mathbb{Z}/p\mathbb{Z})^\times$, the above sequence splits. Since μ_{p-1} lies in the centre of $\Lambda(W)^\times$, the lemma follows. ☐

Let W_1 be a commutative profinite group. We recall the definition of

$$\varphi : \Lambda(W_1) \to \Lambda(W_1), \quad \sigma \mapsto \sigma^p.$$

Let W_2 be a commutative p-adic Lie group with a surjection $\omega : W_2 \to \mathbb{Z}_p$. Since $\varphi(W)$ is an open subgroup of W, $\varphi(S(W)) \subset S(W)$. Hence φ induces the homomorphism

$$\varphi : \Lambda(W)_S \to \Lambda(W)_S.$$

LEMMA 3.27. *For $x \in \Lambda(W_1)$,*

$$x^p \equiv \varphi(x) \bmod p\Lambda(W_1).$$

For $y \in (\Lambda(W_2)_S)^\wedge$,

$$y^p \equiv \varphi(y) \bmod p\left(\Lambda(W_2)_S\right)^\wedge.$$

PROOF. We first assume that W_1 is finite. We get

$$\bar{x}^p = \varphi(\bar{x}) \in \mathbb{F}_p[\![W_1]\!]$$

for the image $\bar{x}$ of x in $\mathbb{F}_p[\![W_1]\!]$. By taking inverse limits, we get the general result for profinite groups.

Note that the second statement is not trivial since $S \cap p\Lambda(W_2)_S = \emptyset$. Let $a \in \Lambda(W_2)$ and $s \in S(W_2)$. Then

$$a^p \equiv \varphi(a) \bmod p\Lambda(W_2) \quad \text{and} \quad s^p \equiv \varphi(s) \bmod p\Lambda(W_2).$$

Since

$$\Lambda(W_2)_S/p\Lambda(W_2)_S \cong (\Lambda(W_2)/p\Lambda(W_2))_S\,,$$

this implies

$$\left(\frac{a}{s}\right)^p \equiv \varphi\left(\frac{a}{s}\right) \bmod p\Lambda(W_2)_S.$$

By passing to the inverse limit, the second assertion follows. □

Lemma 3.28. *Let $R_1 = \Lambda(W_1)$, where W_1 is a p-adic Lie group and let $R_2 = \widehat{\Lambda(W_2)}_S$, where W_2 is a pro-p p-adic Lie group with a surjection $\omega : W_2 \twoheadrightarrow \mathbb{Z}_p$. Put*

$$R_1' := \mathbb{Q}_p[\![W_1]\!] \quad \text{or} \quad R_2' := \mathbb{Q}_p'[\![W_2]\!]_S.$$

then the series

$$\log(1-x) = -\sum_{i\geq 1} \frac{x^i}{i} \in R_i'$$

converges for $x \in J_i := J(R_i)$, $i = 1, 2$.

Proof. Fix $i = 1$ or $i = 2$. We first assume every element of J_i/pR_i is nilpotent. Hence for $x \in J_i$ and $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $x^{kn} \in p^n R_i$. Let $\lfloor * \rfloor$ be the largest integer $\leq *$. Then $x^n \in p^{\lfloor n/k \rfloor} R_i$ and $\frac{x^n}{n} \in \frac{1}{n} p^{\lfloor n/k \rfloor} R_i$. But $v_p(\frac{p^{\lfloor n/k \rfloor}}{n}) \to \infty$ for $n \to \infty$ which proves $\frac{x^n}{n} \to 0$ for $n \to \infty$. (The above argument comes from [**34**, lemma 2.7].)

Clearly, the above assumption is satisfied for $R_1 = \Lambda(W_1)$ when W_1 is a finite group. Let W_2 be a one-dimensional p-adic Lie group with a surjection $\omega : W_2 \twoheadrightarrow \mathbb{Z}_p$ and assume $R_2 = \widehat{\Lambda(W_2)}_S$. By lemma 2.4,

$$\Lambda(W_2)_S/p\Lambda(W_2)_S = Q(\mathbb{F}_p[\![W_2]\!]) =: Q.$$

Since every regular element of $\mathbb{F}_p[\![W_2]\!]$ becomes a unit in Q, every element of $J(Q)$ is a zero divisor. Since $\ker \omega$ is a finite p-group, there is an integer k such that $\langle g^{p^k} | g \in W_2 \rangle \cong \mathbb{Z}_p$. Since $[W_2, W_2] \subset \ker \omega$, the p^k-th power map defines a surjective homomorphism $W \twoheadrightarrow \mathbb{Z}_p$. By corollary 2.6, we may define the ring homomorphism

$$\varphi^k : Q \to Q, \quad g \mapsto g^{p^k} \text{ for } g \in W_2.$$

This map induces a surjective ring homomorphism $Q \twoheadrightarrow Q(\mathbb{F}_p[\![\mathbb{Z}_p]\!])$. Then $\varphi^k(J(Q))$ is isomorphic to a proper ideal of $Q(\mathbb{F}_p[\![\mathbb{Z}_p]\!])$. Since $\mathbb{F}_p[\![\mathbb{Z}_p]\!]$ is an integral domain, this implies $\varphi^k(J(Q)) = 0$. By lemma 3.27, $\varphi^k(x) = x^{p^k}$ for $x \in Q$ and hence R_2 satisfies the above assumption.

Let $\mathcal{U}_1$ be the set of all open normal subgroups of W_1 and let $\mathcal{U}_2$ be the set of all open subgroups of $\ker\omega$ that are normal in W_2. We get the general case by passing to the inverse limits

$$\varprojlim_{U\in\mathcal{U}_1} \Lambda(W_1/U) = \Lambda(W_1), \qquad \varprojlim_{U\in\mathcal{U}_1} \mathbb{Q}_p[\![W_1/U]\!] = \mathbb{Q}_p[\![W_1]\!],$$
$$\varprojlim_{U\in\mathcal{U}_2} (\Lambda(W_2/U)_S)^\wedge = (\Lambda(W_2)_S)^\wedge, \quad \varprojlim_{U\in\mathcal{U}_2} \mathbb{Q}_p'[\![W_2/U]\!]_S^\wedge = \mathbb{Q}_p'[\![W_2]\!]_S^\wedge. \square$$

Let R be a commutative ring and let $R\langle\langle X,Y\rangle\rangle$ be the ring of formal noncommutative power series in two indeterminates. Let $W_n \subset R\langle\langle X,Y\rangle\rangle$ be the set of formal (ordered) monomials of length n in two variables X and Y. Define

$$\delta_n : W_n \to W_n, \quad a_1\dots a_n \mapsto a_n a_1 a_2 \cdots a_{n-1} \text{ for } a_i \in \{X,Y\}.$$

For $v,w\in W_n$, we define the equivalence relation $\sim$ by setting $v\sim w$ if there is a cyclic permutation that transforms v into w, i. e. if there is $l\in\mathbb{N}$ such that $\delta_n^l(v) = w$. We extend this relation linearly to $R\langle\langle X,Y\rangle\rangle$ as follows: Let $W_* := \bigcup_n W_n$ and $\varphi,\psi \in R\langle\langle X,Y\rangle\rangle$. Then $\varphi\sim\psi$ if and only if there is a map $\lambda : W_*\to W_*$ with $\lambda(w)\sim w$ for all $w\in W_*$ and

$$\varphi - \psi = \sum_{w\in W_*} a_w(w-\lambda(w)). \tag{21}$$

Lemma 3.29. *Let R be a commutative topological ring. Let S be a topological R-algebra and let $x,y\in S^\times$ be units. In case of convergence, the evaluation homomorphism*

$$R\langle\langle X,Y\rangle\rangle/\sim\; \to S/\overline{[S,S]}, \quad X\mapsto x, \quad Y\mapsto y$$

does not depend on the choice of representatives modulo $\sim$.

Proof. For $\varphi,\psi\in R\langle\langle X,Y\rangle\rangle$ with $\varphi\sim\psi$, we write $\varphi-\psi$ as in (21). For $w\in W_*$, we can obviously find $\nu\in W_*$ such that $\lambda(w)\nu = \nu w$. Then $\nu(x,y)\in S^\times$ and

$$w(x,y) - \lambda(w)(x,y) = w(x,y) - \nu(x,y)w(x,y)\nu(x,y)^{-1} \in [S,S]$$

Hence $\varphi(x,y)-\psi(x,y)\in\overline{[S,S]}$. $\square$

We call a ring R divisible if $R\to R,\ x\mapsto mx$ is surjective for all $m\in\mathbb{Z}\setminus\{0\}$.

Proposition 3.30. *Let R be a divisible commutative topological ring and define the power series $\log(1+T)\in R[[T]]$ by*

$$\log(1+T) = \sum_{i\geq 1}\frac{(-1)^{i-1}}{i}T^i.$$

Then, in $R\langle\langle X,Y\rangle\rangle$, we have the relation

$$\log[(1+X)(1+Y)] \sim \log(1+X) + \log(1+Y).$$

PROOF. We use the argument given in [**34**, lemma 2.7] to prove our (more general) proposition.

For $w \in W_n$, let $k(w)$ be the number of occurrences of XY in w. Let $r(w)$ be defined by the relation

$$\log(1+X+Y+XY) = \sum_{w\in W_*} r(w)w. \tag{22}$$

Let $r_j(w)$ be defined by the relation $(X+Y+XY)^j = \sum_{w\in W_*} r_j(w)w$. Any summand $w \in W_n$ of $(X+Y+XY)^j$ has exactly $n-j$ factors coming from the XY-summand and $2j-n$ factors coming from the X- or Y-summands of $X+Y+XY$. As there are $\binom{k(w)}{n-j}$ ways of choosing $n-j$ out of the $k(w)$ occurrences of XY in w (and hence expressing w in the above form), we see that $r_j(w) = \binom{k(w)}{n-j}$. Clearly,

$$r(w) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} r_j(w) = \sum_{j=0}^{k(w)} \frac{(-1)^{n-j-1}}{n-j}\binom{k(w)}{j}. \tag{23}$$

For $C \in W_n/\sim$, we define $t, k \in \mathbb{N}$ by

$$\#C = \frac{n}{t} \quad \text{and} \quad k = \max\{k(w) | w \in C\}.$$

Since $\mathbb{Z}/n\mathbb{Z} \cong \{\delta_n^l | l \in \mathbb{Z}\}$ operates transitively on C, we get by the orbit-stabiliser theorem that t is the cardinality of the stabiliser of any element of C. (In particular $t = \frac{n}{\#C} \in \mathbb{N}$.) If $w \in C$ with $k(w) = k$, then we have a t-to-one correspondence of occurrences of $(XY)'s$ in w and elements $Y\cdots X \in C$ with $k(Y\cdots X) = k-1$. Hence

$$\#\{w \in C | k(w) = k-1\} = \frac{k}{t} \qquad \text{and}$$
$$\#\{w \in C | k(w) = k\} = \#C - \frac{k}{t} = \frac{n-k}{t}.$$

Using (23), we get for any $C \in W_n/\sim$

$$\begin{aligned}
\sum_{w\in C} r(w) &= \sum_{j=0}^{k} \frac{(-1)^{n-j-1}}{n-j}\left[\frac{k}{t}\binom{k-1}{j} + \frac{n-k}{t}\binom{k}{j}\right] \\
&= \sum_{j=0}^{k} \frac{(-1)^{n-j-1}}{n-j}\left[\frac{k-j}{t}\binom{k}{j} + \frac{n-k}{t}\binom{k}{j}\right] \\
&= \sum_{j=0}^{k} \frac{(-1)^{n-j-1}}{n-j}\frac{n-j}{t}\binom{k}{j} \\
&= \begin{cases} 0 & \text{if } k>0 \\ \frac{(-1)^{n-1}}{n} & \text{if } k=0 \quad (\text{so } t=n). \end{cases}
\end{aligned}$$

For every $C \in W_*/\sim$, we choose an element $w_C \in C$. Combining the above result with (22), we get

$$\begin{aligned}
\log(1+X+Y+XY) \sim \sum_{C\in W_*/\sim}\left(\sum_{w\in C} r(w)\right) w_C \\
= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(X^n+Y^n) \\
= \log(1+X)+\log(1+Y).
\end{aligned}$$

□

In the further argument, we will not need the following interesting corollary to the above proposition.

Corollary 3.31. *Let R be a divisible commutative topological ring. Let $U \subset R\langle\langle X,Y\rangle\rangle^{\times}$ be the subgroup of power series with constant term 1 (i. e. $U = \{w \in R\langle\langle X,Y\rangle\rangle^{\times} | w(0,0) = 1\}$). The power series* log *induces a homomorphism*

$$\log : U \to R\langle\langle X,Y\rangle\rangle/\sim$$

Proof. We endow $R\langle\langle X,Y\rangle\rangle$ with the natural topology. For $\psi \in U$, clearly $\log(\psi)$ converges. By lemma 3.29, the map

$$\log : U \to R\langle\langle X,Y\rangle\rangle/\overline{[R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle]}, \quad \psi \mapsto \overline{\log\circ\psi}$$

does not depend on the choice of a representative of

$$[\log] \in R\langle\langle X,Y\rangle\rangle/\sim .$$

By proposition 3.30, it is a homomorphism. By lemma 3.4 (applied to the monoid W_*), we have

$$\langle v-w | v,w\in W_*, v\sim w\rangle_R = [R\langle X,Y\rangle, R\langle X,Y\rangle],$$

where $R\langle X, Y\rangle = R[W_*]$ is the ring of noncommutative polynomials in two variables. Taking quotients modulo the topological closure of this group in $R\langle\langle X, Y\rangle\rangle$, we get

$$R\langle\langle X, Y\rangle\rangle/\!\sim\, \cong R\langle\langle X, Y\rangle\rangle/\overline{[R\langle\langle X, Y\rangle\rangle, R\langle\langle X, Y\rangle\rangle]}.$$

This finishes the proof. □

Corollary 3.32. *Let W be a pro-p p-adic Lie group. There is a homomorphism*

$$\log : K_1(\Lambda(W)) \to \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!], \quad [1 - x] \mapsto [-\sum_i \frac{x^i}{i}].$$

If $I \subset J := J(\Lambda(W))$ is a two sided ideal, we get a homomorphism

$$\log^I : K_1(\Lambda(W), I) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I/[\Lambda(W), I].$$

Proof. Let $U \subset \mathbb{Z}_p\langle\langle X, Y\rangle\rangle^\times$ be the subgroup of power series with constant term 1. By lemma 3.29, in case of convergence, evaluation at $x, y \in 1 + J$ or $x, y \in 1 + I$ induces the maps

$$\begin{aligned}\mathbb{Z}_p\langle\langle X, Y\rangle\rangle/\!\sim\, &\to \mathbb{Q}_p[\![W]\!]/\overline{[\mathbb{Q}_p[\![W]\!], \mathbb{Q}_p[\![W]\!]]} \cong \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!]\\ \mathbb{Z}_p\langle\langle X, Y\rangle\rangle/\!\sim\, &\to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I/\overline{[\Lambda(W), I]}.\end{aligned}$$

By lemma 3.28, log converges on $1 + J$ and by proposition 3.30, we get homomorphisms

$$\begin{aligned}\log : 1 + J &\to \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!]\\ \log^I : 1 + I &\to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I/[\Lambda(W), I].\end{aligned}$$

We put $\log(\zeta) = 0$ for $\zeta \in \mu_{p-1}$. (This is the unique homomorphism $\mu_{p-1} \to \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!]$.) By lemma 3.26, this induces the homomorphism

$$\log : \Lambda(W)^\times \to \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!].$$

Since the images of log and $\log^I$ are commutative groups, the homomorphism log (respectively $\log^I$) is well-defined on $K_1(\Lambda(W))$ (respectively $K_1(\Lambda(W), I)$). □

Proposition 3.33. *Let R be a divisible commutative topological ring and let U be as above. We define the power series*

$$\exp(T) := \sum_{i=0}^{\infty} \frac{1}{i!} T^i \in R[[T]].$$

Then

$$\exp(X) \exp(Y) \exp(-X - Y) \in \overline{[U, U]} \subset R\langle\langle X, Y\rangle\rangle.$$

Proof. We define the Campbell-Hausdorff series

$$\Phi(X,Y) \in R\langle\langle X,Y\rangle\rangle$$

and the commutator Campbell-Hausdorff series

$$\Psi(X,Y) \in R\langle\langle X,Y\rangle\rangle$$

by the relations

$$\begin{aligned}\exp(\Phi(X,Y)) &= \exp(X)\exp(Y) \qquad \text{and}\\ \exp(\Psi(X,Y)) &= \exp(-X)\exp(-Y)\exp(X)\exp(Y).\end{aligned}$$

(For a proof of existence and uniqueness of these series, see [**13**].) Clearly, $\exp(\Psi(X,Y)) \in [U,U]$ and

$$\begin{aligned}\Phi(X,\Phi(Y,Z)) &= \Phi(\Phi(X,Y),Z)\\ \Phi(X,-X) &= 0\\ \Phi(X,0) &= X.\end{aligned}$$

By [**13**, theorem 6.28],

$$\begin{aligned}\Phi(X,Y) - X - Y &\in [R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle] \qquad \text{and}\\ \Psi(X,Y) &\in [R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle].\end{aligned} \tag{24}$$

For $n \in \mathbb{N}$, let $\deg n \subset R\langle\langle X,Y\rangle\rangle$ be the R-module generated by the monomials of degree $\geq n$. Then

$$\begin{aligned}\Phi(X,Y) &\equiv X + Y \bmod \deg 2 \qquad \text{and}\\ \Psi(X,Y) &\equiv XY - YX \bmod \deg 3.\end{aligned} \tag{25}$$

For any power series

$$p(X,Y) = \sum_i [q_i, r_i] \in [R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle], \quad q_i, r_i \in R\langle\langle X,Y\rangle\rangle,$$

there is a power series $\Psi(p) \in R\langle\langle X,Y\rangle\rangle$ such that $\Psi(p)(X,Y) = \sum_i \Psi(q_i, r_i)$. If $p \in \deg n$, $n \geq 1$, then by (24) and (25)

$$\Phi(p, -\Psi(p)) \in [R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle] \cap \deg(n+1).$$

Hence for each p as above, there is

$$q \in [R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle] \cap \deg(n+1)$$

such that

$$\exp(p)\exp(q)^{-1} = \exp(\Phi(p,-q)) \in [U,U].$$

By induction, we get $\exp(p) \in \overline{[U,U]}$.

Since $\Phi(\Phi(X,Y), -X-Y) \in [R\langle\langle X,Y\rangle\rangle, R\langle\langle X,Y\rangle\rangle]$, this implies

$$\begin{aligned}\exp(X)\exp(Y)\exp(-X-Y) &= \exp\Big(\Phi(\Phi(X,Y), -X-Y)\Big)\\ &\in \overline{[U,U]}.\end{aligned}$$

□

COROLLARY 3.34. *The power series* exp *induces a homomorphism*

$$\exp : R\langle\langle X, Y\rangle\rangle \to U/\overline{[U, U]}.$$

LEMMA 3.35. *Let W be a pro-p p-adic Lie group and let $I \subset \Lambda(W)$ be a both sided ideal and assume that there is $\xi \in Z(\Lambda(W))$ such that $I \subset \xi\Lambda(W)$ and $\xi^p \in p\xi\Lambda(W)$. Then the logarithm series induces the homomorphism*

$$\log : K_1(\Lambda(W), I) \to I/[\Lambda(W), I]$$

If $I^p \subset pIJ$, $J := J(\Lambda(W))$, then log *is an isomorphism whose inverse map*

$$\exp : I/[\Lambda(W), I] \to K_1(\Lambda(W), I).$$

is induced by the exponential series exp.

(This is a slight generalisation of [**34**, theorem 2.8]*.)*

PROOF. By assumption, $I^p \subset \xi^p\Lambda(W) \cap I \subset pI$ and hence $I^n \subset nI$ for all $n \geq 1$. This implies $\log(1 + I) \subset I$ and proves the first part of this lemma.

Now assume $I^p \subset pIJ$. We show convergence of the exponential series. Let $n \geq 1$ be a natural number. The identity

$$v_p(n!) = \sum_{l=1}^{\infty} \left\lfloor \frac{n}{p^l} \right\rfloor$$

is well known and can be verified easily. Let $k \in \mathbb{N}$ be the number such that $p^k \leq n < p^{k+1}$. Then

$$I^n \subset p^{\lfloor \frac{n}{p} \rfloor} I^{\lfloor \frac{n}{p} \rfloor} J \subset \ldots \subset p^{\sum_{l=1}^{k} \lfloor \frac{n}{p^l} \rfloor} IJ^k \subset n! IJ^k.$$

Thus $\frac{1}{n!}I^n \subset I$ and since $\Lambda(W)$ is noetherian (cf. corollary 1.11) $\bigcap_n \frac{1}{n!}I^n \subset \bigcap_n IJ^n = 0$. Hence $\exp(x)$ converges in $1 + I$ for $x \in I$. □

Let P be a compact p-adic Lie group with a surjection $\omega : P \twoheadrightarrow \mathbb{Z}_p$.

LEMMA 3.36. *Assume that $Z := Z(P) \subset P$ is an open subgroup. Every unit $x \in \widehat{\Lambda(P)_S}^\times$ can be written as a product $x = uv$ with $u \in \widehat{\Lambda(Z)_S}^\times$ and $v \in \Lambda(P)^\times$.*

PROOF. By lemma 2.10, we can write every element of $\widehat{\Lambda(P)_S}$ as a product $x = uv$ with $u \in \widehat{\Lambda(Z)_S}$ and $v \in \Lambda(P)$. When x is a unit, then u, v are units, too (u is a right unit and since it is central, it is also a left unit). □

The following proposition defines the *integral logarithm*.

PROPOSITION 3.37. *Let P be a compact p-adic Lie group. There is a well-defined group homomorphism*

$$\mathscr{L}_P : K_1(\Lambda(P)) \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!], \quad x \mapsto (1 - p^{-1}\varphi) \circ \log(x)$$

Assume that there is a surjective homomorphism $\omega : P \twoheadrightarrow \mathbb{Z}_p$. Let $\mathfrak{W}_P$ be as in lemma 3.8. Then we can define

$$\mathscr{L}_{P,S} : K_1((\Lambda(P)_S)^\wedge) \to \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]_S^\wedge,$$

to be the composition of the natural homomorphism

$$K_1((\Lambda(P)_S)^\wedge) \to \varprojlim_{W \in \mathfrak{W}_P} K_1((\Lambda(P/W)_S)^{\langle p \rangle})$$

and the homomorphism

$$\varprojlim_{W \in \mathfrak{W}_P} K_1((\Lambda(P/W)_S)^{\langle p \rangle}) \to \varprojlim_{W \in \mathfrak{W}} \left(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]_S\right)^{\langle p \rangle},$$

which is induced by the maps

$$\begin{aligned} K_1((\Lambda(P/W)_S)^{\langle p \rangle}) &\to \left(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]_S\right)^{\langle p \rangle}, \\ [x]_{(\Lambda(P/W)_S)^{\langle p \rangle}} &\mapsto \frac{1}{p}\log(u^p\varphi(u)^{-1}) + (1 - \frac{1}{p}\varphi) \circ \log([v]_{\Lambda(P/W)}), \end{aligned}$$

for all $W \in \mathfrak{W}_P$, where we write an element $x \in \left((\Lambda(P/W)_S)^{\langle p \rangle}\right)^\times$ in the form $x = uv$ with $u \in \left((\Lambda(Z(P/W))_S)^{\langle p \rangle}\right)^\times$ and $v \in \Lambda(P/W)^\times$.

REMARK. If P is commutative, the integral logarithm has the form

$$\mathscr{L}_P : \Lambda(P)^\times \to \Lambda(P) \qquad \mathscr{L}_{P,S} : \widehat{\Lambda(P)_S}^\times \to \widehat{\Lambda(P)_S}.$$

PROOF. Note that by lemma 3.36, the decomposition $x = uv$ exists and by lemma 3.27, $u^p\varphi(u)^{-1} \in 1 + p\,(\Lambda(Z(P/W))_S)^{\langle p \rangle}$. Since φ is a continuous ring homomorphism, we get

$$\frac{1}{p}\log(u^p\varphi(u)^{-1}) = (1 - \frac{1}{p}\varphi) \circ \log(u).$$

for $u \in \left((\Lambda(Z(P/W))_S)^{\langle p \rangle}\right)^\times \cap \Lambda(P/W)^\times$. Hence $\mathscr{L}_{P/W,S}$ is independent of the decomposition $x = uv$.

It suffices to prove the existence of $\mathscr{L}_{P,S}$ in case $Z(P) \subset P$ is an open subgroup. Since

$$\log(1 + p\,(\Lambda(Z(P/W))_S)^{\langle p \rangle}) \subset p\,(\Lambda(Z(P/W))_S)^{\langle p \rangle},$$

it suffices to show that $\mathscr{L}_P$ is integral. Put $\mathfrak{R} := \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]$.

For any $x \in J$, we have

$$\mathscr{L}_P(1-x) = -\sum_{i \geq 1} \frac{x^i}{i} + \sum_{i \geq 1} \frac{\varphi(x^i)}{ip}.$$

For $i \in \mathbb{Z} \setminus p\mathbb{Z}$, $n \geq 1$, clearly $\frac{x^i}{i} \in \mathfrak{R}$ and $\frac{x^{p^n}}{p^n} - \varphi(\frac{x^{p^{n-1}}}{p^{n-1}}) \in \mathfrak{R}$ implies $\frac{x^{p^n i}}{p^n i} - \varphi(\frac{x^{p^{n-1} i}}{p^{n-1} i}) \in \mathfrak{R}$. Hence, we only have to show

$$p^n \left| [x^{p^n} - \varphi(x^{p^{n-1}})] \right.$$

for all $n \geq 1$, $x \in J$. We write the image of x in $\mathfrak{R}$ in the form

$$\bar{x} = \sum_{i \in I} r_i \mathrm{class}(g_i), \quad r_i \in \Lambda(Z(P)), \quad g_i \in P,$$

where I is a finite index set. We put $q := p^n$. Then

$$\overline{x^q} = \sum_{i \in I^q} r_{i_1} \cdots r_{i_q} \mathrm{class}(g_{i_1} \cdots g_{i_q}) =: \sum_{i \in I^q} s_i \in \mathfrak{R}.$$

Let $\delta_q : I^q \to I^q$ be the map defined by

$$\delta_q((i_1, \dots, i_q)) = (i_q, i_1 \dots, i_{q-1}).$$

Let $C \in I^q/\sim$ be an equivalence class, where the relation $\sim$ is defined by $i \sim j \Leftrightarrow s_i = s_j$. Define $t = t(C) \in \mathbb{N}$ by $\#C = p^{n-t}$. Then $s_{\delta_q^t(i)} = s_i$ for all $s \in C$ and hence

$$\begin{aligned}
\overline{x^q} &= \sum_{C \in I^q/\sim} p^{n-t(C)} \left(\prod_{j=1}^{p^{n-t(C)}} r_{i_j} \right)^{p^{t(C)}} \mathrm{class}\left(\left(g_{i_1} \cdots g_{i_{p^{n-t(C)}}} \right)^{p^{t(C)}} \right) \\
&=: \sum_{C \in I^q/\sim} p^{n-t(C)} \hat{r}_C^{p^{t(C)}} \mathrm{class}\left(\hat{g}_C^{p^{t(C)}} \right) \in \mathfrak{R}
\end{aligned}$$

with $\hat{r}_C \in \Lambda(Z(P))$, $\hat{g}_C \in P$. For $t(C) = 0$, we get $p^n | \sum_{y \in C} y$. We calculate

$$\begin{aligned}
\varphi(\overline{x^{p^{n-1}}}) &= \sum_{i \in I^{p^{n-1}}} \varphi(r_{i_1} \cdots r_{i_{p^{n-1}}}) \mathrm{class}\left(\left(g_{i_1} \cdots g_{i_{p^{n-1}}} \right)^p \right) \\
&= \sum_{\substack{C \in I^q/\sim \\ t(C) > 0}} p^{n-t(C)} \varphi(\hat{r}_C)^{p^{t(C)-1}} \mathrm{class}\left(\hat{g}_C^{p^{t(C)}} \right).
\end{aligned}$$

Thus, we only need to show

$$p^n \left| [p^{n-t} \hat{r}^{p^t} \mathrm{class}\left(\hat{g}^{p^t} \right) - p^{n-t} \varphi(\hat{r})^{p^{t-1}} \mathrm{class}\left(\hat{g}^{p^t} \right)] \right.$$

for $t > 0$. This follows from lemma 3.27. $\square$

LEMMA 3.38. *Let P be a compact p-adic Lie group. Then*

$$K_1(\Lambda(P)) \cong \varprojlim_{U} K_1(\Lambda(P/U))$$

where the limit is over all open normal subgroups U of P. Assume that there is a surjective homomorphism $\omega : P \twoheadrightarrow \mathbb{Z}_p$. Then

$$K_1((\Lambda(P)_S)^{\wedge}) \twoheadrightarrow \varprojlim_{W \in \mathfrak{W}_P} K_1((\Lambda(P/W)_S)^{\langle p \rangle})$$

is surjective.

PROOF. The isomorphism follows from proposition 1.19. (The assumptions of proposition 1.19 are satisfied by [**16**, 1.4.2].) Since for the ring $(\Lambda(P)_S)^{\wedge}$ is semi-local by lemma 3.14, the second assertion follows from lemma 1.20. □

4. Kernel and Cokernel of the Integral Logarithm

We use techniques developed by R. Oliver to prove the exactness of the sequence

$$1 \to \mu_{p-1} \times P^{ab} \to K_1(\Lambda(P)) \xrightarrow{\mathscr{L}_P} \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to P^{ab} \to 1,$$

where P is a pro-p p-adic Lie group. This is a consequence of the following theorem:

THEOREM 3.39. *Let W be a (finite) p-group. Define*

$$\omega = \omega_W : \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \to W^{ab}$$

to be the group homomorphism induced by $\mathrm{class}_W(\sigma) \mapsto [\sigma]$ for $\sigma \in W$. Then, the sequence

$$1 \to K_1(\Lambda(W))_{tors} \to K_1(\Lambda(W)) \xrightarrow{\mathscr{L}_W} \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \xrightarrow{\omega} W^{ab} \to 1$$

is exact.

PROPOSITION 3.40. *Let W be a p-group. Let $z \in Z(W)$ be an element of order p. Put $L := (1-z)\Lambda(W)$. Then the sequence*

$$1 \to \langle z \rangle \to K_1(\Lambda(W), L) \overset{\log}{\to} L/[\Lambda(W), L] \overset{\alpha''}{\to} \mathbb{Z}/p\mathbb{Z} \to 1 \tag{26}$$

is exact (where the map α'' will be constructed below) and
(27)
$$[(1-z)\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] : \mathscr{L}_W(1+L)] = \begin{cases} 1 & \text{if } z \text{ is a commutator} \\ p & \text{otherwise} \end{cases}.$$

Proof. We give a more detailed version of the rather short proof in [**34**, prop. 6.4].

We have

$$1 = z^p = [1-(1-z)]^p \equiv 1-(1-z)^p \bmod pL$$

and hence $(1-z)^p \in pL$. By lemma 3.35, there is a homomorphism

$$\log^L : K_1(\Lambda(W), L) \to L/[\Lambda(W), L]$$

and an isomorphism

$$\log^{LJ} : K_1(\Lambda(W), LJ) \xrightarrow{\cong} LJ/[\Lambda(W), LJ].$$

The following diagram is commutative:

$$\begin{tikzcd} & K_1(\Lambda(W), LJ) \arrow[r] \arrow[d, "\cong"', "\log^{LJ}"] & K_1(\Lambda(W), L) \arrow[r] \arrow[d, "\log^L"] & K_1(\frac{\Lambda(W)}{LJ}, \frac{L}{LJ}) \arrow[r] \arrow[d, "\log_0"] & 1 \\ 0 \arrow[r] & LJ/[\Lambda(W), LJ] \arrow[r] & L/[\Lambda(W), L] \arrow[r] & \frac{L}{LJ}/[\frac{\Lambda(W)}{LJ}, \frac{L}{LJ}] \arrow[r] & 0, \end{tikzcd} \tag{28}$$

where $\log_0$ is the homomorphism induced by $\log^L$. The first row is exact by [**30**, remark 6.6]. From the inclusion

$$\begin{aligned} [\Lambda(W), \Lambda(W)] &= \langle gh - hg | g, h \in W\rangle_{\mathbb{Z}_p} \\ &= \langle (1-g)h - h(1-g) | g, h \in W\rangle_{\mathbb{Z}_p} \subset [\Lambda(W), J], \end{aligned}$$

we get

$$LJ \cap [\Lambda(W), L] = [\Lambda(W), LJ]$$

and hence the second row is exact.

Since $\mathbb{Z}/p\mathbb{Z}$ is commutative, we can define

$$\begin{aligned} \alpha' &: K_1(\Lambda(W), L) \to \mathbb{Z}/p\mathbb{Z}, \quad 1 + (1-z)\sum_i r_i g_i \mapsto \sum_i \bar{r}_i \\ \alpha'' &: L/[\Lambda(W), L] \to \mathbb{Z}/p\mathbb{Z}, \quad (1-z)\sum_i r_i g_i \mapsto \sum_i \bar{r}_i, \end{aligned}$$

where $r_i \in \mathbb{Z}_p$, $g_i \in W$ and $\bar{r}_i$ is the image of r_i in $\mathbb{Z}/p\mathbb{Z}$. By Vaserstein's identity (cf. [**34**, theorem 1.15]) and the description of J in lemma 3.26, these maps induce the isomorphisms

$$K_1\left(\frac{\Lambda(W)}{LJ}, \frac{L}{LJ}\right) \cong \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad \frac{L}{LJ}\Big/\left[\frac{\Lambda(W)}{LJ}, \frac{L}{LJ}\right] \cong \mathbb{Z}/p\mathbb{Z}.$$

We will now evaluate $\alpha''(\log^L(1+(1-z)rg))$, where $r \in \mathbb{Z}_p$, $g \in W$. Since $(1-z)^p \in pL$, we get for $n > p$

$$\frac{(1-z)^n}{n} \in p(1-z)\Lambda(W)$$

and hence $\alpha''(\frac{(1-z)^n}{n} r^n g^n) = 0$. For $n > 1$, $p \nmid n$, clearly

$$\alpha''(\frac{(1-z)^n}{n} r^n g^n) = 0.$$

It remains to calculate $\alpha''(\frac{(1-z)^p}{p} r^p g^p)$. The set

$$\frac{1}{p}((1-X)^p - 1 + X^p) + (1 - X^p)\mathbb{Z}_p[X] \subset \mathbb{Z}_p[X]$$

is the set of all polynomials such that evaluation for $X = z$ gives $\frac{(1-z)^p}{p}$. The image of

$$\begin{aligned}&\frac{\frac{1}{p}((1-X)^p - 1 + X^p)}{1-X} + \frac{1-X^p}{1-X}\mathbb{Z}_p[X] \\ =&\frac{1}{p}((1-X)^{p-1} - 1 - \cdots - X^{p-1}) + (1 + \cdots + X^{p-1})\mathbb{Z}_p[X]\end{aligned}$$

under the evaluation homomorphism $\mathbb{Z}_p[X] \to \mathbb{Z}/p\mathbb{Z}$, $X \mapsto 1$ is $\{-1\}$. Hence[1]

$$\alpha''(\log^L(1 + (1-z)rg)) = \alpha''((1-z)(rg - r^p g^p)) = \bar{r} - \bar{r}^p = 0.$$

Thus, the diagram (28) is just

$$\begin{tikzcd} & K_1(\Lambda(W), LJ) \arrow[r] \arrow[d, "\log^{LJ}", "\cong"'] & K_1(\Lambda(W), L) \arrow[r] \arrow[d, "\log^L"] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "0"] & 1 \\ 0 \arrow[r] & LJ/[\Lambda(W), LJ] \arrow[r] & L/[\Lambda(W), L] \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Now the snake lemma yields the isomorphisms

$$\ker(\log^L) \cong \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad \operatorname{coker}(\log^L) \cong \mathbb{Z}/p\mathbb{Z}.$$

Since $\alpha'(z) = -1$ and $z \in \ker(\log^L)$ ($L/[\Lambda(W), L]$ is p-torsion free), we have $\ker(\log^L) = \langle z \rangle$. Hence the sequence (26) is exact.

Let

$$p : L/[\Lambda(W), L] \twoheadrightarrow (1-z)\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \subset \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]$$

be the natural projection. For $\xi \in \Lambda(W)$,

$$\log^L(1 + (1-z)\xi) = (1-z)\eta$$

for some $\eta \in \Lambda(W)$. Since $\varphi((1-z)\eta) = (1-z^p)\varphi(\eta) = 0$, we get

$$\mathscr{L}_W(1 + (1-z)\xi) = p \circ \log^L(1 + (1-z)\xi).$$

By (26), the group

$$(1-z)\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]/\mathscr{L}_W(1+L) = (1-z)\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!]/p \circ \log^L(1+L)$$

[1] Note that $(1-z)^2 \nmid \frac{(1-z)^p}{p}$. The above argument does not work for $\frac{1}{p}(1-X)^p \in \mathbb{Q}_p[X]$.

is a quotient of $\mathbb{Z}/p\mathbb{Z}$. It is the trivial group if and only if $\ker p \neq 0$. But this is equivalent to the existence of some $x \in \Lambda(W)$ with

$$x \notin [\Lambda(W), \Lambda(W)] \qquad \text{and} \qquad (1-z)x \in [\Lambda(W), \Lambda(W)].$$

Equivalently, z is a commutator. This proves (27). □

Lemma 3.41. *Let W be a p-group. Then $K_2(\Lambda(W))$ is torsion.*

Proof. By [**26**, theorem 7.2.7], $K_2(\mathbb{Z}[W])$ is finite. By [**26**, theorem 7.2.2], the homomorphism

$$K_2(\mathbb{Z}[W]) \to K_2(\mathbb{Q}[W])$$

induced by the natural inclusion $\mathbb{Z}[W] \hookrightarrow \mathbb{Q}[W]$ has finite kernel and torsion cokernel. Hence $K_2(\mathbb{Q}[W])$ is a torsion group.

By Maschke's theorem, $\mathbb{Q}[W]$ is a semisimple $\mathbb{Q}$-algebra. As it is finite dimensional, it is Artinian. Hence we get the Wedderburn decomposition

$$\mathbb{Q}[W] \cong \prod_{i=1}^{r} M_{n_i}(D_i),$$

where D_i are finite dimensional skew fields over $\mathbb{Q}$. By [**39**, Satz 2], we get that the D_i are actually fields. By tensoring with $\mathbb{Q}_p$ and using the isomorphisms $D_i \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{v|p}(D_i)_v$, where the product is over all primes v of D_i lying over p, we get the isomorphism

$$\mathbb{Q}_p[W] \cong \prod_{i=1}^{r} M_{n_i}(D_i \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cong \prod_{i=1}^{r} \prod_{v|p} M_{n_i}((D_i)_v)$$

of $\mathbb{Q}_p$-algebras.

Using the fact that for two rings R, S, we have $K_2(R \times S) = K_2(R) \oplus K_2(S)$ (cf. [**28**, proposition 12.8]) and Morita invariance, we get

$$K_2(\mathbb{Q}[W]) \cong \bigoplus_{i=1}^{r} K_2(D_i)$$
$$K_2(\mathbb{Q}_p[W]) \cong \bigoplus_{i=1}^{r} \bigoplus_{v|p} K_2((D_i)_v).$$

By [**2**, theorem 5.2], the homomorphism

$$K_2(D_i) \to \bigoplus_{v} K_2((D_i)_v),$$

where the direct sum is over all primes v of D_i, has finite cokernel. Hence $K_2(\mathbb{Q}_p[W])$ is a torsion group.

$\mathbb{Q}_p[W]$ is a semisimple $\mathbb{Q}_p$-algebra that contains the maximal order $\mathbb{Z}_p[W]$. Hence, by [**26**, theorem 7.1.1 (c)]

$$K_2(\mathbb{Z}_p[W]) \hookrightarrow K_2(\mathbb{Q}_p[W])$$

is injective. Therefore, we get that $K_2(\mathbb{Z}_p[W])$ is a torsion group. $\square$

Let A be a semisimple $\mathbb{Q}_p$-algebra and let $\Lambda \subset A$ be a $\mathbb{Z}_p$-order (i. e. a subring Λ of A with $A = \mathbb{Q}_p\Lambda$ which is finitely generated over $\mathbb{Z}_p$). We define

$$SK_1(\Lambda) := \ker(K_1(\Lambda) \to K_1(A)),$$

where the homomorphism on the right hand side is induced by the natural inclusion $\Lambda \subset A$.

THEOREM 3.42 (Wall). *Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$ and let W be a finite group. Then*

$$K_1(\Lambda_{\mathcal{O}}(W))_{tors} = \mu(\mathcal{O}) \times W^{ab} \times SK_1(\Lambda_{\mathcal{O}}(W)),$$

where $\mu(\mathcal{O}) := \mathcal{O}^\times_{tors}$ is the group of roots of unity of $\mathcal{O}$.

PROOF. [**34**, Theorem 7.3] $\square$

LEMMA 3.43. *Let W be a p-adic Lie group. The map*

$$1 - \frac{1}{p}\varphi : \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!] \to \mathbb{Q}_p[\![\mathrm{Conj}(W)]\!]$$

is injective and hence $\ker(\mathscr{L}_W) = \ker(\log)$.

PROOF. There is a descending sequence of subsets

$$\mathrm{Conj}(W) \supset \varphi(\mathrm{Conj}(W)) \supset \varphi^2(\mathrm{Conj}(W)) \supset \dots$$

with $\bigcap_n \varphi^n(\mathrm{Conj}(W)) = 1$. Let x be an element of the kernel of $1 - \frac{1}{p}\varphi$. If $x \notin \mathbb{Q}_p 1_W$, then there is a maximal n such that $x \in \mathbb{Q}_p[\![\varphi^n(\mathrm{Conj}(W))]\!]$. Then

$$\frac{1}{p}\varphi(x) \in \mathbb{Q}_p[\![\varphi^{n+1}(\mathrm{Conj}(W))]\!] \quad \text{and} \quad x \notin \mathbb{Q}_p[\![\varphi^{n+1}(\mathrm{Conj}(W))]\!].$$

This is a contradiction to the identity $x - \frac{1}{p}\varphi(x) = 0$. The restriction of $1 - \frac{1}{p}\varphi(x)$ to $\mathbb{Q}_p 1_W$ is trivially injective. $\square$

PROOF OF THEOREM 3.39. We extend the rather short proof given in [**34**, theorem 6.6]. In particular, we provide a proof for the exactness of the left column of diagram (29).

Firstly, we consider the case $W = 1$. We have $\log(1 + p\mathbb{Z}_p) = p\mathbb{Z}_p$ and hence $\log(K_1(\Lambda(1))) = p\mathbb{Z}_p$. This implies

$$\begin{aligned}\mathscr{L}_1(K_1(\Lambda(1))) &= (1 - \frac{1}{p}\varphi)(\log(K_1(\Lambda(1)))) = (1 - \frac{1}{p})(p\mathbb{Z}_p)\\ &= \mathbb{Z}_p = \ker \omega_1.\end{aligned}$$

Clearly, $\ker \mathscr{L}_1 = \mu_{p-1} = K_1(\Lambda(1))_{tors}$.

We will show, that $\omega_W \circ \mathscr{L}_W = 1$ for a p-group $W \neq 1$. Commutativity of the diagram

$$\begin{tikzcd} K_1(\Lambda(W)) \arrow[r, "\mathscr{L}_W"] \arrow[d] & \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \arrow[r, "\omega_W"] \arrow[d] & W^{ab} \arrow[r] \arrow[d, equal] & 1 \\ K_1(\Lambda(W^{ab})) \arrow[r, "\mathscr{L}_{W^{ab}}"] & \mathbb{Z}_p[\![\mathrm{Conj}(W^{ab})]\!] \arrow[r, "\omega_{W^{ab}}"] & W^{ab} \arrow[r] & 1 \end{tikzcd}$$

implies that we may assume $W = W^{ab}$ without loss of generality.

Let $I = I(W)$ be the augmentation ideal of $\Lambda(W)$. Since

$$K_1(\Lambda(W)) \cong \mu_{p-1} \times \Big(1 + I + p\Lambda(W)\Big)$$

by lemma 3.26, it suffices to show $\mathscr{L}_W(1 + I) \subset \ker \omega_W$ and $\mathscr{L}_W(1 + p\Lambda(W)) \subset \ker \omega_W$. We get the latter inclusion from the inclusion $\log(1 + p\Lambda(W)) \subset p\Lambda(W)$ and the fact that $\omega_W(x) = \omega_W(\frac{1}{p}\varphi(x))$ for $x \in p\Lambda(W)$.

Let $u \in 1 + I$ and write $u = 1 + \sum_{i=1}^n r_i(1 - \tau_i)\sigma_i$ with $r_i \in \mathbb{Z}_p$ and $\sigma_i, \tau_i \in W$. Using the congruence

$$\tau_i^p = [1 - (1 - \tau_i)]^p \equiv 1 - p(1 - \tau_i) - (1 - \tau_i)^p \bmod pI^2$$

we get

$$\begin{aligned}u^p &\equiv 1 + p\sum_i r_i(1 - \tau_i)\sigma_i + \sum_i r_i^p(1 - \tau_i)^p\sigma_i^p \bmod pI^2\\ &\equiv 1 + p\sum_i r_i(1 - \tau_i)\sigma_i + \sum_i r_i(1 - \tau_i)^p\sigma_i^p \bmod pI^2\\ &\equiv 1 + p\sum_i r_i(1 - \tau_i)\sigma_i + \sum_i r_i[(1 - \tau_i^p) - p(1 - \tau_i)]\sigma_i^p\\ &\equiv \underbrace{1 + \sum_i r_i(1 - \tau_i^p)\sigma_i^p}_{=\varphi(u)} + \underbrace{p\sum_i r_i(1 - \tau_i)(\sigma_i - \sigma_i^p)}_{\in pI^2}.\end{aligned}$$

Consequently, $u^p \equiv \varphi(u) \bmod pI^2$. Since pI^2 is an ideal, this is equivalent to $\frac{u^p}{\varphi(u)} \in 1 + pI^2$. Then

$$\begin{aligned}\mathscr{L}_W(u) &= \frac{1}{p}\log(u^p) - \frac{1}{p}\varphi(\log(u)) \\ &= \frac{1}{p}\log(\frac{u^p}{\varphi(u)}) \in I^2.\end{aligned}$$

On the other hand, we get for $r \in \mathbb{Z}_p$, $\tau_1, \tau_2, \tau_3 \in W$

$$\begin{aligned}\omega_W(r(1-\tau_1)(1-\tau_2)\tau_3) &= \omega_W(r(\tau_3 - \tau_1\tau_3 - \tau_2\tau_3 + \tau_1\tau_2\tau_3)) \\ &= \tau_3^r(\tau_1\tau_3)^{-r}(\tau_2\tau_3)^{-r}(\tau_1\tau_2\tau_3)^r = 1 \in W.\end{aligned}$$

Hence $I^2 \subset \ker \omega_W$ and $\mathscr{L}_W(1+I) \subset \ker \omega_W$.

Let $z \in Z(W)$ be an element of order p which is a commutator if W is not abelian. (The existence of z follows from theorem [**34**, prop. 6.5].) Define $\hat{W} := W/\langle z \rangle$.

We will prove the theorem by induction on the order of W. For $W = 1$, we have already shown everything. Now, we assume that we have already proven the theorem for $\hat{W}$. Let

$$\alpha : W \to \hat{W}$$

be the natural projection. For an abelian group K define $K/tors = K/K_{tors}$. Put

$$\begin{aligned} L &:= (1-z)\Lambda(W) \subset \Lambda(W) \quad \text{and} \\ L_{conj} &:= (1-z)\mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \subset \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!].\end{aligned}$$

We consider the following commutative diagram (where the maps $\mathscr{L}_0$ and ω_0 are induced by $\mathscr{L}_W$ and ω_W, respectively):

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & K_1(\Lambda(W), L)/tors \arrow[r, "\mathscr{L}_0"] \arrow[d] & L_{conj} \arrow[r, "\omega_0"] \arrow[d] & \ker \alpha^{ab} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_1(\Lambda(W))/tors \arrow[r, "\mathscr{L}_W"] \arrow[d, "K(\alpha)"] & \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \arrow[r, "\omega_W"] \arrow[d, "H(\alpha)"] & W^{ab} \arrow[r] \arrow[d, "\alpha^{ab}"] & 1 \\
1 \arrow[r] & K_1(\Lambda(\hat{W}))/tors \arrow[r, "\mathscr{L}_{\hat{W}}"] \arrow[d] & \mathbb{Z}_p[\![\mathrm{Conj}(\hat{W})]\!] \arrow[r, "\omega_{\hat{W}}"] \arrow[d] & \hat{W}^{ab} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{29}$$

We show that the left column is exact. First note that by [**3**, corollary III.2.9], $GL(\Lambda(W)) \to GL(\Lambda(\hat{W}))$ is surjective and hence induces the surjections

$$K_1(\Lambda(W)) \twoheadrightarrow K_1(\Lambda(\hat{W})) \qquad \text{and}$$
$$K_1(\Lambda(W))/tors \twoheadrightarrow K_1(\Lambda(\hat{W}))/tors.$$

If W is noncommutative, the kernel of $W \to \hat{W}$ is generated by a commutator and hence [**35**, lemma 14] implies that

$$SK_1(\Lambda(W)) \to SK_1(\Lambda(\hat{W}))$$

is surjective. Since $SK_1(\Lambda(W)) = 1$ if W is commutative (cf. [**10**, proposition 45.12]), this map is also surjective for commutative groups W. Hence by Wall's theorem (cf. theorem 3.42), the upper row in the commutative diagram

$$\begin{tikzcd}
 & K_1(\Lambda(W),L)_{tors} \arrow[r] \arrow[d] & K_1(\Lambda(W))_{tors} \arrow[r] \arrow[d, hook] & K_1(\Lambda(\hat{W}))_{tors} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & K_1(\Lambda(W),L)/K_2 \arrow[r] & K_1(\Lambda(W)) \arrow[r] & K_1(\Lambda(\hat{W})) \arrow[r] & 1,
\end{tikzcd}$$

is exact. The lower row, where K_2 denotes the image of $K_2(\Lambda(\hat{W}))$ in $K_1(\Lambda(W), L)$, is also exact. Since $K_2(\Lambda(\hat{W}))$ is torsion by lemma 3.41, the snake lemma yields the exact sequence

$$1 \to K_1(\Lambda(W), L)/tors \to K_1(\Lambda(W))/tors \to K_1(\Lambda(\hat{W}))/tors \to 1.$$

The right column in the diagram (29) is exact by definition of α^{ab}. The exactness of the middle column follows from the exactness of the sequence

$$1 \to L \to \Lambda(W) \to \Lambda(\hat{W}) \to 1.$$

Obviously, ω_0 is surjective. By proposition 3.40, $\mathscr{L}_0$ is injective. We have already shown that $\operatorname{im}(\mathscr{L}_0) \subset \ker(\omega_0)$. By proposition 3.40, we get that

$$\#\operatorname{coker}\mathscr{L}_0 = \left\{ \begin{array}{ll} 1 & \text{if } z \text{ is a commutator} \\ p & \text{otherwise} \end{array} \right\} = \#\ker\alpha^{ab},$$

i. e. the upper row is exact. By our induction hypothesis, the lower row is exact. From $\omega_W \circ \mathscr{L}_W = 1$ and the 3×3 lemma[2] (cf. [**49**, Exercise 1.3.2]), we deduce that the middle row is exact. $\square$

Corollary 3.44. *Let W be a pro-p p-adic Lie group with*

$$SK_1(\mathbb{Z}_p[W/U]) = 1$$

for every open normal subgroup U of W. Then the sequence

$$1 \to \mu_{p-1} \times W^{ab} \to K_1(\Lambda(W)) \xrightarrow{\mathscr{L}_W} \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \xrightarrow{\omega_W} W^{ab} \to 1$$

is exact.

Proof. Write $W = \varprojlim_U W/U$ where the limit is over all open normal subgroups U of W. Then

$$W^{ab} = W/[W, W] = \varprojlim_U (W/U)/[W/U, W/U] = \varprojlim_U (W/U)^{ab}.$$

Since $SK_1(\mathbb{Z}_p[W/U]) = 1$, Wall's theorem (cf. theorem 3.42) implies

$$K_1(\Lambda(W/U))_{tors} = \mu_{p-1} \times (W/U)^{ab}.$$

Since the inverse limit functor is left exact, we get from lemma 3.38 and theorem 3.39 the isomorphism

$$\ker \mathscr{L}_W = \varprojlim_U \ker \mathscr{L}_{W/U} \cong \mu_{p-1} \times W^{ab}.$$

Theorem 3.39 implies that the sequences

$$1 \to K_1(\Lambda(W/U))/tors \xrightarrow{\mathscr{L}_{W/U}} \mathbb{Z}_p[\![\mathrm{Conj}(W/U)]\!] \xrightarrow{\omega_{W/U}} (W/U)^{ab} \to 1$$

are exact for all open normal subgroups U of W. Since $\mathbb{Z}_p[\![\mathrm{Conj}(W/U)]\!]$ is compact for all U, we get the short exact sequence

$$1 \to \varprojlim_U K_1(\Lambda(W/U))/tors \xrightarrow{\mathscr{L}_W} \mathbb{Z}_p[\![\mathrm{Conj}(W)]\!] \xrightarrow{\omega} W^{ab} \to 1$$

(cf. [**51**, lem. 15.16]). $\square$

[2] The 3×3 lemma says that if in the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

in an abelian category, all columns, the top row and the bottom row are exact and the composition $A_2 \to B_2 \to C_2$ is zero, then the middle row is exact.

COROLLARY 3.45. *Let P be a pro-p p-adic Lie group that satisfies assumption 3.1. Then, the sequence*

$$1 \to \mu_{p-1} \times P^{ab} \to K_1(\Lambda(P)) \xrightarrow{\mathscr{L}_P} \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \xrightarrow{\omega_P} P^{ab} \to 1$$

is exact.

PROOF. For every open subgroup U of P, $[P/U, P/U]$ is the central cyclic group generated by the image of γ in P/U. Then a theorem of R. Oliver (cf. [**34**, theorem 8.10 (ii)]) implies $SK_1(\mathbb{Z}_p[P/U]) = 1$. $\square$

5. The Multiplicative Homomorphism θ

Let P be a group that satisfies assumption 3.1. In this subsection, we will define subsets $\Psi \subset \prod_n \Lambda(U_n/V_n)^\times$ and $\Psi_S \subset \prod_n \Lambda(U_n/V_n)_S^\times$ and show that $\operatorname{im}\theta = \Psi$ and $\operatorname{im}\theta_S \subset \Psi_S$ for the homomorphisms $\theta : K_1(\Lambda(P)) \to \prod_n \Lambda(U_n/V_n)^\times$ and $\theta_S : K_1(\Lambda(P)_S) \to \prod_n \Lambda(U_n/V_n)_S^\times$.

We use the relation $\mathrm{Tr} \circ \log = \log \circ \mathrm{N}$ to prove the following relations of the integral logarithm with the homomorphisms τ and θ for all $n \geq 1$:

$$\tau_n(\mathscr{L}(x)) = \log(\theta_n(x)\varphi(\theta_{n-1}(x))^{-1}) \quad \text{for all } x \in K_1(\Lambda(P))$$

$$\tau_{n,S}(\mathscr{L}_S(x)) = \log(\theta_{n,S}(x)\varphi(\theta_{n-1,S}(x))^{-1}) \quad \text{for all } x \in K_1(\widehat{\Lambda(P)_S}).$$

To prove the inclusions

$$\operatorname{im}\theta \subset \Psi \qquad \operatorname{im}\theta_S \subset \Psi_S,$$

we need the facts that $1 + I_n$ is a group and

$$\log : 1 + I_n \to I_n$$

is an isomorphism. We show that θ surjects onto Ψ. As we have already pointed out, we do this by proving that diagram (12) is commutative with exact rows.

Define $\theta_n : K_1(\Lambda(P)) \to \Lambda(U_n/V_n)^\times$ to be the composition of

$$N : K_1(\Lambda(P)) \to K_1(\Lambda(U_n)) \qquad \text{and}$$
$$p_* : K_1(\Lambda(U_n)) \to K_1(\Lambda(U_n/V_n)) = \Lambda(U_n/V_n)^\times$$

For $m \leq n$, $m, n \in \underline{\underline{c}}$, let $\mathrm{N}_{m,n} : \Lambda(U_m/V_m)^\times \to \Lambda(U_n/V_m)^\times$ be the norm map and $p_{n,m} : \Lambda(U_n/V_n)^\times \to \Lambda(U_n/V_m)^\times$ be the projection map. For $n \geq 1$, define

$$\varphi : U_{n-1}/V_{n-1} \to U_n/V_n, \quad \sigma \mapsto \sigma^p.$$

From

$$(\sigma\tau)^p = \sigma^p\tau^p[\tau,\sigma]^{\frac{1}{2}p(p-1)} = \sigma^p\tau^p \in U_n/V_n \quad \text{ for } \sigma, \tau \in U_{n-1}/V_{n-1}$$

we get that φ is a continuous group homomorphisms. Let

$$\varphi : \Lambda(U_{n-1}/V_{n-1}) \to \Lambda(U_n/V_n)$$

be the continuous ring homomorphism induced by the group homomorphism φ. We define

$$\begin{aligned}\Psi := \{(x_n) \in \prod_{n\in\underline{\underline{c}}} &\Lambda(U_n/V_n)^\times \mid \\ &(i) \quad \mathrm{N}_{m,n}(x_m) = p_{n,m}(x_n) \text{ for } m \le n,\ m,n \in \underline{\underline{c}} \\ &(ii) \quad x_n\varphi(x_{n-1})^{-1} \in 1 + I_n \text{ for } n \ge 1,\ n \in \underline{\underline{c}}\}.\end{aligned}$$

For $n \ge m, n, m \in \underline{\underline{c}}$, let $\mathrm{N}_{m,n} : \Lambda(U_m/V_m)_S^\times \to \Lambda(U_n/V_m)_S^\times$ be the norm map and $p_{n,m} : \Lambda(U_n/V_n)_S^\times \to \Lambda(U_n/V_m)_S^\times$ be the projection map.

By corollary 2.6, the group homomorphism φ induces the continuous ring homomorphism

$$\varphi : \Lambda(U_{n-1}/V_{n-1})_S \to \Lambda(U_n/V_n)_S.$$

We define

$$\begin{aligned}\Psi_S := \{(x_n)_n \in \prod_{n\in\underline{\underline{c}}} &\Lambda(U_n/V_n)_S^\times \mid \\ &(i) \quad \mathrm{N}_{m,n}(x_m) = p_{n,m}(x_n) \text{ for } m \le n,\ m,n \in \underline{\underline{c}} \\ &(ii) \quad x_n\varphi(x_{n-1})^{-1} \in 1 + I_{n,S} \text{ for } n \ge 1,\ n \in \underline{\underline{c}}\}.\end{aligned}$$

Define $\theta_{n,S} : K_1(\Lambda(P)_S) \to \Lambda(U_n/V_n)_S^\times$ to be the composition of

$$\begin{aligned}&\mathrm{N} : K_1(\Lambda(P)_S) \to K_1(\Lambda(U_n)_S) \quad \text{and} \\ &p_* : K_1(\Lambda(U_n)_S) \to K_1(\Lambda(U_n/V_n)_S) \cong \Lambda(U_n/V_n)_S^\times.\end{aligned}$$

Define

$$\theta_S : K_1(\Lambda(P)_S) \to \prod_{n\ge 0} \Lambda(U_n/V_n)_S^\times, \quad x \mapsto (\theta_{n,S}(x))_{n\ge 0}.$$

Lemma 3.46. *The map $\varphi^* : P \to \Psi, \quad g \mapsto (\varphi^n(g))_{n\ge 0}$ is a well-defined multiplicative map with $\varphi^*(g) = \theta([g]_{\Lambda(P)})$ for all $g \in P$.*

Proof. Let $(1, \beta^{p^m}, \ldots, \beta^{(p^{n-m}-1)p^m})$ be a basis of $\Lambda(U_m/V_m)$ over $\Lambda(U_n/V_m)$. Then

$$\mathrm{N}_{m,n}(\beta^{p^m}) = \det \begin{pmatrix} 0 & & & \beta^{p^n} \\ 1 & \ddots & & 0 \\ & \ddots & 0 & \vdots \\ & & 1 & 0 \end{pmatrix} = (-1)^{p^{n-m}-1}\beta^{p^n} = \beta^{p^n}.$$

From $p_{n,m}(\beta^{p^m}) = \beta^{p^m}$ and $\beta^{p^n} = \varphi(\beta^{p^{n-1}})$, we get that $\varphi^*(\beta) = (\beta^{p^n})_{n \in \underline{\underline{c}}} \in \Psi$. Obviously, $\varphi^*(g) \in \Psi$ for $g \in U_\infty$ and hence

$$\varphi^*(g) = (g^{p^n})_n = (\theta_n([g]))_n$$

for all $g \in P$ and $\varphi^*(P) \subset \Psi$. $\square$

The main result of this chapter is the following theorem:

THEOREM 3.47. *The homomorphisms θ_n, $n \in \mathbb{N}$ ($0 \le n \le c$ in case γ is of finite order p^c) induce an isomorphism*

$$\theta : K_1(\Lambda(G)) \xrightarrow{\cong} \Psi, \quad x \mapsto (\theta_n(x))_n.$$

REMARKS.
- If G is abelian, $K_1(\Lambda(G)) = \Lambda(G)^\times = \Psi$ (note $c = 0$ and $V_0 = 1$) and
$$\theta = \mathrm{id} : \Lambda(G)^\times \to \Lambda(G)^\times$$
is the identity isomorphism.
- It is not obvious that Ψ and Ψ_S are groups ($I_n \subset \Lambda(U_n/V_n)$ and $I_{n,S} \subset \Lambda(U_n/V_n)_S$ are not ideals, generally!) and that $\operatorname{im}\theta \subset \Psi$, $\operatorname{im}\theta_S \subset \Psi_S$.

LEMMA 3.48. *Let W_1 be a p-adic Lie group and let W_2 be an open subgroup of index n of W_1. Assume that W_2 is commutative. Let*

$$\mathrm{Tr} : \mathbb{Q}_p[\![\mathrm{Conj}(W_1)]\!] \to \mathbb{Q}_p[\![W_2]\!]$$

be the trace homomorphism induced by $\mathrm{Tr} : \mathbb{Q}_p[\![W_1]\!] \to \mathbb{Q}_p[\![W_2]\!]$. *Then the following diagram is commutative:*

$$\begin{tikzcd} K_1(\Lambda(W_1)) \arrow[r, "\log"] \arrow[d, "\mathrm{N}"] & \mathbb{Q}_p[\![\mathrm{Conj}(W_1)]\!] \arrow[d, "\mathrm{Tr}"] \\ \Lambda(W_2)^\times \arrow[r, "\log"] & \mathbb{Q}_p[\![W_2]\!]. \end{tikzcd}$$

In particular, the following diagram is commutative for any $n \in \mathbb{N}$:

$$\begin{tikzcd} K_1(\Lambda(P)) \arrow[r, "\log"] \arrow[d, "\theta_n"] & \mathbb{Q}_p[\![\mathrm{Conj}(P)]\!] \arrow[d, "\tau_n"] \\ \Lambda(U_n/V_n)^\times \arrow[r, "\log"] & \mathbb{Q}_p[\![U_n/V_n]\!]. \end{tikzcd} \tag{30}$$

PROOF. We use parts of the proof of [**34**, theorem 6.2].

We assume first that W_1 is a finite group. Since W_2 is commutative, we have the description (3) of the norm map. Hence

$$\mathrm{N}(1 + p^n x) \equiv 1 + p^n \mathrm{Tr}(x) \bmod p^{2n-1}$$

for $x \in \Lambda(W_1)$ and $n > 0$. Thus

$$\begin{aligned}\log(\mathrm{N}(1+p^n x)) &\equiv \log(1+p^n \mathrm{Tr}(x)) \\ &\equiv p^n \mathrm{Tr}(x) \equiv \mathrm{Tr}(\log(1+p^n x)) \bmod p^{2n-1}\end{aligned}$$

Let $u \in 1 + J(\Lambda(W_1))$. Then the image of u in $\mathbb{F}_p[\![W_1]\!]$ is of finite order and hence there is $k \in \mathbb{N}$ such that $u^{p^k} \in 1 + p\Lambda(W_1)$. Then $u^{p^{k+n}} \in 1 + p^{n+1}\Lambda(W_1)$ for all $n > 0$. Thus we get for $n \geq k$

$$\begin{aligned}\log(\mathrm{N}(u)) &= p^{-n-k} \log\left(\mathrm{Tr}(u^{p^{k+n}})\right) \\ &\equiv p^{-n-k}\mathrm{Tr}\left(\log(u^{p^{k+n}})\right) \bmod \frac{p^{2(n+1)-1}}{p^{n+k}} = p^{n-k+1} \\ &= \mathrm{Tr}(\log(u))\end{aligned}$$

Since this holds for any $n \geq k$, we get the equality

$$\log(\mathrm{N}(u)) = \mathrm{Tr}(\log(u)).$$

We prove this identity for not necessarily finite groups W_1 by passing to the inverse limit over all finite quotients of W_1 and W_2.

From the commutativity of the diagram

$$\begin{tikzcd} \Lambda(P) \arrow[r] \arrow[d] & M_n(\Lambda(U_n)) \arrow[d] \\ \Lambda(P/V_n) \arrow[r] & M_n(\Lambda(U_n/V_n)), \end{tikzcd}$$

we get that the diagram

$$\begin{tikzcd} K_1(\Lambda(P)) \arrow[r, "\mathrm{N}"] \arrow[d, "p_*"] & K_1(\Lambda(U_n)) \arrow[d, "p_*"] \\ K_1(\Lambda(P/V_n)) \arrow[r, "\mathrm{N}"] & K_1(\Lambda(U_n/V_n)) \end{tikzcd} \tag{31}$$

is commutative. Recall that by lemma 3.20, τ_n can be written as the composition

$$\tau_n : \mathbb{Q}_p[\![\mathrm{Conj}(P)]\!] \xrightarrow{\pi} \mathbb{Q}_p[\![\mathrm{Conj}(P/V_n)]\!] \xrightarrow{\mathrm{Tr}} \mathbb{Q}_p[\![U_n/V_n]\!].$$

We have already shown that the diagram

$$\begin{tikzcd} K_1(\Lambda(P/V_n)) \arrow[r, "\log"] \arrow[d, "\mathrm{N}"] & \mathbb{Q}_p[\![\mathrm{Conj}(P/V_n)]\!] \arrow[d, "\mathrm{Tr}"] \\ \Lambda(U_n/V_n)^\times \arrow[r, "\log"] & \mathbb{Q}_p[\![U_n/V_n]\!] \end{tikzcd}$$

commutes. Since $p_*(\log(x)) = \log(p_*(x))$, this implies the commutativity of (30). $\square$

LEMMA 3.49. *For $m, n \in \underline{\underline{c}}$, $m < n$, There are the following commutative diagrams of $\mathbb{Z}_p$-modules:*

$$\begin{tikzcd}
\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \arrow[r, "\varphi"] \arrow[d, "p\cdot\tau_{n-1}"] & \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \arrow[d, "\tau_n"] \\
\Lambda(U_{n-1}/V_{n-1}) \arrow[r, "\varphi"] & \Lambda(U_n/V_n)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda(U_{m-1}/V_{m-1}) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Tr}_{m-1,n-1}"] & \Lambda(U_m/V_m) \arrow[d, "\mathrm{Tr}_{m,n}"] \\
\Lambda(U_{n-1}/V_{m-1}) \arrow[r, "\varphi"] & \Lambda(U_n/V_m)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^\wedge_S \arrow[r, "\varphi"] \arrow[d, "p\cdot\tau_{n-1,S}"] & \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!]^\wedge_S \arrow[d, "\tau_{n,S}"] \\
(\Lambda(U_{n-1}/V_{n-1})_S)^\wedge \arrow[r, "\varphi"] & (\Lambda(U_n/V_n)_S)^\wedge
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda(U_{m-1}/V_{m-1})_S \arrow[r, "\varphi"] \arrow[d, "\mathrm{Tr}_{m-1,n-1}"] & \Lambda(U_m/V_m)_S \arrow[d, "\mathrm{Tr}_{m,n}"] \\
\Lambda(U_{n-1}/V_{m-1})_S \arrow[r, "\varphi"] & \Lambda(U_n/V_m)_S
\end{tikzcd}$$

PROOF. We have

$$\beta^{p^{n-1}}\sigma^p\beta^{-p^{n-1}} = \sigma^p \in \Lambda(U_n/V_n)$$

for $\sigma \in U_n$ since $[\alpha^p, \beta^{p^{n-1}}] \in V_n$. Thus for $\sigma \in U_{n-1}$,

$$\begin{aligned}
\tau_n \circ \varphi(\mathrm{class}(\sigma)) &= \sum_{i=0}^{p^n-1} \mathrm{class}(\beta^i\sigma^p\beta^{-i}) = p\sum_{i=0}^{p^{n-1}-1} \mathrm{class}(\beta^i\sigma^p\beta^{-i}) \\
&= p\cdot\varphi\left(\sum_{i=0}^{p^{n-1}-1} \mathrm{class}(\beta^i\sigma\beta^{-i})\right) \\
&= p\cdot\varphi\circ\tau_{n-1}(\mathrm{class}(\sigma)).
\end{aligned}$$

For $\sigma \in P \setminus U_{n-1}$, clearly

$$\tau_n \circ \varphi(\mathrm{class}(\sigma)) = 0 = p\cdot\varphi\circ\tau_{n-1}(\mathrm{class}(\sigma)).$$

Since τ_n, τ_{n-1} and φ are $\mathbb{Z}_p$-linear and continuous, we have proven the commutativity of the top left diagram. Since τ_S is $\Lambda(Z(P))_S$-linear for groups P with open centre, this also implies the commutativity of the bottom left diagram.

Since

$$\varphi : \Lambda(U_{m-1}/V_{m-1}) \to \Lambda(U_m/V_m)$$

sends the $\Lambda(U_{n-1}/V_{m-1})$-basis $(1, \beta^{p^{m-1}}, \ldots, \beta^{p^{n-1}-1})$ of the domain to the $\Lambda(U_n/V_m)$-basis $(1, \beta^{p^m}, \ldots, \beta^{p^m-1})$ of the codomain, the top right diagram is commutative. Using the same argument, we get the commutativity of the bottom right diagram. $\square$

LEMMA 3.50. *For $n \in \underline{\underline{c}}$, $n \geq 1$, we have*

$$\tau_n(\mathscr{L}_P(x)) = \log(\theta_n(x)\varphi(\theta_{n-1}(x))^{-1}) \quad \textit{for all } x \in K_1(\Lambda(P)),$$

$$\tau_{n,S}(\mathscr{L}_{P,S}(x)) = \log(\theta_{n,S}(x)\varphi(\theta_{n-1,S}(x))^{-1}) \quad \textit{for all } x \in K_1(\widehat{\Lambda(P)_S}).$$

PROOF. We first prove the second equation. Since φ is a continuous ring homomorphism,

$$\log \varphi(1-y) = -\sum_{i\geq 1} \frac{\varphi(y)^i}{i} = \varphi(\log(1-y))$$

for $y \in J\left((\Lambda(U_n/V_n))_S)^\wedge\right)$.

For $x \in \widehat{\Lambda(P)_S}^\times$, we write $x = uv$ with $u \in \left((\Lambda(Z(P))_S)^\wedge\right)^\times$ and $v \in \Lambda(P)^\times$ (cf. lemma 3.36). Note that $Z(P) \subset U_n$ and hence (by the remark after the definition of the norm map)

$$\theta_{n,S}(u) = \bar{u}^{p^n} \in \left((\Lambda(U_n/V_n)_S)^\wedge\right)^\times .$$

Using lemma 3.48 and lemma 3.49, we get

$$\begin{aligned}
&\log(\theta_{n,S}([x])\varphi(\theta_{n-1,S}([x]))^{-1}) \\
=&\log(\bar{u}^{p^n}\varphi(\bar{u}^{p^{n-1}})^{-1}) + \log\theta_{n,S}([v]) - \log\varphi\theta_{n-1,S}([v]) \\
=&\frac{1}{p}\log\theta_{n,S}((\bar{u}^p\varphi(\bar{u})^{-1})) + \tau_{n,S}\log([v]) - \varphi\tau_{n-1,S}\log([v]) \\
=&\tau_{n,S}\circ\frac{1}{p}\log(\bar{u}^p\varphi(\bar{u})^{-1}) + \tau_{n,S}\log([v]) - \tau_{n,S}\circ\frac{1}{p}\varphi\circ\log([v]) \\
=&\tau_{n,S}\circ\mathscr{L}_{P,S}([u]) + \tau_{n,S}\circ(1-\frac{1}{p}\varphi)\circ\log([v]) \\
=&\tau_{n,S}\circ\mathscr{L}_{P,S}([x]).
\end{aligned}$$

A similar argument proves the first part of the lemma. $\square$

PROPOSITION 3.51. *Let $n \in \underline{\underline{c}}$, $n \geq 1$ be an integer. Then*

(1) *$1 + I_n$ is a multiplicative group.*
(2) *The logarithm induces an isomorphism*

$$\log : 1 + I_n \xrightarrow{\cong} I_n.$$

Our proof of the proposition uses K. Kato's sketch in [**25**].

We will first provide some lemmata, which we require for the proof of proposition 3.51. We recall the definition of $h_{n,i}$ before lemma 3.19.

LEMMA 3.52. *If $0 \leq i \leq j \leq n$,*

$$h_{n,i}h_{n,j} = p^{n-j}h_{n,i}.$$

PROOF. Define

$$h_{n,s}(T) := (T^{p^n} - 1)(T^{p^s} - 1)^{-1} = \sum_{t=0}^{p^{n-s}-1} T^{p^s t} \in \mathbb{Z}[T]$$

for $0 \leq s \leq n$. Since $i \leq j$, we have $(T^{p^i} - 1) | (T^{p^j t} - 1)$ for any $t > 0$, and hence

$$h_{n,j}(T) - p^{n-j} = \sum_{t=0}^{p^{n-j}-1} (T^{p^j t} - 1) \in (T^{p^i} - 1).$$

Obviously, $h_{n,i}(T) \in (\frac{T^{p^n}-1}{T^{p^i}-1})$. Therefore,

$$h_{n,i}(T) h_{n,j}(T) - p^{n-j} h_{n,i}(T) \in (T^{p^n} - 1).$$

Substituting γ for T (and noting $\gamma^{p^n} \in V_n$), we get $h_{n,i} h_{n,j} = p^{n-j} h_{n,i}$. □

LEMMA 3.53. *For $i \leq j \leq n$, we have $h_{n,j} | h_{n,i}$ in $\Lambda(U_n/V_n)$.*

PROOF.

$$\begin{aligned} h_{n,i} = \sum_{k=0}^{p^{n-i}-1} \gamma^{p^i k} &= \sum_{r=0}^{p^{n-j}-1} \sum_{s=0}^{p^{j-i}-1} \gamma^{p^i (r p^{j-i} + s)} \\ &= h_{n,j} \sum_{s=0}^{p^{j-i}-1} \gamma^{p^i s} \end{aligned}$$

□

LEMMA 3.54. *Let G and H be profinite groups with normal subgroups $G = G_1 \supset G_2 \supset \dots$ and $H = H_1 \supset H_2 \supset \dots$ such that $\bigcap_{i \geq 1} G_i = 1$, $\bigcap_{i \geq 1} H_i = 1$. (Note that this implies $G = \varprojlim_n G/G_n$ and $H = \varprojlim_n H/H_n$, see* [**36**, corollary 1.1.8]*.) Let*

$$\varphi : G \to H$$

be a continuous group homomorphism such that $\varphi(G_i) \subset H_i$ for all $i \geq 1$, and the induced maps

$$\varphi_i : G_i/G_{i+1} \to H_i/H_{i+1} \quad \forall i \geq 1$$

are isomorphisms. Then φ is an isomorphism.

PROOF. Let $x \in \ker \varphi$, $x \neq 1$ and $i \geq 1$ such that $x \in G_i$, $x \notin G_{i+1}$. Then $\varphi_i(x) = 1 \in H_i/H_{i+1}$ and hence $x = 1 \in G_i/G_{i+1}$, contradicting $x \notin G_{i+1}$.

Let $x = x_1 \in H$. We inductively define sequences $(x_k)_{k \geq 1}$, $x_k \in H_k$ and $(y_k)_{k \geq 1}$, $y_k \in G_k$ such that

- $x_k \varphi(y_k)^{-1} \in H_{k+1}$ (The existence of y_k for a given x_k is clear since φ_k is bijective.)
- $x_{k+1} = x_k \varphi(y_k)^{-1}$.

Then $\varphi(y_k \cdots y_1) = x_{k+1}^{-1} x_k \cdots x_2^{-1} x_1 = x_{k+1}^{-1} x_1$. Since $(x_k)_{k \geq 1}$ and $(y_k)_{k \geq 1}$ converge to 1, $y := \lim_{k \to \infty} y_k \cdots y_1$ exists and $\varphi(y) = x$. □

Proof of proposition 3.51. (1) For $i \leq j$, let $x_1 \in U_{n,i}$, $x_1 \notin U_{n,i+1}$ if $i < n$, $x_2 \in U_{n,j}$, $x_2 \notin U_{n,j+1}$ if $j < n$. From lemma 3.52 and lemma 3.53, we get

$$p^i h_{n,i} x_1 \cdot p^j h_{n,j} x_2 = p^n \cdot p^i h_{n,i} x_1 x_2 \in I_n$$

and hence $I_n I_n \subset I_n$. Thus

$$(1 + I_n)(1 + I_n) = 1 + 2I_n + I_n^2 \subset 1 + I_n,$$

i. e. $1 + I_n$ is multiplicatively closed. For $x \in I_n$, we have

$$(1 - x)^{-1} = \sum_{i \geq 0} x^i \in 1 + I_n,$$

i. e. every element of $1+I_n$ is invertible. So $1+I_n$ is a subgroup of $\Lambda(U_n/V_n)^\times$.

(2) By corollary 3.32, log is well-defined on $1+I_n$ and $\log(1+I_n) \subset \mathbb{Q}_p[\![U_n/V_n]\!]$. For $x \in I_n$, $n \geq 1$, we need to show $x^k/k \in I_n$ for all $k \geq 1$, or equivalently

$$x^k \in p^{v_p(k)} I_n \quad \text{for all } k \geq 1.$$

For $x \in I_n$, x^k is a $\Lambda(U_{n,n}/V_n)$-linear combination of elements of the form

$$\prod_{r=1}^{k} p^{i_r} h_{n,i_r} x_r, \quad x_r \in U_{n,i_r}, \quad x_r \notin U_{n,i_r+1} \text{ if } i_r < n.$$

We may assume $i_0 \leq i_1 \leq \ldots \leq i_k$. Then, by lemma 3.52,

$$\prod_{r=1}^{k} p^{i_r} h_{n,i_r} x_r = p^{(k-1)n} \cdot p^{i_0} h_{n,i_0} x$$

with $x := \prod_{r=1}^{k} x_r \in U_{n,i_0}$. Thus

$$x^k \in p^{(k-2)n} I_n \subset p^{v_p(k)} I_n.$$

Let

$$\log_n : (1 + I_n^i)/(1 + I_n^{i+1}) \to I_n^i / I_n^{i+1}$$

be the homomorphism induced by log. We get

$$\log_n(1 - x) = -x - x^2 \sum_{k \geq 2} \frac{x^{k-2}}{k} = -x.$$

Hence the maps $\log_n$ are isomorphisms for all $n \geq 1$. By lemma 3.54, log is an isomorphism. $\square$

We will denote the inverse map of log by

$$\exp : I_n \to 1 + I_n.$$

Lemma 3.55. *Let $I''_n = \langle h_{n,i} x_1,\, p x_2 |\, x_1 \in U_{n,i}/V_n,\, x_1 \notin U_{n,i+1}/V_n,\, 0 \leq i < n,\, x_2 \in U_{n,n}/V_n \rangle$ as $\Lambda(U_{n,n}/V_n)$-submodule of $\Lambda(U_n/V_n)$. Then we have:*

(1) $I_n \subset I''_n$
(2) $I''_n I''_n \subset I''_n$
(3) $1 + I''_n$ *is a group.*
(4) $I''_n I_n \subset I_n$

PROOF. (1) This is obvious.

(2) This follows from the fact that the elements

$$h_{n,i}h_{n,j} = p^{n-j}h_{n,i} \quad (\text{for } i \leq j), \qquad ph_{n,j}, \qquad p^2 \quad (\text{for } i = n)$$

are divisible by p and $h_{n,k}$ for $i \leq k < n$.

(3) The inclusion (2) shows that $1 + I''_n$ is multiplicatively closed. Since $(1-x)^{-1} = \sum_{i\geq 0} x^i \in 1 + I''_n$ for $x \in I''_n$, we get that $1 + I''_n$ is a subgroup of $\Lambda(U_n/V_n)^\times$.

(4) Let $h_{n,j}x_1$, px_2 be generators of I''_n and $p^i h_{n,i}y$ be a generator of I_n (terminology as in the definition of I''_n and I_n). Then, using lemma 3.52, we get for $i \leq j$:

$$(h_{n,j}x_1)(p^i h_{n,i}y) = p^{n-j} \cdot p^i h_{n,i}x_1 y \in I_n.$$

For $i > j$, we have:

$$(h_{n,j}x_1)(p^i h_{n,i}y) = p^{n-i} \cdot p^i h_{n,j}x_1 y \in I_n.$$

Obviously, $(px_2)(p^i h_{n,i}y) \in I_n$. Therefore, $I''_n I_n \subset I_n$. □

The following lemma will be needed in the next chapter, where we calculate certain congruences for zeta functions.

LEMMA 3.56. *Let $(x_n) \in \prod_{n\geq 0} \Lambda(U_n/V_n)^\times$ be an element that satisfies condition (i) in the definition of Ψ. Then $x_n \equiv \varphi(x_{n-1}) \bmod I_n$ for all $n \geq 1$ if and only if $x_n\varphi(x_{n-1})^{-1} \in 1 + I_n$ for all $n \geq 1$.*

PROOF. "$\Rightarrow$" Since $\varphi(h_{k,i}) = \sum_{j=0}^{p^{k-i}-1} \gamma^{p^{i+1}j} = h_{k+1,i+1}$ for all $k \in \underline{\underline{c}}$ with $k \geq 1$, $i \leq k$, it follows that $\varphi(I''_k) \subset I''_{k+1}$. Since $x_k \equiv \varphi(x_{k-1}) \bmod I''_k$ for all $k \geq 1$ (cf. lemma 3.55), we get

$$x_n - \varphi^n(x_0) = \sum_{k=1}^{n} \varphi^{n-k}(x_k - \varphi(x_{k-1})) \in I''_n.$$

Since $\varphi^n(x_0) \in \Lambda(U_{n,n}/V_n)$ and since $1+I''_n$ is a group (cf. lemma 3.55), we have the equivalences

$$\begin{aligned} & x_n \equiv \varphi^n(x_0) \bmod I''_n \\ \Leftrightarrow\quad & x_n\varphi^n(x_0^{-1}) \in 1 + I''_n \\ \Leftrightarrow\quad & x_n^{-1}\varphi^n(x_0) \in 1 + I''_n \\ \Leftrightarrow\quad & x_n^{-1} \equiv \varphi^n(x_0^{-1}) \bmod I''_n \end{aligned}$$

Hence,

$$\begin{aligned}
&x_n^{-1}\varphi(x_{n-1}) - 1\\
=&-(x_n^{-1} - \varphi^n(x_0^{-1}))(x_n - \varphi(x_{n-1})) - \varphi^n(x_0^{-1})(x_n - \varphi(x_{n-1}))\\
\in& I_n'' I_n + I_n = I_n,
\end{aligned}$$

or equivalently (using proposition 3.51) $x_n\varphi(x_{n-1})^{-1} \in 1 + I_n$.

"$\Leftarrow$" Since φ is a ring homomorphism, $\varphi(I_k'') \subset I_{k+1}''$ implies $\varphi(1+I_k'') \subset 1 + I_{k+1}''$. Using lemma 3.55, we get

$$x_n\varphi^n(x_0)^{-1} = \prod_{k=1}^{n} \varphi^{n-k}(x_k\varphi(x_{k-1})^{-1}) \in 1 + I_n''.$$

Since $\varphi^n(x_0) \in \Lambda(U_{n,n}/V_n)$ and since $1 + I_n$ is a group, we get

$$\begin{aligned}
&\varphi(x_{n-1}) - x_n\\
=&(x_n - \varphi^n(x_0))(x_n^{-1}\varphi(x_{n-1}) - 1) + \varphi^n(x_0)(x_n^{-1}\varphi(x_{n-1}) - 1)\\
\in& I_n'' I_n + I_n = I_n
\end{aligned}$$

□

Proposition 3.57. *$\theta(K_1(\Lambda(P))) \subset \Psi$ and $\theta_S(K_1(\Lambda(P)_S)) \subset \Psi_S$*

Proof. Let $x \in K_1(\Lambda(P))$. Then $\theta(x)$ satisfies the first condition in the definition of Ψ since the diagram

(32)
$$\begin{tikzcd}
 & K_1(\Lambda(U_n)) \arrow[r, "p_*"] & \Lambda(U_n/V_n)^\times \arrow[dr, "p_*"] & \\
K_1(\Lambda(P)) \arrow[ur, "\mathrm{N}"] \arrow[urr, "\theta_n"] \arrow[dr, "\mathrm{N}"'] \arrow[drr, "\theta_m"'] & & & \Lambda(U_n/V_m)^\times \\
 & K_1(\Lambda(U_m)) \arrow[r, "p_*"'] & \Lambda(U_m/V_m)^\times \arrow[ur, "\mathrm{N}"'] &
\end{tikzcd}$$

commutes (see diagram (31)). A similar diagram shows that for $y \in K_1(\Lambda(P)_S)$, $\theta_S(y)$ satisfies the first condition in the definition of Ψ_S.

Define

$$q_n : K_1(\Lambda(P)) \to \Lambda(U_n/V_n)^\times, \quad x \mapsto \theta_n(x)\varphi(\theta_{n-1}(x))^{-1}.$$

By lemma 3.50, $\tau_n \circ \mathscr{L}_P(x) = \log(q_n(x))$ for all $x \in K_1(\Lambda(P))$. We will show that $q_n(x) \in 1 + I_n$, i. e. the second condition in the definition of Ψ is satisfied. It suffices to show that

$$\log : \mathrm{im}\,(q_n) \to I_n$$

is injective, since then (by proposition 3.51) $q_n(x) = \exp \circ \log(q_n(x)) \in 1 + I_n$.

Assume that $\log \circ q_n(x) = 0$. Then $\tau_n \circ \mathscr{L}_P(x) = 0$. Using the fact that τ_n is an isomorphism and the description of $\ker \mathscr{L}_P$ in corollary 3.45, we deduce

$$x \in \ker \mathscr{L}_P = \mu_{p-1} \times P^{ab}.$$

But then clearly $q_n(x) = 0$.

We will now prove that

$$\theta_{n,S}(y) \cdot \varphi(\theta_{n-1,S}(y))^{-1} \in 1 + I_{n,S}$$

for $y \in K_1(\Lambda(P)_S)$. For $W \in \mathfrak{W}_{U_n}$, let $\bar{y}$ be the image of y in $K_1\left((\Lambda(P/W)_S)^{\langle p\rangle}\right)$. Let $\theta_{n,S}(P/W)$ (respectively $I_{n,S}(P/W)$) be the homomorphism $\theta_{n,S}$ (respectively the module $I_{n,S}$) assigned to P/W. Since the diagram

$$\begin{tikzcd}
K_1(\Lambda(P)_S) \arrow[r, "\mathrm{N}"] \arrow[d] & K_1(\Lambda(U_n)_S) \arrow[r, "p_*"] \arrow[d] & \Lambda(U_n/V_n)_S^\times \arrow[d] \\
K_1(\Lambda(P/W)_S) \arrow[r, "\mathrm{N}"] & K_1(\Lambda(U_n/W)_S) \arrow[r, "p_*"] & \Lambda(U_n/WV_n)_S^\times,
\end{tikzcd}$$

is commutative, the image of $\theta_{n,S}(P)(y)$ in $\Lambda(U_n/WV_n)_S^\times$ is $\theta_{n,S}(P/W)(\bar{y})$. Using the fact that

$$I_{n,S} = \varprojlim_{W \in \mathfrak{W}_{U_n}} \tau_{n,S}\left(\mathbb{Z}_p[\![\mathrm{Conj}(P/W)]\!]_S\right) = \varprojlim_{W \in \mathfrak{W}_{U_n}} I_{n,S}(P/W),$$

we get that it suffices to show that

$$\theta_{n,S}(P/W)(\bar{y}) \cdot \varphi\left(\theta_{n-1,S}(P/W)(\bar{y})\right)^{-1} \in 1 + I_{n,S}(P/W). \tag{33}$$

By lemma 3.36, we can write $\bar{y} = uv$ with $u \in \Lambda(Z(P/W))_S^\times$ and $v \in \Lambda(P/W)^\times$. We have already shown that equation (33) holds for $\bar{y} = v$. Since $\theta_{n,S}(P/W)(u) = u^{p^n}$ and

$$p^n \Lambda(Z(P/W)_S) \subset I_{n,S}(P/W),$$

it suffices to show that $u^{p^n} \varphi(u^{p^{n-1}})^{-1} \in 1 + p^n \Lambda(Z(P/W))_S$. But this follows from lemma 3.27. $\square$

Proof of theorem 3.47. We define

$$\widetilde{\mathscr{L}_P} : \Psi \to \Omega, \quad (x_n)_n \mapsto (y_n)_n,$$

where $y_0 = \mathscr{L}_{U_0/V_0}(x_0)$ and $y_n = \log(x_n \varphi(x_{n-1})^{-1})$ for $n \geq 1$. We need to show that this map is well-defined, i. e. that

- $y_n \in I_n$ for all $n \in \underline{\underline{c}}$
- $\mathrm{Tr}_{m,n}(y_m) = p_{n,m}(y_n)$ for all $m \leq n$, $m, n \in \underline{\underline{c}}$.

The first condition follows from lemma 3.51 and the definition of Ψ (recall that $I_0 = \Lambda(U_0/V_0)$). By lemma 3.48 and lemma 3.49, we get for $m, n \in \underline{\underline{c}}$, $1 \le m < n$

$$\begin{aligned}\mathrm{Tr}_{m,n}(y_m) &= \mathrm{Tr}_{m,n} \circ \log(x_m) - \mathrm{Tr}_{m,n} \circ \varphi \circ \log(x_{m-1}) \\ &= \log \circ \mathrm{N}_{m,n}(x_m) - \varphi \circ \mathrm{Tr}_{m-1,n-1} \circ \log(x_{m-1}) \\ &= \log \circ \mathrm{N}_{m,n}(x_m) - \varphi \circ \log(\mathrm{N}_{m-1,n-1}(x_{m-1})) \\ &= p_{m,n} \circ \log(x_n) - \varphi \circ p_{n-1,m-1} \circ \log(x_{n-1}) \\ &= p_{m,n} \circ \log(x_n \varphi(x_{n-1})) = p_{n,m}(y_n).\end{aligned}$$

Now, it suffices to prove the second condition for $n > 0$, $m = 0$. Since $\mathrm{N}_{0,n}(x_0) = p_{n,0}(x_n)$, we get (using lemma 3.48) that

$$\mathrm{Tr}_{0,n}(\log(x_0)) = p_{n,0}(\log(x_n)).$$

Using lemma 3.49 and the above equation, we get

$$\begin{aligned}\mathrm{Tr}_{0,n}\left(\frac{1}{p}\log \circ \varphi(x_0)\right) &= \varphi\Big(\mathrm{Tr}_{0,n-1}(\log(x_0))\Big) \\ &= \varphi\Big(p_{n-1,0}(\log(x_{n-1}))\Big) \\ &= p_{n,0}\Big(\log \circ \varphi(x_{n-1})\Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathrm{Tr}_{0,n}(y_0) &= \mathrm{Tr}_{0,n}\Big(\log(x_0) - \frac{1}{p}\log(\varphi(x_0))\Big) \\ &= p_{n,0}\Big(\log(x_n) - \log(\varphi(x_{n-1}))\Big) = p_{n,0}(y_n).\end{aligned}$$

We define the continuous continuous group homomorphisms

$$\begin{aligned}&\tilde{\omega} : \Omega \to U_0/V_0, \quad (x_n)_{n \ge 0} \mapsto x_0 \text{ if } x_0 \in U_0/V_0 \subset \Lambda(U_0/V_0) \\ &\tilde{\theta} : \mu_{p-1} \times P^{ab} \to \Psi, \quad (\zeta, g) \mapsto (\zeta g^{p^n})_{n \in \underline{\underline{c}}}.\end{aligned}$$

By lemma 3.46, $\tilde{\theta}$ is well-defined.

Now, we claim that the following diagram is commutative with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \mu_{p-1} \times P^{ab} \arrow[r] \arrow[d, equal] & K_1(\Lambda(P)) \arrow[r, "\mathscr{L}_P"] \arrow[d, "\theta"] & \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \arrow[r, "\omega"] \arrow[d, "\tau", "\cong"'] & P^{ab} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mu_{p-1} \times P^{ab} \arrow[r, "\tilde{\theta}"] & \Psi \arrow[r, "\widetilde{\mathscr{L}_P}"] & \Omega \arrow[r, "\tilde{\omega}"] & U_0/V_0 \arrow[r] & 1 \end{tikzcd} \tag{34}$$

The left square is commutative by lemma 3.46. Commutativity of the middle square follows from lemma 3.50. The right square is trivially commutative. The upper row is exact by corollary 3.45. We need to show that the lower row is exact.

Injectivity of $\tilde{\theta}$ is obvious.

The exactness of the upper row in diagram 34 implies $\operatorname{im}\tilde{\theta} \subset \ker \widetilde{\mathscr{L}_P}$. Let $x \in \ker \widetilde{\mathscr{L}_P}$. Since $\log : 1 + I_n \to I_n$ is injective for all $n \geq 1$ (cf. proposition 3.51), we can write $x = (\varphi^n(x_0))_{n\geq 0}$ for some $x_0 \in \Lambda(U_0/V_0)^\times$. Since $x \in \Psi$, the image of $\varphi^m(x_0)$ under $N : \Lambda(U_m/V_m) \to \Lambda(U_n/V_m)$ coincides with $\varphi^n(x_0) \in \Lambda(U_n/V_m)$ for all $m \leq n$. Let y be an inverse image of x_0 under $p_* : K_1(\Lambda(P)) \to \Lambda(U_0/V_0)^\times$. Then the commutativity of the diagram

$$\begin{tikzcd} K_1(\Lambda(P)) \arrow[r, "\mathrm{N}"] \arrow[d, "p_*"] & K_1(\Lambda(U_n)) \arrow[d, "p_*"] \\ \Lambda(U_0/V_0)^\times \arrow[r, "\mathrm{N}"] & \Lambda(U_n/V_0)^\times \end{tikzcd}$$

implies $\theta_n(y) = p_* \circ \mathrm{N}(y) = \mathrm{N} \circ p_*(y) = \mathrm{N}(x_0) = \varphi^n(x_0)$, i. e. $\theta(y) = x$. Since $x \in \ker \widetilde{\mathscr{L}_P}$ and τ is an isomorphism, we get $y \in \ker \mathscr{L}_P$, i. e. $y = [\zeta g]$, $\zeta \in \mu_{p-1}$, $g \in P$, and this gives rise to an inverse image of x under $\tilde{\theta}$. Hence $\operatorname{im}\tilde{\theta} = \ker \widetilde{\mathscr{L}_P}$.

We now show exactness at Ω. By corollary 3.45, $\operatorname{im} \mathscr{L}_{P^{ab}} = \ker \omega_{P^{ab}}$. This clearly implies $\operatorname{im} \widetilde{\mathscr{L}_P} \subset \ker \tilde{\omega}$. Since τ is an isomorphism,

$$\tau(\operatorname{im} \mathscr{L}_P) = \operatorname{im}(\widetilde{\mathscr{L}_P} \circ \theta) \subset \operatorname{im} \widetilde{\mathscr{L}_P}$$

and $\tau(\ker \omega) = \ker \tilde{\omega}$, and we get the inclusion $\ker \tilde{\omega} \subset \operatorname{im} \widetilde{\mathscr{L}_P}$.

Surjectivity of $\tilde{\omega}$ is obvious.

By the five lemma, θ is an isomorphism. □

Corollary 3.58. *Let P be a p-adic Lie group that satisfies assumption 3.1. Then*

$$K_1(\Lambda(P)) \subset K_1(\Lambda(P)_S).$$

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd} K_1(\Lambda(P)_S) \arrow[r] & \Psi_S \\ K_1(\Lambda(P)) \arrow[r, "\cong"] \arrow[u] & \Psi. \arrow[u, hook] \end{tikzcd}$$

□

Corollary 3.59.

$$\theta_S|_{K_1(\Lambda(P))} = \theta$$

PROOF. By corollary 2.11 the norm on $K_1(\Lambda(P))$ is the restriction of the norm on $K_1(\Lambda(P)_S)$. $\square$

LEMMA 3.60.
$$\Psi_S \cap \prod_{n \in \underline{\underline{c}}} \Lambda(U_n/V_n)^\times = \Psi$$

PROOF. This follows from the fact that the homomorphisms N, p_* and φ in the definition of Ψ are restrictions of the corresponding homomorphisms in the definition of Ψ_S. $\square$

LEMMA 3.61. *For every irreducible Artin representation ρ of P, there is $n \in \underline{\underline{c}}$ such that ρ is induced by a one dimensional representation χ of U_n. (Then ρ maps γ to a p^n-th root of unity.)*

PROOF. Let $\rho : P \to GL_k(\overline{\mathbb{Q}})$ be the irreducible Artin representation. Let $W \subset P$ be an open normal subgroup of P such that ρ factors through P/W. Put $W_i := W \cap U_i$ for $i \in \mathbb{N} \cup \{\infty\}$. Then

$$P/W = U_\infty/W_\infty \rtimes \langle\beta\rangle/\langle\beta^{p^n}\rangle \tag{35}$$

for some $n \in \mathbb{N}$. By [**42**, proposition 25], $\rho = \mathrm{ind}_{U_n}^P(\chi)$ is induced by an irreducible representation χ of U_∞/W_∞.

Since U_∞ is abelian, $\chi : U_\infty/W_\infty \to \overline{\mathbb{Q}}^\times$ is a character. Assume that n in (35) is minimal. Then $U_\infty/W_\infty = U_n/W_n$. By composing χ with the projection
$$U_n \to U_n/W_n = U_\infty/W_\infty,$$
we can regard χ as a character of U_n.

Since $\gamma^{p^n} = [\alpha, \beta^{p^n}] \in W \subset \ker\rho$, the image of γ under ρ is a p^n-th root of unity. $\square$

We summarise what we have proved in the following theorem:

THEOREM 3.62. *Let P be a p-adic Lie group that satisfies assumption 3.1. Then the set $\mathcal{I}$ of pairs (U_n, V_n), $n \in \underline{\underline{c}}$ and the subgroups Ψ and Ψ_S satisfy property 2.39.*

CHAPTER 4

Hilbert Modular Forms

Deligne-Ribet [**12**], Wiles [**52**] and Kakde [**24**] have proven existence and uniqueness of the p-adic zeta functions

$$\xi_n = \xi_\Sigma(F_{V_n}|F_{U_n}) \in \Lambda(U_n/V_n)_S^\times, \qquad n \in \underline{\underline{c}}$$

for $F_{V_n}|F_{U_n}$ with respect to Σ (recall definition 2.36). In this chapter, we will show that $(\xi_n)_{n\in\underline{\underline{c}}} \in \Psi_S$. The main difficulty is to prove that

$$\xi_n\varphi(\xi_{n-1})^{-1} \in 1 + I_{n,S}. \tag{36}$$

We will first develop the theory of Hilbert modular forms. For every field F_{U_n}, we will recall the construction of the F_{U_n}-Hilbert Eisenstein series E_n. We will define a restriction homomorphism on the space of Hilbert modular forms. Let g_n be the restriction of E_n to the Hilbert modular variety of F. We will show that the map φ of the previous chapter is the transfer homomorphism and extend it to a map of $\Lambda(U_n/V_n)$-adic Hilbert modular forms. Let $\varphi(g_{n-1})$ be the restriction of $\varphi(E_{n-1})$ to the Hilbert modular variety of F. We will write g_n as a sum of the form

$$g_n = 2^{-r(n)}\xi_n + \sum_{m=0}^{n} \varphi^{n-m}(h_m).$$

Then $g_n - \varphi(g_{n-1}) = 2^{-r(n)}\xi_n - 2^{-r(n-1)}\varphi(\xi_{n-1}) + h_n$. We will show that h_n has coefficients in $I_{n,S}$. Now the q-expansion principle implies $\xi_n \equiv \varphi(\xi_{n-1}) \bmod I_{n,S}$. By lemma 3.56, this is equivalent to (36).

1. Classical Hilbert Modular Forms

Let K be a totally real algebraic number field over $\mathbb{Q}$ and let $r = [K : \mathbb{Q}]$. Let

$$K \to \mathbb{R}, \quad \alpha \mapsto \alpha^{(i)}, \qquad i = 1, \dots, r$$

be the r embeddings of K into $\mathbb{R}$. We write $\alpha \gg 0$ if α is totally positive, i. e. $\alpha^{(i)} > 0$ for $i = 1, \dots, r$. We define the *Hilbert modular group* to be

$$\Gamma_K := SL_2(\mathcal{O}_K)/\{\pm 1\},$$

where $\mathcal{O}_K$ is the ring of integers of K. Let $\Gamma \subset SL_2(K)/\{\pm 1\}$ be a group commensurable with Γ_K. (Two groups Γ_1 and Γ_2 are commensurable if $[\Gamma_1 : \Gamma_1 \cap \Gamma_2] < \infty$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2] < \infty$.)

Let $\mathfrak{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$ be the upper half plane of $\mathbb{C}$. Then

$$GL_2^+(\mathbb{R}) := \{\gamma \in GL_2(\mathbb{R}) | \det(\gamma) > 0\}$$

acts on $\mathfrak{H}$ by linear transformations:

$$\tau \mapsto \gamma\tau := \frac{a\tau + b}{c\tau + d} \qquad \text{for } \tau \in \mathfrak{H} \quad \text{and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

We can define an embedding $GL_2(K) \hookrightarrow GL_2(\mathbb{R})^r$, using the r embeddings of K into $\mathbb{R}$. This induces an operation of $GL_2(K)$ on $\mathfrak{H}^r$.

For a function $f : \mathfrak{H}^r \to \mathbb{C}$, $k = (k_i)_i \in \mathbb{Z}^r$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$, we define $f|_k\gamma : \mathfrak{H}^r \to \mathbb{C}$ by

$$\begin{aligned} f|_k\gamma(\tau) &:= \mathrm{N}((\det\gamma)^{k/2}(c\tau + d)^{-k})f(\gamma\tau) \\ &:= \left[\prod_{i=1}^{r}(a^{(i)}d^{(i)} - b^{(i)}c^{(i)})^{k_i/2}(c^{(i)}\tau_i + d^{(i)})^{-k_i}\right] f(\gamma\tau) \end{aligned}$$

for $\tau = (\tau_1, \ldots, \tau_r) \in \mathfrak{H}^r$. We will write $f|_k\gamma := f|_{(k,\ldots,k)}\gamma$ for $k \in \mathbb{Z}$.

We define an operation of $GL_2(K)$ on $\mathbb{P}^1(K)$:

$$\begin{aligned} &\tau \mapsto \gamma\tau := (a\tau_0 + b\tau_1 : c\tau_0 + d\tau_1) \in \mathbb{P}^1(K) \\ &\text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K), \quad \tau = (\tau_0 : \tau_1) \in \mathbb{P}^1(K) \end{aligned}$$

We define the set of cusps of Γ to be the set $\Gamma\backslash\mathbb{P}^1(K)$.

Remark. There is a natural bijection from $\Gamma\backslash\mathbb{P}^1(K)$ to the ideal class group $Cl(K)$ of K. (See [**17**, proposition 1.1])

Let $f : \mathfrak{H}^r \to \mathbb{C}$ be a holomorphic function. Consider the cusp $\infty = [(1 : 0)]$ of Γ. Define

$$\begin{aligned} M_\infty &:= M_\infty(K, f) := \{b \in K | f(x + b) = f(x), \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma\}, \\ M_\infty^\vee &:= M_\infty^\vee(K, f) := \{x \in K | \mathrm{Tr}(xb) \in \mathbb{Z} \text{ for all } b \in M_\infty\}, \end{aligned}$$

where $\mathrm{Tr}(xb) := \sum_{i=1}^r x^{(i)}b^{(i)}$. For $x \in M_\infty^\vee$, $\tau \in \mathfrak{H}^r$, we put

$$q_K^x(\tau) := \exp(2\pi i \cdot \mathrm{Tr}(x\tau)),$$

where $\mathrm{Tr}(x\tau) := \sum_{i=1}^r x^{(i)}\tau_i$.

Definition 4.1. Assume that $M_\infty(K, f) \neq \{0\}$. Then f can be developed in a Fourier expansion of the form

$$f(\tau) = \sum_{x \in M_\infty^\vee} a(x, f)q_K^x(\tau), \quad a(x, f) \in \mathbb{C},$$

which is called the q-expansion of f at the cusp ∞ with respect to Γ. Let $\kappa = N\infty \in \Gamma\backslash\mathbb{P}^1(K)$ be the translation of the cusp ∞ by some $N \in GL_2(K)$. Then the q-expansion of f at the cusp κ with respect to Γ for $k \in \mathbb{Z}^r$ is defined to be the q-expansion of $f|_k N$ at the cusp ∞ with respect to $N^{-1}\Gamma N$.

We say that f is holomorphic at cusp κ (for k) if $a(x, f) \neq 0$ implies $x \gg 0$ or $x = 0$.

Definition 4.2. For $r > 1$, a *Hilbert modular form* of weight $k \in \mathbb{Z}^r$ on Γ is a holomorphic function $f : \mathfrak{H}^r \to \mathbb{C}$ such that for all $\gamma \in \Gamma$,

$$f|_k\gamma = f.$$

For $r = 1$ (i. e. $K = \mathbb{Q}$), we add the condition that f is holomorphic at the cusps (for k). We denote the vector space of Hilbert modular forms of weight k on Γ by $M_k(\Gamma)$. For any subring $A \subset \mathbb{C}$, we define the subspace

$$M_k(\Gamma, A) := \{f \in M_k(\Gamma) | a(x, f) \in A \text{ for all } x \in M_\infty^\vee\}.$$

Remark. The set $M_\infty(K) = M_\infty(K, f)$ for $f \in M_k(\Gamma)$ is independent of f.

Let $L|K$ be a finite extension of totally real number fields. The containment $K \subset L$ induces the canonical map

$$* : \mathfrak{H}^{[K:\mathbb{Q}]} \hookrightarrow \mathfrak{H}^{[L:\mathbb{Q}]} \quad (\tau_1, \dots, \tau_r) \mapsto (\underbrace{\tau_1, \dots, \tau_1}_{[L:K] \text{ times}}, \dots, \underbrace{\tau_r, \dots, \tau_r}_{[L:K] \text{ times}}).$$

Let Γ be a group commensurable with Γ_L. Then $\Gamma(K) := \Gamma \cap GL_2(K)$ is commensurable with Γ_K. For $f \in M_k(\Gamma)$, we define

$$\mathrm{res}_{L|K} f : \mathfrak{H}^{[K:\mathbb{Q}]} \to \mathbb{C}, \quad \tau \mapsto f(\tau^*).$$

Lemma 4.3. *Let $k \in \mathbb{Z}$ be an integer. Then $res_{L|K}$ defines a homomorphism*

$$res_{L|K} : M_k(\Gamma) \to M_{[L:K]\cdot k}(\Gamma(K)).$$

If the q-expansion of f is

$$f(\tau) = \sum_{y \in M_\infty^\vee(L)} a(y, f) q_L^y(\tau), \quad \tau \in \mathfrak{H}^{[K:\mathbb{Q}]}$$

then

$$\begin{aligned} res_{L|K} f(\tau) &= \sum_{y \in M_\infty^\vee(L)} a(y, f) q_K^{\mathrm{Tr}_{L|K}(y)}(\tau) \\ &= \sum_{x \in M_\infty^\vee(K)} a_*(x, f) q_K^x(\tau), \quad \tau \in \mathfrak{H}^{[K:\mathbb{Q}]}, \end{aligned}$$

with $a_*(x, f) := \sum_{y:\mathrm{Tr}_{L|K}(y)=x} a(y, f)$.

PROOF. Let

$$L \to \mathbb{R}, \quad \alpha \mapsto \alpha^{(i,j)}, \quad i = 1, \dots, [K:\mathbb{Q}],\ j = 1, \dots, [L:K]$$

be the embeddings of L in $\mathbb{R}$ such that $\alpha^{(i,j)} = \alpha^{(i,j')}$ for $\alpha \in K$ and all i, j, j'. For $\tau = (\tau_i)_i \in \mathfrak{H}^{[K:\mathbb{Q}]}$, we write $\tau^* = (\tau^*_{i,j})_{i,j} \in \mathfrak{H}^{[L:\mathbb{Q}]}$ such that $\tau^*_{i,j} = \tau_i$ for all i, j. For $y \in M^\vee_\infty(L)$, we have

$$\begin{aligned} q_L^y(\tau^*) &= \exp\left(2\pi i \sum_{i=1}^{[K:\mathbb{Q}]} \sum_{j=1}^{[L:K]} \tau^*_{i,j} y^{(i,j)}\right) \\ &= \exp\left(2\pi i \sum_{i=1}^{[K:\mathbb{Q}]} \tau_i (\mathrm{Tr}_{L|K} y)^{(i)}\right) = q_K^{\mathrm{Tr}_{L|K} y}(\tau). \end{aligned}$$

Let $\gamma \in \Gamma(K)$ and let γ^* be its image in Γ. Then

$$\begin{aligned} (\mathrm{res}_{L|K} f)|_{[L:K]k}\gamma(\tau) &= \mathrm{N}_K\Big(\det(\gamma)^{k/2\cdot[L:K]}(c\tau+d)^{-k\cdot[L:K]}\Big)\mathrm{res}_{L|K} f(\gamma\tau) \\ &= \mathrm{N}_L\Big(\det(\gamma^*)^{k/2}(c^*\tau^*+d^*)^{-k}\Big) f(\gamma^*\tau^*) \\ &= f|_k\gamma^*(\tau^*) = f(\tau^*) = \mathrm{res}_{L|K} f(\tau). \end{aligned}$$

□

2. Λ-adic Hilbert Modular Forms

Let K be a number field and let $L|K$ be a field extension such that $W := G(L|K)$ is a compact p-adic Lie group. Assume that there is a surjective homomorphism $\omega : W \to \mathbb{Z}_p$ and put $S := S(W, \omega) \subset \Lambda(W)$ (cf. definition 2.2). Let

$$f = \sum_{x \in M^\vee_\infty} a(x, f) q_K^x$$

be a formal sum with coefficients $a(x, f) \in K_1(\Lambda(W)_S)$. We fix an isomorphism $\mathbb{C}_p \cong \mathbb{C}$. We define the evaluation of f at a continuous representation $\rho : W \to GL_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$, by

$$f(\rho) := \sum_{x \in M^\vee_\infty} a(x, f)(\rho) q_K^x,$$

where $a(x, f)(\rho) \in \mathbb{C}$ is the evaluation defined in definition 2.13.

DEFINITION 4.4. Let Γ be a group commensurable with Γ_K. With the notation as above, we define f to be a $\Lambda(W)$-adic K-Hilbert modular form with respect to Γ if

$$f(\rho\kappa^k) \in M_k(\Gamma, \mathcal{O}_{\mathbb{C}_p})$$

for all but finitely many even $k \geq 1$ and all Artin representations $\rho : W \to GL_n(\mathbb{C})$. We denote the set of $\Lambda(W)$-adic Hilbert modular forms by

$$M(\Gamma, L|K).$$

Let K' be a finite extension of K contained in L. Let Γ be a group commensurable with $\Gamma_{K'}$. By lemma 4.3, we get a homomorphism

$$\begin{gathered}\mathrm{res}_{K'|K} : M(\Gamma, L|K') \to M(\Gamma(K), L|K),\\ \sum_{x \in M_\infty^\vee(K')} a(x,f) q_{K'}^x \mapsto \sum_{x \in M_\infty^\vee(K')} a(x,f) q_K^{\mathrm{Tr}_{K'|K}(x)}.\end{gathered}$$

Let $* : K \hookrightarrow K'$ be the canonical homomorphism. Let N, N' be the maximal abelian extensions contained in L of K, K' respectively. Let Γ be a group commensurable with Γ_K. Let $\mathrm{Ver} : G(N|K) \to G(N'|K')$ be the transfer homomorphism. This induces the homomorphisms

$$\Lambda(G(N|K)) \to \Lambda(G(N'|K')) \subset \Lambda(G(N'|K))$$

and

$$K_1(\Lambda(G(N|K)_S) \to K_1(\Lambda(G(N'|K)_S)).$$

We define

$$\begin{aligned}\mathrm{Ver} : M(\Gamma, N|K) &\to M(\Gamma, N'|K),\\ \sum_{x \in M_\infty^\vee(K)} a(x,f) q_K^x &\mapsto \mathrm{res}_{K'|K}\left(\sum_{x \in M_\infty^\vee(K)} \mathrm{Ver}(a(x,f)) q_{K'}^{x^*}\right)\\ &= \sum_{x \in M_\infty^\vee(K)} \mathrm{Ver}(a(x,f)) q_K^{[K':K]\cdot x}.\end{aligned}$$

We show that this is well-defined. Let $\chi : G(N'|K') \to \overline{\mathbb{Q}}^\times$ be a character and let $\chi_{\mathrm{Ver}} = \chi \circ \mathrm{Ver} : G(N|K) \to \overline{\mathbb{Q}}^\times$ be the character induced by χ. Then

$$\mathrm{Ver}(f)(\chi\kappa_{K'}^k)(\tau) = f(\chi_{\mathrm{Ver}}(\kappa_{K'})_{\mathrm{Ver}}^k)([K':K]\tau)$$

for the cyclotomic character $\kappa_{K'}$ of K', for any even $k \geq 2$ and for all $\tau \in \mathfrak{H}^r$. By [**33**, ch. I §5], Ver is the the corestriction

$$\mathrm{cor}_{G(L|K)}^{G(L|K')} : H_1(G(L|K), \mathbb{Z}_p) \to H_1(G(L|K'), \mathbb{Z}_p).$$

By [**33**, cor. 1.5.7],

$$\mathrm{cor}_{G(L|K)}^{G(L|K')} \circ \mathrm{res}_{G(L|K')}^{G(L|K)} = [K':K].$$

Since

$$\begin{gathered}\mathrm{res}_{G(L|K')}^{G(L|K)}(\sigma)(\zeta) = \sigma(\zeta)\\ \mathrm{cor}_{G(L|K)}^{G(L|K')} \circ \mathrm{res}_{G(L|K')}^{G(L|K)}(\sigma)(\zeta) = \sigma(\zeta)^{[K':K]}\end{gathered}$$

for all $\zeta \in \mu_{p^\infty}$, this implies $(\kappa_{K'})_{\mathrm{Ver}} = (\kappa_K)^{[K':K]}$. Hence

$$\mathrm{Ver}(f)(\chi\kappa_{K'}^k)(\tau) = f(\chi_{\mathrm{Ver}}\kappa_K^{[K':K]\cdot k})([K':K]\tau)$$

for $f \in M(\Gamma, N|K)$ and all even $k \geq 2$ and thus $\mathrm{Ver}(f)(\chi\kappa_{K'}^k) \in M_k(\Gamma, \mathcal{O}_K)$.

3. Existence of the p-adic Zeta Function

We keep the notation from the previous section. Assume that $F_\infty|F$ and $G = G(F_\infty|F)$ satisfy assumption 2.1 and 3.1 and that Ψ and Ψ_S satisfy property 2.39.

THEOREM 4.5. *Let ξ_n be the p-adic zeta function for $F_{V_n}|F_{U_n}$. Then $(\xi_n)_n \in \Psi_S$.*

COROLLARY 4.6. *The p-adic zeta function for $F_\infty|F$ with respect to Σ exists and in this case, the main conjecture is true.*

PROOF. See theorem 2.40. □

For the proof of the above theorem, we will need the Λ-adic Eisenstein series, which we will now define. Let $n \in \underline{\underline{c}}$ be a fixed integer. Henceforth, we write F_n for F_{U_n} and K_n for F_{V_n}. Let

$$\kappa_n : G(F_n(\mu_{p^\infty})|F_n) \to \mathbb{Z}_p^\times$$

be the cyclotomic character. Put $r(n) := [F_n : \mathbb{Q}]$. Then $r(n) = p^n r$, where $r := [F : \mathbb{Q}]$.

Let A_n be the monoid of non-zero integral ideals of F_n prime to Σ. Let $\sigma_{\mathfrak{a}} \in U_n/V_n$ be the Artin symbol of $\mathfrak{a} \in A_n$. We define

$$\mathcal{R}(F_m) := \{(\mathfrak{a}, x) | \mathfrak{a} \in A_m,\ x \in \mathcal{O}_{F_m}^{\gg 0} \cap \mathfrak{a}\},$$

where $\mathcal{O}_{F_m}^{\gg 0} := \{x \in \mathcal{O}_{F_m} | x \gg 0\}$. Put $\mathcal{R} := \mathcal{R}(F_n)$.

Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_{F_n}$. We define

$$\Gamma_{00}(\mathfrak{f}) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F_n) | a, d \in 1 + \mathfrak{f}, b \in \mathfrak{D}^{-1}, c \in \mathfrak{f}\mathfrak{D}\},$$

where $\mathfrak{D}$ is the different[1] of F_n. Then $\Gamma_{00}(\mathfrak{f})$ is commensurable with Γ_{F_n}.

[1]The different of F_n is defined to be the inverse of the fractional ideal $\{x \in F_n | \mathrm{Tr}(xb) \in \mathbb{Z}$ for all $b \in \mathcal{O}_{F_n}\}$.

LEMMA 4.7. *There is an ideal $\mathfrak{f}_n \subset \mathcal{O}_{F_n}$ with all its prime factors in Σ such that the series*

$$E_n := 2^{-r(n)}\xi_n + \sum_{(\mathfrak{a},x)\in\mathcal{R}} \kappa_n(\sigma_\mathfrak{a})^{-1}\sigma_\mathfrak{a} q_{F_n}^x \in M(\Gamma_{00}(\mathfrak{f}_n), K_n|F_n)$$

is a $\Lambda(U_n/V_n)$-adic Hilbert modular form with respect to $\Gamma_{00}(\mathfrak{f}_n)$. We call it the $\Lambda(U_n/V_n)$-adic Eisenstein series*.*

PROOF. Let $\chi_n : U_n/V_n \to \mathbb{C}^\times$ be an even (i. e. $\chi_n(g) = \chi_n(g^{-1})$ for all $g \in U_n/V_n$) Artin character. Then χ_n induces a homomorphism

$$\chi_n : A_n \to \mathbb{C}^\times, \quad \chi_n(\mathfrak{a}) := \chi_n(\sigma_\mathfrak{a}).$$

There is a norm homomorphism defined by

$$\mathfrak{N} : A_n \to \mathbb{C}^\times, \quad \mathfrak{N}(\mathfrak{a}) := (\mathcal{O}_{F_n} : \mathfrak{a}) = \kappa_n(\sigma_\mathfrak{a}).$$

By [**37**, proposition 8], there is an integral ideal $\mathfrak{f}_n$ in F_n with all its prime factors in Σ and a (classical) modular form $G_{k,\chi_n} \in M_k(\Gamma_{00}(\mathfrak{f}_n))$ with standard q-expansion

$$G_{k,\chi_n} = 2^{-r(n)}L_\Sigma(1-k,\chi_n) + \sum_{x\in\mathcal{O}_{F_n}^{\gg 0}} \left(\sum_{x\in\mathfrak{a}\in A_n} \chi_n(\mathfrak{a})\mathfrak{N}(\mathfrak{a})^{k-1}\right) q_{F_n}^x.$$

Using the interpolation property of ξ, we get

$$\begin{aligned}
E_n(\chi_n\kappa_n^k) &= 2^{-r(n)}\xi(\chi_n\kappa_n^k) + \sum_{(\mathfrak{a},x)\in\mathcal{R}} \chi_n\kappa_n^{k-1}(\sigma_\mathfrak{a})q_{F_n}^x \\
&= 2^{-r(n)}L_\Sigma(1-k,\chi_n) + \sum_{x\in\mathcal{O}_{F_n}^{\gg 0}} \left(\sum_{x\in\mathfrak{a}\in A_n} \chi_n(\mathfrak{a})\mathfrak{N}(\mathfrak{a})^{k-1}\right) q_{F_n}^x \\
&= G_{k,\chi_n} \in M_k(\Gamma_{00}(\mathfrak{f}_n)).
\end{aligned}$$

Thus E_n is a $\Lambda(U_n/V_n)$-adic F_n-Hilbert modular form. $\square$

We recall the following well-known facts on the transfer homomorphism:

LEMMA 4.8. *The transfer homomorphism $Ver_{m,n} : U_m/V_m \to U_n/V_n$, $m \leq n$, induces the homomorphism*

$$\varphi^{n-m} : \Lambda(U_m/V_m) \to \Lambda(U_n/V_n), \quad \sigma \mapsto \sigma^{p^{n-m}} \text{ for } \sigma \in U_m/V_m.$$

For $\mathfrak{a} \in A_m$, we have

$$\varphi^{n-m}(\sigma_\mathfrak{a}) = \sigma_{\mathfrak{a}\mathcal{O}_{F_n}}.$$

PROOF. For $\sigma \in U_m$ we write $\sigma = \alpha^i \beta^{p^n j + p^m k} z$ with $i, j \in \mathbb{Z}_p$, $k \in \{0, \dots, p^{n-m} - 1\}$, $z \in Z(U_m)$. The explicit description of the transfer homomorphism (cf. [**33**, ch. I, §5]) yields

$$\begin{aligned} \mathrm{Ver}_{m,n}(\sigma V_m) &= \prod_{l=0}^{p^{n-m}-1} \beta^{p^m l} \sigma \beta^{-p^m(l+k)} V_n \\ &= (\alpha^i \beta^{p^n j} z)^{p^{n-m}} V_n = \sigma^{p^{n-m}} V_n \end{aligned}$$

(note that $(\beta^{p^m l} U_n)(\alpha^i \beta^{p^n j + p^m k} z) = \beta^{p^m(l+k)} U_n)$.

The second statement is part of class field theory. $\square$

PROOF OF THEOREM 4.5. For $P \in Ob(\mathcal{S}_{\mathbb{Z}_p})$, let $\mathcal{I}(P)$, $\Psi_S(P)$ be the sets defined in the previous section. Since

$$K_1(\Lambda(G)) = \varprojlim_n K_1(\Lambda(G/V_n))$$

(cf. lemma 3.38), we get $\Psi_S(G) = \varprojlim_n \Psi_S(G/V_n)$. Since $\mathcal{I}(G) = \bigcup_n \mathcal{I}(G/V_n)$, it suffices to prove the theorem for all groups G/V_n, $n \in \mathbb{N}$. Since the image of γ in G/V_n is of finite order, we may assume that γ is of finite order.

We need to prove the conditions (i) and (ii) in the definition of Ψ_S. Let $\mathrm{N}_{m,n} : \Lambda(U_m/V_m)_S^\times \to \Lambda(U_n/V_m)_S^\times$ be the norm map and $p_{n,m} : \Lambda(U_n/V_n)_S^\times \to \Lambda(U_n/V_m)_S^\times$ be the projection map. For (i), we need to show

$$\mathrm{N}_{m,n}(\xi_m) = p_{n,m}(\xi_n) \text{ for } m \leq n,\ m, n \in \underline{\underline{c}}.$$

By definition 2.36 and lemma 2.14, this is equivalent to the equation

$$\xi_m(\mathrm{ind}_{U_m/V_m}^{U_n/V_m}(\chi \kappa_n^k)) = \xi_n(\mathrm{inf}_{U_n/V_n}^{U_n/V_m}(\chi \kappa_n^k))$$

for all characters $\chi : U_n/V_m \to \overline{\mathbb{Q}}^\times$ and all $k \geq 1$, where $\mathrm{inf}_{U_n/V_n}^{U_n/V_m}$ is defined by

$$\mathrm{inf}_{U_n/V_n}^{U_n/V_m}(\chi \kappa_n^k) : U_n \to \mathbb{C}^\times, \quad \sigma \mapsto (\chi \kappa_n^k)(\sigma V_n).$$

Equivalently,

$$L_\Sigma(1-k, \mathrm{ind}_{U_m/V_m}^{U_n/V_m}(\chi)) = L_\Sigma(1-k, \mathrm{inf}_{U_n/V_n}^{U_n/V_m}(\chi)).$$

But this is true by proposition 2.34.

We will now prove condition (ii):

$$\xi_n \varphi(\xi_{n-1})^{-1} \in 1 + I_{n,S} \quad \text{for all } n \in \underline{\underline{c}}$$

For $0 \leq m \leq n$, we put $\mathfrak{f}_m := \mathfrak{f}_n \cap \mathcal{O}_{F_m}$. Then

$$\Gamma_{00}(\mathfrak{f}_m) = \Gamma_{00}(\mathfrak{f}_n) \cap F_m.$$

We define

$$g_m := \mathrm{res}_{F_m|F}(E_m) \in M(\Gamma_{00}(\mathfrak{f}_0), K_m|F) \qquad \text{and}$$
$$\varphi(g_{n-1}) := \mathrm{res}_{F_{n-1}|F}(\mathrm{Ver}(E_{n-1})) \in M(\Gamma_{00}(\mathfrak{f}_0), K_n|F).$$

REMARK. If $g_{n-1} = \sum_x a_x q_F^x$, then $\varphi(g_{n-1}) = \sum_x \varphi(a_x) q_F^{px}$.

We will now determine the q-expansion of $g_n - \varphi(g_{n-1})$ by writing $\mathcal{R}$ as a disjoint union of subsets and by calculating the corresponding sums separately.

We define

$$\mathcal{R}_m := \Big\{(\mathfrak{a}, x) \in \mathcal{R} \Big| G(F_n|F_m) = \{\sigma \in G(F_n|F) | \sigma(\mathfrak{a}) = \mathfrak{a},\ \sigma(x) = x\}\Big\}.$$

Then $\mathcal{R} = \dot\bigcup_m \mathcal{R}_m$. For $1 \le m \le n$, define

$$\mathcal{R}'_m := \{(\mathfrak{b}, y) \in \mathcal{R}(F_m) | (\mathfrak{b} \ne \mathfrak{c}\mathcal{O}_{F_m} \text{ for all } \mathfrak{c} \in A_{m-1}) \text{ or } y \notin F_{m-1}\}.$$

and put $\mathcal{R}'_0 := \mathcal{R}(F_0)$. Let $(\mathfrak{b}, y) \in \mathcal{R}'_m$. For all $\sigma \in G(F_n|F_m)$, we have $\sigma(\mathfrak{b}\mathcal{O}_{F_n}) = \mathfrak{b}\mathcal{O}_{F_n}$ and $\sigma(y) = y$. If $m \ge 1$, there is $\sigma \in G(F_n|F_{m-1})$ such that $\sigma(\mathfrak{b}\mathcal{O}_{F_n}) \ne \mathfrak{b}\mathcal{O}_{F_n}$ or $\sigma(y) \ne y$. Hence we can define the map

$$\mathcal{R}'_m \to \mathcal{R}_m,\ (\mathfrak{b}, y) \mapsto (\mathfrak{b}\mathcal{O}_{F_n}, y).$$

By the above considerations, this map is bijective.

Define $\mathcal{R}''_m \subset \mathcal{R}'_m$ to be a set of representatives of $G(F_m|F)\backslash\mathcal{R}'_m$. Let $l = l(\mathfrak{b})$ be the largest element of $\{0, \dots, m\}$ such that $\sigma_{\mathfrak{b}} \in U_{m,l}/V_m$. Then

$$\beta\sigma_{\mathfrak{b}}\beta^{-1} = \gamma^{p^l t}\sigma_{\mathfrak{b}}$$

for some $t \in \mathbb{Z}_p^\times$. (We may assume $\sigma_{\mathfrak{b}} = \alpha^{p^l i}\beta^j z$ with $i \in \mathbb{Z}_p^\times$, $j \in p^m\mathbb{Z}_p$ and $z \in Z(U_{m,l}/V_m)$ and get the above identity from a direct calculation.) For $s \in G(F_m|F)$, clearly $\sigma_{s(\mathfrak{b})} = s\sigma_{\mathfrak{b}}s^{-1}$. Recall

$$G(F_m|F) = G/U_m = \{1, \bar\beta, \dots, \bar\beta^{p^m-1}\},$$

where $\bar\beta$ is the image of β in $G(F_m|F)$. Hence we get

$$\sum_{s \in G(F_m|F)} \sigma_{s(\mathfrak{b})} = \sum_{i=0}^{p^m-1} \beta^i\sigma_{\mathfrak{b}}\beta^{-i} = \sum_{i=0}^{p^m-1} \sigma_{\mathfrak{b}}\gamma^{p^l ti} = p^l\sigma_{\mathfrak{b}}h_{m,l},$$

where $h_{m,l} \in \Lambda(U_m/V_m)$ is defined as in the previous section. By lemma 4.8, $\varphi^{n-m}(\sigma_{\mathfrak{b}}) = \sigma_{\mathfrak{b}\mathcal{O}_{F_n}}$ for $\mathfrak{b} \in A_m$. Put $\mathrm{Tr}_n := \mathrm{Tr}_{F_n|F}$. Then

$$\begin{aligned} g_n =& 2^{-r(n)}\xi_n + \sum_{m=0}^{n} \sum_{(\mathfrak{b},y)\in\mathcal{R}''_m} \sum_{s\in G(F_m|F)} \kappa_n(\sigma_{\mathfrak{b}\mathcal{O}_{F_n}})^{-1}\sigma_{s(\mathfrak{b}\mathcal{O}_{F_n})} q_F^{\mathrm{Tr}_n(s(y))} \\ =& 2^{-r(n)}\xi_n + \sum_{m=0}^{n} \sum_{(\mathfrak{b},y)\in\mathcal{R}''_m} \sum_{s\in G(F_m|F)} \varphi^{n-m}\Big(\kappa_m(\sigma_{\mathfrak{b}})^{-p^{n-m}}\sigma_{s(\mathfrak{b})}\Big) q_F^{\mathrm{Tr}_n(s(y))} \\ =& 2^{-r(n)}\xi_n + \sum_{m=0}^{n} \sum_{(\mathfrak{b},y)\in\mathcal{R}''_m} p^{l(\mathfrak{b})}\varphi^{n-m}\Big(\kappa_m(\sigma_{\mathfrak{b}})^{-p^{n-m}}\sigma_{\mathfrak{b}} h_{m,l(\mathfrak{b})}\Big) q_F^{\mathrm{Tr}_n(y)}, \end{aligned}$$

Thus we get

$$\begin{aligned} g_n - \varphi(g_{n-1}) =& 2^{-r(n)}\xi_n - 2^{-r(n-1)}\varphi(\xi_{n-1}) \\ &+ \sum_{(\mathfrak{b},y)\in\mathcal{R}''_n} p^{l(\mathfrak{b})} h_{n,l(\mathfrak{b})}\kappa_n(\sigma_{\mathfrak{b}})^{-1}\sigma_{\mathfrak{b}} q_F^{\mathrm{Tr}_n(y)}. \end{aligned}$$

Here, all the non-constant coefficients in the q-expansion are elements of $I_{n,S}$. The q-expansion principle (cf. [**12**]) implies $\xi_n \equiv \varphi(\xi_{n-1}) \bmod I_{n,S}$. By lemma 3.56, this is equivalent to $\xi_n\varphi(\xi_{n-1})^{-1} \in 1+I_{n,S}$. Hence $(\xi_n)_n \in \Psi_S$. $\square$

Appendix: Commutative Main Conjecture

Assume that $F_\infty|F$ is an abelian field extension that satisfies assumption 2.1. Then the validity of the main conjecture is well known ([**52**] and [**24**]). We show here that our formulation is equivalent to another well-known formulation.

We assume $F = \mathbb{Q}$ and $F_\infty = \mathbb{Q}(\mu_{p^\infty})^+$. Then

$$G = G(F_\infty|F) \cong \mathbb{Z}_p^\times/\{\pm 1\} \cong (\mu_{p-1}/\{\pm 1\}) \times (1 + p\mathbb{Z}_p)$$

and hence G has no element of order p. Hence $\mathcal{H}_S^\Lambda(G) = \Lambda(G)\text{-mod}_{S\text{-tors}}$ and $K_0(\Lambda(G), \Lambda(G)_S) = K_0(\Lambda(G)\text{-mod}_{S\text{-tors}})$. The exact sequence (4) becomes

$$0 \to \Lambda(G)^\times \to \Lambda(G)_S^\times \xrightarrow{\partial} K_0(\Lambda(G)\text{-mod}_{S\text{-tors}}) \to 0,$$

where $\partial(f) = [\Lambda(G)/\Lambda(G)g] + [\Lambda(G)/\Lambda(G)s]$ for $f \in \Lambda(G)_S$ with $f = gs^{-1}$, $g \in \Lambda(G)$, $s \in S$. Then ∂ induces the isomorphism

$$\overline{\partial} : \Lambda(G)_S^\times/\Lambda(G)^\times \xrightarrow{\cong} K_0(\Lambda(G)\text{-mod}_{S\text{-tors}}).$$

We will construct an inverse homomorphism of $\overline{\partial}$. Let M be a finitely generated projective S-torsion module. By the structure theory of finitely generated torsion $\Lambda(G)$-modules, there is an exact sequence

$$0 \to \bigoplus_{i=1}^{r} \Lambda(G)/\Lambda(G)f_i \to M \to D \to 0,$$

where f_i are non-zero divisors of $\Lambda(G)$ and D is a module of finite cardinality. Let $F_M := f_1 \cdots f_r \in \Lambda(G)$ be the characteristic element of M. (F_M is defined up to units.)

We show that $[D] = 0 \in K_0(\Lambda(G)\text{-mod}_{S\text{-tors}})$. There is a commutative diagram

$$\begin{tikzcd}
K_1(\Lambda(G)_S) \arrow[r, "{[(M,f)]\mapsto[(pM,f|_{pM})]}"] \arrow[d] & K_1(\Lambda(G)_S) \arrow[d] \\
K_0(\Lambda(G)\text{-mod}_{S\text{-tors}}) \arrow[r, "{[M]\mapsto[pM]}"] & K_0(\Lambda(G)\text{-mod}_{S\text{-tors}}).
\end{tikzcd}$$

Since $p\Lambda(G)_S \cong \Lambda(G)_S$ and $K_0(\Lambda(G)_S)$ is generated by elements of the form $[(\Lambda(G)_S^n, f)]$, the top arrow is the identity. Hence the lower row is an isomorphism. Since there is $n \in \mathbb{N}$ such that $p^n D = 0$, we get that $[D] = 0$. Therefore,

$$\partial(F_M) = \sum_{i=1}^{r} \partial(f_i) = \left[\bigoplus_{i=1}^{r} \Lambda(G)/\Lambda(G) f_i \right] = [M]$$

and hence the inverse homomorphism of $\overline{\partial}$ is

$$K_0(\Lambda(G)\text{-mod}_{S\text{-tors}}) \to \Lambda(G)_S^\times/\Lambda(G)^\times, \quad [M] \mapsto F_M \bmod \Lambda(G)^\times.$$

Let $\mathrm{ch}_G(M) = F_M \Lambda(G)$ be the characteristic ideal of M. Then

$$[\Lambda(G)/\mathrm{ch}_G(M)] = \partial(F_M) = [M] \in K_0(\Lambda(G)\text{-mod}_{S\text{-tors}}).$$

Since G is abelian, every irreducible Artin representation ρ is already a character. It is well known that $F_\infty | F$ is unramified outside p. Hence $\Sigma \subset \mathbb{Z}$ is a finite set of primes with $p \in \Sigma$. We can write

$$L_\Sigma(s, \rho) = \prod_{q \notin \Sigma} (1 - \rho(\sigma_q) q^{-s})^{-1},$$

where $\sigma_q \in G$ is the element such that $\sigma_q \zeta = \zeta^q$ for all roots of unity $\zeta \in F_\infty$.

Let $X = X_\Sigma(F_\infty|F)$ be the Galois group of the maximal abelian p-extension of F_∞ unramified outside Σ. Let $I(G)$ the kernel of the augmentation map $\Lambda(G) \to \mathbb{Z}_p$. We call an element $f \in Q(G)$ a pseudo-measure if $(g-1)f \in \Lambda(G)$ for all $g \in G$. For the definition of the integral

$$\int_G \rho(g) df(g)$$

of an Artin character $\rho : G \to \overline{\mathbb{Q}}^\times$ against a pseudomeasure f, we refer to [**9**, §3.2]. We denote by

$$\kappa : G(\mathbb{Q}(\mu_{p^\infty})|\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z}_p^\times$$

the cyclotomic character.

Main Conjecture 5.1.

- *There is a unique element $\xi = \xi_\Sigma \in \Lambda(G)_S^\times$ such that*

$$\xi_\Sigma(\rho\kappa^r) = L_\Sigma(1 - r, \rho) \tag{37}$$

  for all continuous characters $\rho : G \to \overline{\mathbb{Q}}^\times$ and all even integers $r \geq 2$.
- $\partial(\xi_\Sigma) = [X_\Sigma] - [\mathbb{Z}_p]$

Main Conjecture 5.2.

- *There is a unique pseudo-measure $\xi \in Q(G)^\times$ such that*

$$\int_G \kappa(g)^r d\xi(g) = (1-p^{r-1})\zeta(1-r) \tag{38}$$

  for all even integers $r \geq 2$.
- $\mathrm{ch}_G(X) = I(G)\xi$

Remarks.
- In the proof of one of the above two conjectures, the second part constitutes the main difficulty. (The name "Main Conjecture" often denotes only this part.)
- For a proof of main conjecture 5.2, see [**9**].

Proposition 5.3. *Main conjecture 5.1 implies that main conjecture 5.2 holds.*

Proof. Assume that equation (37) holds for $\rho = \mathbf{1}$ (the trivial character) and $\Sigma = \{p\}$.

We first show that $\xi = \xi_{\{p\}}$ is a pseudo-measure. Let θ be a generator of $I(G)$ as a $\Lambda(G)$-module. Then

$$\partial(\theta) = [\Lambda(G)/I(G)] = [\mathbb{Z}_p],$$

and hence (by the assumption on ξ)

$$\partial(\xi\theta) = \partial(\xi) + \partial(\theta) = [X] = \partial(F_X).$$

Therefore $\frac{\xi\theta}{F_X} \in \Lambda(G)^\times$ and thus $\xi\theta \in \Lambda(G)$. Since

$$I(G) = \langle g-1 | g \in G\rangle_{\Lambda(G)},$$

we have

$$(g-1)\xi = \theta^{-1}(g-1)\cdot\xi\theta \in \Lambda(G) \quad \text{for all } g \in G,$$

i. e. ξ is a pseudo-measure.

By assumption, $\xi(\kappa^r) = L_\Sigma(1-r,\mathbf{1}) = (1-p^{r-1})\zeta(1-r)$. By definition of the integral, $\xi(\kappa^r) = \int_G \kappa^r\, d\xi$ for all positive even integers. By [**9**, lemma 4.2.2], this determines ξ uniquely.

We get the second statement of main conjecture 5.2 from the equivalences

$$\begin{aligned} [X] = [\mathbb{Z}_p] + \partial(\xi) \quad &\Leftrightarrow \quad \partial(F_X) = \partial(\theta\xi) \\ &\Leftrightarrow \quad F_X \equiv \theta\xi \bmod \Lambda(G)^\times \\ &\Leftrightarrow \quad \mathrm{ch}_G(X) = I(G)\xi. \end{aligned} \tag{39}$$

□

Proposition 5.4. *Assume that main conjecture 5.2 is true. Then equation* (37) *holds for* $\rho = \mathbf{1}$ *and* $\Sigma = \{p\}$. *The element* $\xi \in \Lambda(G)_S^\times$ *is determined uniquely by its values on* κ^r *for positive even integers* r. *The equation*

$$\partial(\xi) = [X_{\{p\}}] - [\mathbb{Z}_p]$$

holds.

Proof. Let g be a topological generator of G. Then the $\Lambda(H)$-module

$$\Lambda(G)/\Lambda(G)(g-1) \cong \mathbb{Z}_p$$

is finitely generated and hence $g - 1 \in S$. Since $\mu(F_\infty|F) = 0$ (see [**15**]) and since X and $\Lambda(G)/(g-1)\xi\Lambda(G)$ have the same μ-invariant, this implies that $p \nmid (g-1)\xi$. By lemma 2.4, this implies $(g-1)\xi \in S$. Therefore, $\xi \in \Lambda(G)_S^\times$.

Since $(1 - p^{r-1})\zeta(1-r) = L_{\{p\}}(1-r, \mathbf{1})$, we get

$$\xi(\kappa^r) = L_{\{p\}}(1-r, \mathbf{1})$$

for all positive even integers r. By assumption, these equations determine ξ uniquely.

The second statement of main conjecture 5.1 for $\Sigma = \{p\}$ follows from the equivalences (39). $\square$

Remark. In the above argument, we used the fact that $\mu(F_\infty|F) = 0$. This is only known when $F|\mathbb{Q}$ is abelian. For more general fields, we need the assumption $\mu(F_\infty|F) = 0$ to deduce proposition 5.4.

Proposition 5.5. *Let* $q \notin \Sigma$ *be a prime number and put* $\Sigma' := \Sigma \cup \{q\}$.

- *There is an element* $\pi_q \in Q(G)^\times$ *such that*

$$\pi_q(\rho\kappa^r) = 1 - \rho(\sigma_q)q^{r-1}$$

for all even integers $r \geq 2$.
- $\partial(\pi_q) = [X_{\Sigma'}] - [X_\Sigma]$

Proof. We put $\pi_q := 1 - \frac{1}{q}\sigma_q$. We only need to show that $\partial(\pi_q) = [X_{\Sigma'}] - [X_\Sigma]$.

Let $F_\Sigma := (F_\infty)_\Sigma(p)$ be the maximal pro-p extension of F_∞ unramified outside Σ. Recall that $G_\Sigma = G(F_\Sigma|F_\infty)$ and put $G_\Sigma^{\Sigma'} := G(F_{\Sigma'}|F_\Sigma)$. Since $H^2(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = H^{-2}(C^\bullet)^\vee = 0$ (see lemma 2.19), we get the exact five term sequence

$$0 \to H^1(G_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(G_{\Sigma'}, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(G_\Sigma^{\Sigma'}, \mathbb{Q}_p/\mathbb{Z}_p)^{G_\Sigma} \to 0.$$

Put $X_\Sigma^{\Sigma'} := H^1(G_\Sigma^{\Sigma'}, \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. We dualise the above exact sequence to get the exact sequence

$$0 \to (X_\Sigma^{\Sigma'})_{G_\Sigma} \to X_{\Sigma'} \to X_\Sigma \to 0.$$

Hence,

$$[X_{\Sigma'}] - [X_\Sigma] = [(X_\Sigma^{\Sigma'})_{G_\Sigma}].$$

By [**33**, 10.5.4],

$$H^1(G_\Sigma^{\Sigma'}, \mathbb{Q}_p/\mathbb{Z}_p) \cong \varinjlim_{F'} \bigoplus_{\mathfrak{q}|q} H^1(G_{F'_\mathfrak{q}}, \mathbb{Q}_p/\mathbb{Z}_p),$$

where the limit is over all finite extensions F' of F inside F_Σ, $\mathfrak{q}$ runs through the prime ideals of F' that divide q and $G_{F'_\mathfrak{q}}$ is the absolute Galois group of $F'_\mathfrak{q}$. Recall that $H_1(W, \mathbb{Z}_p) = W^{ab}(p)$ is the p-component of the abelianisation of W for any profinite group W. Then

$$X_\Sigma^{\Sigma'} = \varprojlim_{F'} \bigoplus_{\mathfrak{q}|q} G^{ab}_{F'_\mathfrak{q}}(p) = \bigoplus_{\mathfrak{q}|q} G^{ab}_{F_{\Sigma,\mathfrak{q}}}(p)$$

Since $F_{\Sigma,\mathfrak{q}}|F_q$ is unramified and $F_q^{ab}(p)|F_{\Sigma,\mathfrak{q}}$ is totally ramified ($F_q^{ab}(p)$ is the maximal abelian pro-p extension of the q-completion of F), we get that

$$G^{ab}_{F_{\Sigma,\mathfrak{q}}}(p) = T(F_q^{ab}(p)|F_q) =: T_q$$

is the inertia group of $F_q^{ab}(p)|F_q$.

Put $K_\Sigma := G(F_\Sigma|F)$ and $K_{\Sigma,\mathfrak{q}} := G(F_{\Sigma,\mathfrak{q}}|F_q)$. Let

$$G_\mathfrak{q} = G(F_{H,\mathfrak{q}'}|F_q) \subset \Gamma$$

be the decomposition group of $\mathfrak{q}' := \mathfrak{q} \cap F_H$ over F. There are the short exact sequences

$$1 \to G_\Sigma \to K_\Sigma \to G \to 1$$
$$1 \to G_{F_{H,\mathfrak{q}'}} \to K_{\Sigma,\mathfrak{q}} \to G_\mathfrak{q} \to 1.$$

We may regard $X_\Sigma^{\Sigma'}$ as a $\Lambda(K_{\Sigma,\mathfrak{q}})$-module and as a $\Lambda(K_\Sigma)$-module. Then

$$\begin{aligned}\bigoplus_{\mathfrak{r}|q} G^{ab}_{F_{\Sigma,\mathfrak{r}}}(p) \otimes_{\Lambda(K_\Sigma)} \Lambda(G) &= G^{ab}_{F_{\Sigma,\mathfrak{q}}}(p) \otimes_{\Lambda(K_{\Sigma,\mathfrak{q}})} \Lambda(G) \\ &= T_q \otimes_{\Lambda(K_{\Sigma,\mathfrak{q}})} \Lambda(G_q) \otimes_{\Lambda(G_q)} \Lambda(G)\end{aligned}$$

Equivalently,

$$(X_\Sigma^{\Sigma'})_{G_\Sigma} = \mathrm{ind}_G^{G_q}\left((T_q)_{G_{F_{H,\mathfrak{q}'}}}\right).$$

The ramification group of $F_q^{ab}(p)|F_q$ is pro-p and pro-q, hence it is trivial. Since $v_\mathfrak{q}(F_q^{ab}(p)^\times)/v_q(F_q^\times) = \mathbb{Q}_p/\mathbb{Z}_p$, [**31**, II, 9.15] implies

$$T_q \cong \mathrm{Hom}\left(\mathbb{Q}_p/\mathbb{Z}_p, \mu(F_q^{ab}(p))\right).$$

By lemma 2.26, $G_{\mathfrak{q}}$ is an open subgroup of Γ and hence $\langle\sigma_q\rangle = G_q \cong \mathbb{Z}_p$. Thus, since $\mu_{p^\infty} \subset F_q^{ab}(p)$ and $\mu_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}_p(1)$,

$$T_q = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathbb{Z}_p(1).$$

Since $\pi_q(x) = 0$ for $x \in \mathbb{Z}_p(1)$, we get

$$\begin{aligned} \partial(\pi_q) &= [\Lambda(G)/\pi_q\Lambda(G)] = [\mathbb{Z}_p(1) \otimes_{\Lambda(G_q)} \Lambda(G)] \\ &= [\operatorname{ind}_G^{G_{\mathfrak{q}}}(\mathbb{Z}_p(1)_{G_{F_{H,\mathfrak{q}'}}})] = [(X_\Sigma^{\Sigma'})_{G_\Sigma}]. \end{aligned}$$ □

Proposition 5.6. *Assume that the main conjecture 5.1 is true for $\Sigma = \{p\}$. Then it is true for any finite set Σ of primes of F with $p \in \Sigma$.*

Proof. Assume that we have proven this for some set Σ and set $\Sigma' = \Sigma \cup \{q\}$ for a prime $q \notin \Sigma$. By proposition 5.5, there is an element $\pi_q \in Q(G)^\times$ such that

$$\pi_q(\rho\kappa^r) = L_\Sigma(1-r,\rho)/L_{\Sigma'}(1-r,\rho)$$

for all even $r \geq 2$. Define $\xi_{\Sigma'} := \xi_\Sigma \pi_q^{-1} \in Q(G)^\times$. Then $\xi_{\Sigma'}(\rho\kappa^r) = L_{\Sigma'}(1-r,\rho)$ for all even $r \geq 2$.

The second assertion follows from

$$\partial(\xi_{\Sigma'}) = \partial(\xi_\Sigma) - \partial(\pi_q) = [X_{\Sigma'}] - [\mathbb{Z}_p]$$

(cf. proposition 5.5). □

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Erklärung

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