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Diplomarbeit

Syntomic Cohomology

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Contents

Preface

Modern algebraic geometry knows a large number of different topologies. Most of them have been developed to compute certain cohomology groups. The syntomic cohomology is one of the less known topologies. It has proven useful in various cases, as it is able to catch more details without becoming impossible to calculate. It was Mazur who first came up with the notion of syntomic morphisms, probably while looking for alternatives to the flat cohomology that are fine enough to catch interesting structure but still not too complicated to calculate. Fontaine and Messing used the syntomic topology in [FM87] to prove some results on p-adic periods. In [Bau92] Bauer gives a proof for the conjecture of Birch and Swinnerton-Dyer for abelian varieties with good reduction everywhere over function fields in characteristic p. Kato extended this notion to the log-syntomic topology to generalize Bauer's result to handle bad reduction as well in [KT03]. Today, syntomic cohomology is used, e.g., in the theory of syntomic regulators. They are an p-adic analogue of Beilinson's regulators and give raise to connections between syntomic cohomology and K-theory (see, e.g., [Niz12])

However, the basic foundations of this topology have not yet been fully collected in a concise manner. This thesis aims at presenting the foundations of syntomic topology and cohomology at least as far as they are necessary for the proof of the conjecture of Birch and Swinnerton-Dyer in [Bau92].

This thesis has three main parts: The first one consists of chapters 1–3. Here the notion of syntomic morphisms is developed. It forms the constitutive element of the topology underlying the syntomic cohomology. This part uses mainly material from [SGA6, Exp. VII], [EGAIV.4] and [Koe89]. It is extended by some propositions pointing out the simplest applications of the definitions that should make understanding the basic ideas easier (e.g., 1.1.17, 1.1.20, and 1.2.11). Furthermore, a classification of generating sequences of a Koszul-regular ideal has been proven (1.1.14 and 1.1.15) and some propositions on the behaviour of sequence-regular immersions on stalks have been given (1.2.13, 1.2.15).

Chapter 1 defines and explores *regular immersions*: A special kind of immersion, which has some good behaviour, e.g., with respect to codimension. There are two different types of regular immersions – here called *Koszul-regular immersion* and *sequence-regular immersion* – which are compared. In some cases they coincide. For this fact a proof will be given, as well as counter examples for other cases.

Chapter 2 is about *local complete intersections*. Local complete intersections are morphisms which factor through a regular immersion and a smooth morphism. In particular, syntomic morphisms are local complete intersections. As in the case of regular immersions, there are two different definitions which will be compared. For syntomic morphisms they will turn out to coincide.

Chapter 3 finally collects the notions introduced so far in order to define syntomic morphisms and gives some important properties that will be used later. In particular, some criteria for a morphism to be syntomic will be given. These criteria will be crucial later to see that important morphisms are syntomic coverings, e.g., the p-multiplication on a smooth group scheme (if it is faithfully flat).

Chapters 4–6 constitute the second part of this thesis. Here the notions of syntomic sites and of syntomic cohomology are developed. Chapter 4 collects some important statements about Grothendieck topologies that will be employed later to understand syntomic sites. These statements originate mainly from [SGA3.1] and are reformulated here using the nowadays more popular notion of covering families instead of sieves. Additionally, some propositions on the relation between covering morphisms and surjective morphisms of sheaves have been added.

Chapter 5 introduces the different syntomic sites. Apart from the usual distinction between big and small sites there will be the syntomic site of a scheme S and the crystalline-syntomic site, the first one being based on the category $\mathrm{Sch}(S)$ like the étale and the flat site, the latter one being a refinement of the crystalline site. Also, a comparison morphism will be constructed from the crystalline-syntomic to the syntomic topos. This will allow us to calculate crystalline cohomology using the syntomic site. One of the most important features of the syntomic sites will be that the Kummer sequence is exact in characteristic p, too. The material in this chapter as well as in the next one is mostly based on [Koe89], but has been extended and generalized to allow a broader range of base schemes as is needed for the application in the conjecture of Birch and Swinnerton-Dyer later on: [Koe89] mostly works on the syntomic site of $\operatorname{Spec} W_s(k)$ for k a perfect field of characteristic p. This is generalized here to the site of syntomic schemes over $W_s(k)$ and in some parts even to syntomic schemes over a noetherian ring R.

Chapter 6 defines the crucial syntomic sheaves $\mathcal{O}_n^{\mathrm{cris}}$ using the comparison morphism of the previous chapter. The construction of these sheaves, resembling universal crystalline coverings, will be given in detail. Also, theses sheaves will be used to construct some important exact sequences.

The last part, chapter 7, outlines the proof of the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic p as given by [Bau92]. It shows how the syntomic sheaves $\mathcal{O}_n^{\mathrm{cris}}$ are used to calculate the cohomology groups appearing in the L-function of an abelian variety.

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Notation: In this thesis, p always denotes a prime, which will be arbitrary apart from the last chapter, where it will be assumed to be odd. Let G be an abelian group. Then we define ${}_{p^n}G := \ker(p^n : G \to G)$, the p-part $G(p) := \varinjlim_n {}_{p^n}G$ where the limit is taken over the inclusion maps ${}_{p^n}G \hookrightarrow {}_{p^{n+1}}G$, and $T_pG := \varprojlim_n {}_{p^n}G$ where the limit is taken over the maps $p : {}_{p^{n+1}}G \to {}_{p^n}G$. $\operatorname{Tor} G$ denotes the torsion subgroup and we let $G_{\mathrm{Tor}} := G/\operatorname{Tor} G$. All rings are commutative. For the sake of readability, the Spec functor is often omitted. For example, by $X \times_Y \kappa(y)$ we denote $X \times_Y \operatorname{Spec} \kappa(y)$.

1. Regular Immersions

1.1. Koszul-Regular Immersions

The key element in the definition of the syntomic site is the notion of *regular immersion*. It generalizes the idea of factoring out a non-zero divisor and in nice cases it is well-behaved with respect to codimension (1.2.10). In order to define it in the generality as needed for our purpose, the *Koszul complex* – a tool from homological algebra – is used. The Koszul complex was first introduced by the French mathematician Jean-Louis Koszul (a member of the second generation of Bourbaki). He originally used the Koszul complex to construct a cohomology theory for Lie-algebras. As a reminder the definition of the Koszul complex and some important facts will be given. For a more complete introduction see [EGAIII.1, 1.1], [Eis99, 17], and [Sta, 12.23ff.].

Definition 1.1.1 ([SGA6, Exp. VII, 1.1])**.** Let R be a ring, E a projective finitely generated R-module, and $u : E \to R$ an R-linear homomorphism, i.e., an element of the dual space $E^\vee$. Then the *Koszul complex* $K_\bullet(u)$ associated to u is defined as follows: Its i-th degree is defined to be the i-th grade of the exterior algebra $\bigwedge(E)$, i.e., $K_i(u) = \bigwedge^i(E)$. Its boundary operator ∂ is defined to be the inner multiplication by u where u is considered as element of the dual space $E^\vee$, i.e., $x_1 \wedge \cdots \wedge x_n \mapsto \sum_{k=1}^n (-1)^{k+1} u(x_k) x_1 \wedge \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_n$. A short calculation shows that $\partial^2 = 0$. This is an antiderivation of degree -1 on $\bigwedge(E)$ and is equal to u on $\bigwedge^1(E) = E$. The Koszul complex is generalized in the obvious way to ringed spaces.

In many cases the following definition is sufficiently flexible and eases notation a lot:

Definition 1.1.2. Let R be a ring and $f = (f_1, \dots, f_r)$ a sequence[1] of elements in R. The Koszul complex $K_\bullet(f, R)$ is defined as the Koszul complex of $E = R^r$ and $u : R^r \to R$ given by $e_i \mapsto f_i$ with the canonical base $\{e_1, \dots, e_r\}$ of R^r: We let $K_\bullet(f, R) := K_\bullet(u)$.

Now let M be an R-module. Then we define $K_\bullet(f, M) := K_\bullet(f, R) \otimes_R M$. This is in accordance with the explicit definition given in [EGAIII.1, 1.1.1]. The homology groups $H_i(K_\bullet(f, M))$ are denoted by $H_i(f, M)$.

Remark 1.1.3 ([EGAIII.1, 1.1.3.5])**.** One has

$$H_0(f, M) = M/(\sum_i f_i M) = R/(f) \otimes_R M.$$

[1] Of course, such an object should be called r-tuple more appropriately. However, in the context of regular immersions, the notion of sequences has become common for these objects. Thus, this thesis will stick with this notation. Also, the enclosing brackets will often be suppressed to ease notation.

The Koszul complex has the following structure:

Remark 1.1.4. Let $f = (f_1, \ldots, f_r)$ be a sequence of elements in R, $E = R^r$ with the canonical base $\{e_1, \ldots, e_r\}$, and u as above. Then we have $K_i(f, R) = \bigwedge^i(R^r)$. The boundary operator ∂ is

$$\begin{aligned} \partial : \bigwedge^{n}(E) &\to \bigwedge^{n-1}(E) \\ x_1 \wedge \cdots \wedge x_n &\mapsto \sum_{k=1}^{n} (-1)^{k+1} u(x_k) x_1 \wedge \ldots \widehat{x_k} \cdots \wedge x_n. \end{aligned}$$

In particular, we have $\partial(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \sum_{k=1}^{n} (-1)^{k+1} f_{i_k} e_{i_1} \wedge \ldots \widehat{e_{i_k}} \cdots \wedge e_{i_n}$.
In the special case $f = (f_1)$ we get the Koszul complex

$$0 \longrightarrow R \xrightarrow{f_1 \cdot} R \longrightarrow 0.$$

With this considerations, one can verify that for general $f = (f_1, \ldots, f_r)$ one has an isomorphism $K_\bullet(f, R) \cong K_\bullet(f_1, R) \otimes \cdots \otimes K_\bullet(f_r, R)$. In this sense, the Koszul complex is independent of the ordering of f.

Useful for inductions involving the Koszul complex is the following homological lemma:

Lemma 1.1.5. *Let $K_\bullet$ be a chain complex of free R-modules, concentrated in degrees 0 and 1. Then for every chain complex $L_\bullet$ of R-modules and for every index $k \geq 0$ there is an exact sequence*

$$0 \longrightarrow H_0(K_\bullet \otimes H_k(L_\bullet)) \longrightarrow H_k(K_\bullet \otimes L_\bullet) \longrightarrow H_1(K_\bullet \otimes H_{k-1}(L_\bullet)) \longrightarrow 0,$$

where we set $H_{-1}(L_\bullet) = 0$.

Proof. See [EGAIII.1, 1.1.4.1] or [Wei94, 4.5.3].

Now the necessary notions have been defined and we can continue with the definition of Koszul-regularity for morphisms, sequences, and ideals. The term *Koszul-regular* goes back to T. Kabele in [Kab71].

Definition 1.1.6.

(i) Let R be a ring, E a projective finitely generated R-module, and $u : E \to R$ an R-linear homomorphism. Then u is called *Koszul-regular* – or simply *regular* if no ambiguity is possible – if the Koszul complex $K_\bullet(u)$ is acyclic in positive degrees and thus is a resolution of $B = R/u(E)$.

(ii) Let $f = (f_1, \ldots, f_r)$ a sequence of elements in R, and M an R-module. The sequence f is called *M-Koszul-regular*, if its Koszul complex $K_\bullet(f, M)$ is acyclic positive degrees. For $M = R$, the term *Koszul-regular* is used.

(iii) An ideal $I \subset R$ is called *Koszul-regular* if it is generated by a Koszul-regular sequence. Then the Koszul complex of the generating sequence is a resolution of R/I.

Lemma 1.1.7. *An ideal $I \subset R$ is Koszul-regular if and only if there is a finitely generated free R-module E and a Koszul-regular homomorphism $u : E \longrightarrow R$ with* $\operatorname{im}(u) = I$. *In fact, this is the definition of Koszul-regular ideals given in [SGA6, 1.4].*

Proof. Let I be Koszul-regular. By definition, there is a Koszul-regular sequence $f_1, \ldots, f_r \in R$. By choosing $E = R^r$ with the canonical base $\{e_1, \ldots, e_r\}$ and $u : e_i \mapsto f_i$, the morphism u is Koszul-regular by 1.1.6 and by construction one has $\operatorname{im}(u) = I$.

Let E be a finitely generated free R module and $u : E \to R$ Koszul-regular with $\operatorname{im}(u) = I$. Let $\{e_1, \ldots, e_r\}$ be a basis of E. Define $f_i := u(e_i)$. Then the f_i generate I and they form a Koszul-regular sequence by the definition of Koszul-regular sequence. □

Example 1.1.8. Let $f = (f_1)$. Then f is M-Koszul-regular if and only if $H_1(f, M) = 0$. But as we have $H_1(f, M) = ker(\cdot f_1 : M \to M)$ (1.1.4), this is the case if and only if f_1 is a non zero-divisor in M. For f of length greater than one, this can be extended only to a sufficient condition (see 1.3.1), the *sequence-regular sequences* as defined in 1.2.1.

Lemma 1.1.9 ([Sta, 12.23.4])**.** *Let R be a ring, $f_1, \ldots, f_r \in R$, and (c_{ij}) an invertible $r \times r$-matrix with coefficients in R. Then the Koszul complexes $K_\bullet(f_1, \ldots, f_r)$ and $K_\bullet(\sum_i c_{i1} f_1, \ldots, \sum_i c_{ir} f_r)$ are isomorphic.*

Lemma 1.1.10 ([Sta, 12.24.4])**.** *Let R, S be rings, and $\varphi : R \to S$ a flat map of rings. If a sequence $(f_1, \ldots, f_r)$ in R is Koszul-regular, then so is the sequence $(\varphi(f_1), \ldots, \varphi(f_r))$ in S.*

Now we want to prove a result on the classification of Koszul-regular generating sequences of an ideal. This final result will be in 1.1.14 and 1.1.15. .

Lemma 1.1.11. *Let $I \subset R$ be an proper ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$. Then I/I^2 is a free R/I-module of rank r and the images of $f_1, \ldots, f_r$ in I/I^2 form a base of this R/I-module.*

Proof. Let $f = (f_1, \ldots, f_r)$. Since f is Koszul-regular, we have $H_1(f) = 0$. This shows that the following part of the Koszul complex $K_\bullet(f)$ is exact:

$$\bigwedge^2(R^r) \xrightarrow{\partial_2} R^r \xrightarrow{\partial_1} R,$$

where $\partial_2(e_i \wedge e_j) = f_i e_j - f_j e_i$ and $\partial_1(e_i) = f_i$ with the canonical base $\{e_1, \ldots, e_r\}$ of R^r. The image of ∂_1 equals I. Therefore, this induces an exact sequence of R-modules

$$\bigwedge^2(R^r) \xrightarrow{\partial_2} R^r \xrightarrow{\partial_1} I \longrightarrow 0.$$

Tensoring with R/I, we get an exact sequence of R/I-modules

$$\bigwedge^2(R^r) \otimes_R R/I \xrightarrow{\partial_2'} R^r \otimes_R R/I \xrightarrow{\partial_1'} I \otimes_R R/I \longrightarrow 0.$$

Here we have for $a \in R/I$ that $\partial_2'(e_i \wedge e_j \otimes a) = e_i f_j \otimes a - e_j f_i \otimes a = e_i \otimes f_j a - e_j \otimes f_i a = 0$ and therefore ∂_1' is an isomorphism $R^r \otimes_R R/I \to I \otimes_R R/I$. Since the tensor product commutes with direct sums, we have $R^r \otimes_R R/I \cong (R/I)^r$ and by A.1.1 we get $I \otimes_R R/I \cong I/I^2$. Therefore, I/I^2 is a free R/I-modules of rank r. The images of $f_1, \ldots, f_r$ form a base, as we have $\partial_1(e_i) = f_i$. □

Corollary 1.1.12. *Two Koszul-regular sequences generating the same ideal have the same length.*

Lemma 1.1.13 ([Sta, 12.24.13])**.** *Let R be a ring, $I \subset R$ an ideal generated by $f_1, \ldots, f_r$. If I can be generated by a Koszul-regular sequence of length r, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.*

Corollary 1.1.14. *Every set of generators for a Koszul-regular ideal is Koszul-regular if and only if it has the minimal possible length (of all sets of generators).*

Proof. Let I be an Koszul-regular ideal generated by a Koszul-regular sequence $g_1, \ldots, g_s$. Furthermore, let $f_1, \ldots, f_r$ be another sequence generating I. As the images of the elements f_i in I/I^2 form a generating system of I/I^2, by 1.1.11 we have $r \geq s$. Since the sequence is assumed to be of minimal length, we have $r \leq s$ and therefore $r = s$. Thus, the sequence $f_1, \ldots, f_r$ is Koszul-regular by 1.1.13.

Let $f_1, \ldots, f_r$ be Koszul-regular. Then by 1.1.12 we have $r = s$. Any generating sequence has to have at least s elements, as the images of the generating sequence in I/I^2 have to generate an free module of rank r. Therefore, the sequence $f_1, \ldots, f_r$ is of the minimal possible length. □

Corollary 1.1.15. *Let $I \subset R$ be an ideal. Then I is Koszul-regular if and only if exactly the generating sequences of minimal possible length are Koszul-regular.*

The definitions of Koszul-regularity can be generalized to ringed spaces and schemes:

Definition 1.1.16 ([SGA6, Exp. VII, 1.4])**.**

(i) Let $(X, \mathcal{O}_X)$ be a locally ringed space, $\mathcal{E}$ an $\mathcal{O}_X$-module which is locally free and finitely generated, and $u : \mathcal{E} \to \mathcal{O}_X$ an $\mathcal{O}_X$-linear homomorphism. Then u is called *Koszul-regular*, if the Koszul complex $K_\bullet(u)$ is acyclic positive degrees.

(ii) Let $\mathcal{J}$ be an ideal in $\mathcal{O}_X$. Then $\mathcal{J}$ is called *Koszul-regular* if there is, locally in X, a locally free and finitely generated $\mathcal{O}_X$-module $\mathcal{E}$ and a surjective Koszul-regular homomorphism $\mathcal{E} \to \mathcal{J}$, i.e., a Koszul-regular homomorphism $\mathcal{E} \to \mathcal{O}_X$ whose image is $\mathcal{J}$.

(iii) Let $i : X \to Y$ be an immersion of schemes, let U be open in Y with $i(X) \subset U$ such that i is a closed immersion from X to U. Then i is called *Koszul-regular*, if the ideal defined by $i(X)$ in $\mathcal{O}_U$ is Koszul-regular (this condition does not depend on U). Let $x \in X$. Furthermore, i is called *Koszul-regular in x* if there is a neighbourhood V of $i(x)$, such that the immersion $i|_{i^{-1}(V)} : i^{-1}(V) \to V$ is Koszul-regular.

Note that an immersion is Koszul-regular if and only if it is Koszul-regular in every point.

Proposition 1.1.17. *Let $Y = \operatorname{Spec} B$ be an affine scheme, $I \subset B$ a finitely generated ideal, and $i : X = \operatorname{Spec} B/I \to Y$ the closed immersion defined by I. If $I \subset B$ is a Koszul-regular ideal (1.1.6), then i is a Koszul-regular immersion.*

Proof. The Koszul complex associated with the ideal sheaf $\tilde{I} \subset \mathcal{O}_Y$ generated by the ideal $I \subset B$ consists of free and finitely generated modules which are therefore quasi-coherent. Thus, whether the Koszul complex is acyclic depends only on the global section ([GW10, 7.14]). □

Remark 1.1.18 ([SGA6, Exp. VII, 1.4.1–3])**.** Let X be a scheme, $\mathcal{J}$ a finitely generated ideal in $\mathcal{O}_X$, and x a point in $\operatorname{supp}(\mathcal{O}_X/\mathcal{J})$. A family of sections $f_1, \dots, f_d$ of $\mathcal{J}$ in a neighbourhood of x is said to be a *minimal generating system of $\mathcal{J}$ in a neighbourhood of x*, if the germs $(f_i)_x$ form a minimal generating system of $\mathcal{J}_x$, i.e., the images in $\mathcal{J}_x \otimes \kappa(x)$ form a base of this $\kappa(x)$-vector space. In this case the sections f_i are said to generate $\mathcal{J}$ in a neighbourhood of x.

With these terms an equivalent criterion for Koszul-regularity can be given: An immersion defined on a certain open subset of X by a finitely generated ideal $\mathcal{J}$ is Koszul-regular if and only if for every point $x \in \operatorname{supp}(\mathcal{O}_X/\mathcal{J})$ every minimal generating system of $\mathcal{J}$ in a neighbourhood of x is Koszul-regular (this follows immediately from 1.1.9).

Given only that $\mathcal{J}$ is finitely generated and $\mathcal{J}/\mathcal{J}^2$ is locally free and finitely generated, one notes that every minimal generating system of $\mathcal{J}$ in a neighbourhood of a point y in Y induces a base of $\mathcal{J}/\mathcal{J}^2$ in a neighbourhood of y. On the other hand, given a base of $\mathcal{J}/\mathcal{J}^2$ in a neighbourhood of y, it can be lifted to a family of sections of J in a neighbourhood of y. These sections generate $\mathcal{J}_y$ by the Nakayama lemma, and therefore they generate $\mathcal{J}$ in a neighbourhood of y, as $\mathcal{J}$ is finitely generated. Furthermore this generating system is minimal because it induces a base of $\mathcal{J}/\mathcal{J}^2$.

Proposition 1.1.19 ([SGA6, Exp VII 1.5])**.** *Let $i : X \to Y$ be an immersion. The following statements are equivalent:*

(a) The immersion i is Koszul-regular.

(b) All faithfully flat base changes of i are Koszul-regular.

(c) The immersion i is Koszul-regular locally in the faithfully flat quasi-compact topology.

Example 1.1.20. Let k be a field. The closed immersion $\operatorname{Spec} k \to \operatorname{Spec} k[X]$ is Koszul-regular as the ideal (X) is generated by a non zero-divisor in $\operatorname{Spec} k[X]$. The closed immersion $\operatorname{Spec} k \to \operatorname{Spec} k[X]/(X^2)$, however, cannot be Koszul-regular: The ideal (X) is generated by a zero-divisor on $\operatorname{Spec} k[X]/(X^2)$ and thus it cannot be

Koszul-regular, as by 1.1.14 every generating sequence of a Koszul-regular ideal which is of minimal length is Koszul-regular. Since $\operatorname{Spec} k[X]/(X^2)$ has no non-trivial open subset, the ideal (X) cannot be generated by a Koszul-regular sequence on an open covering, too.

Definition 1.1.21. Let $i : X \to Y$ be an immersion defined in an open subset $U \subset Y$ by an ideal $\mathcal{J} \subset \mathcal{O}_Y|_U$. The $\mathcal{O}_X$-module $i^*(\mathcal{J}/\mathcal{J}^2)$ (which does not depend on U) is called the *conormal sheaf* of X in Y and denoted by $\mathcal{C}_{X/Y}$. If i is regular, then $\mathcal{C}_{X/Y}$ is locally free and finitely generated and its rank is called the *codimension of* X *in* Y or the *codimension of* i. Note that the sheaf $\mathcal{C}_{X/Y}$ is denoted by $\mathcal{N}_{Y/X}$ in [SGA6, Exp. VII].

Remark 1.1.22. In the noetherian case, the codimension of a regular immersion $X \to Y$ does in fact coincide with the codimension of X in Y (see 1.2.10).

Using the conormal sheaves it is possible to state some facts about composition of regular immersions: Given two closed immersions $X \xrightarrow{j} Y \xrightarrow{i} Z$ the conormal sheaves lead to the exact sequence

$$0 \longrightarrow j^*(\mathcal{C}_{Y/Z}) \longrightarrow \mathcal{C}_{X/Z} \longrightarrow \mathcal{C}_{X/Y} \longrightarrow 0.$$

Proposition 1.1.23 ([SGA6, Exp. VII, 1.7])**.** *Let $j : X \to Y$ and $i : Y \to Z$ be two immersions.*

(i) If i and j are Koszul-regular, then $i \circ j$ is Koszul-regular and the sequence

$$0 \longrightarrow j^*(\mathcal{C}_{Y/Z}) \longrightarrow \mathcal{C}_{X/Z} \longrightarrow \mathcal{C}_{X/Y} \longrightarrow 0$$

is exact. In particular $\operatorname{codim}(i \circ j) = \operatorname{codim}(i) + \operatorname{codim}(j)$.

(ii) If $i \circ j$ and i are Koszul-regular and the above sequence is exact and splits, then j is Koszul-regular.

(iii) If Z is noetherian, and $i \circ j$ as well as j are Koszul-regular, then i is Koszul-regular in the points of $j(X)$.

Remark 1.1.24. It is not easy to find non-trivial examples of sequences that are not Koszul-regular. However, e.g., [Eis99, Exercise 17.2] gives a criterion for disproving Koszul-regularity.

1.2. Sequence-Regular Immersions

There is another definition of regularity, which is in the noetherian case equivalent to Koszul-regularity. It is sometimes easier to work with and it shows the geometrical idea behind regular immersions in a better way. This definition has been given in [EGAIII.1] earlier than the above definition, but behaves in an unwanted way in non-noetherian contexts. This will be discussed later in detail.

Definition 1.2.1. Let R be a ring, $a_1, \ldots, a_n$ a sequence of elements in R, and M an R-module. These elements are called an *M-sequence-regular sequence*, if a_1 is a non zero-divisor in M and for all $i \geq 2$ the element a_i is a non zero-divisor in $M/(a_1M + \cdots + a_{i-1}M)$, i.e., the sequence consisting only of a_i is $M/(a_1M + \cdots + a_{i-1}M)$-regular. The sequence is said to be *sequence-regular* (or just *regular* if there is no ambiguity possible), if it is R-regular.

An ideal $I \subset R$ is called a *sequence-regular ideal*, if it is generated by a sequence-regular sequence.

Remark 1.2.2. Some authors additionally require $M/\sum a_iM \neq 0$ for sequence-regularity, e.g., [Eis99].

Example 1.2.3.

(i) For a ring R, the coordinates $X_1, \ldots, X_n$ form a regular sequence in the polynomial ring $R[X_1, \ldots, X_n]$.

(ii) The sequence consisting only of the element X is regular in $k[X]$, but is not in $k[X]/X^2$, as X is a zero-divisor there.

(iii) Let $A = \mathbb{C}[X, Y]$. Then $a_1 = X$, $a_2 = Y + X$ is a regular sequence, but $b_1 = XY$, $b_2 = X$ is not. Here already a relation to the notion of codimension of schemes can be observed: While $A/(a_1, a_2)$ has codimension 2, $A/(b_1, b_2)$ has codimension 1, although two elements are factored out.

(iv) In a regular noetherian local ring every coordinate system makes up a regular sequence([Liu02, Remark 6.3.2]).

(v) In $\mathbb{Z}$ the sequences $(30, 4)$ and $(4, 30)$ are sequence-regular. But in $\mathbb{Z}/30$ the sequences $(3, 5)$ and $(5, 3)$ both are not sequence-regular, as they consist of zero-divisors.

Remark 1.2.4. While in general the regularity of a sequence depends on its order, this is not the case for a sequence in the maximal ideal of a noetherian local ring (1.3.5).

Lemma 1.2.5 ([Liu02, 6.3.6])**.** *Let R be a ring and $I \subset R$, $I \neq R$ be an ideal*[2] *generated by a sequence-regular sequence $a_1, \ldots, a_n$. Then the images of the elements a_i in I/I^2 form a basis of I/I^2 over A/I. In particular, I/I^2 is a free A/I-module of rank n.*

Corollary 1.2.6. *This shows in particular that for a sequence-regular ideal I all sequence-regular generating sequences have the same length.*

Definition 1.2.7. Let Y be a scheme and $f : X \to Y$ an immersion defined by an ideal $\mathcal{I}$ on an open subscheme of Y. Then f is a *sequence-regular immersion in $x \in X$ (of codimension n in x)*, if there is a neighbourhood U of $f(x)$ such that $\mathcal{I}|_U$ is generated

[2] Actually, [Liu02] includes the case $I = R$, but in this case one gets the zero module over the zero ring and the statement of the lemma does not hold any longer.

by a sequence-regular sequence (of length n) in $\mathcal{O}_Y(U)$. The immersion f is called a *sequence-regular immersion*, if it is sequence-regular in all $x \in X$. The codimension is well-defined by corollary 1.2.6.

Proposition 1.2.8 ([EGAIV.4, 19.1.5])**.**

(i) *An immersion $j : X \longrightarrow Y$ is an open immersion if and only if it is a sequence-regular immersion and of codimension 0 in every point.*

(ii) *Let $f : X \to Y$ be a sequence-regular immersion and $g : Y' \to Y$ flat. Then $f' : X \times_Y Y' \to Y'$ is a sequence-regular immersion as well. The codimensions of f' in $y' \in Y'$ and of f in the corresponding point $y \in Y$ coincide.*

(iii) *Sequence-regular immersions are stable under composition.*

Proposition 1.2.9. *Let S be a scheme and let X, Y be S-schemes with structure morphisms $g : X \to S$ and $h : Y \to S$ locally of finite presentation. Let $u : X \to Y$ be an S-immersion. Let $x \in X$, $y \in Y$ be points with $g(x) = h(y) = s \in S$ and $X_s = g^{-1}(s)$, $Y_s = h^{-1}(s)$ the corresponding fibers. Then the following statements are equivalent:*

(a) *The morphism g is flat in a neighbourhood of x and the immersion u is sequence-regular in x.*

(b) *The morphisms g and h are flat in neighbourhoods of x and y and the induced immersion $u_s : X_s \to Y_s$ is sequence-regular in x.*

Proof. See [EGAIV.4, 19.2.4]. □

The following proposition shows, why the notion of codimension of a regular immersion does in fact correspond to the geometric notion of codimension:

Proposition 1.2.10 ([Liu02, 6.3.11 (b)])**.** *Let $i : X \longrightarrow Y$ be a closed regular immersion of codimension n. Then for any irreducible component Y' of Y one has* $\operatorname{codim}(X \cap Y', Y') = n$ *whenever $X \cap Y' \neq \emptyset$. Furthermore, one has* $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,i(x)} - n$ *for all $x \in X$.*

Example 1.2.11. Let R_1, R_2 be rings and consider the ring $R = R_1 \times R_2$. One has $f = (1, 0) \in R$. Obviously, f is a zero-divisor. However, the closed immersion $i : \operatorname{Spec} R/f \to \operatorname{Spec} R$ is sequence-regular: One has $\operatorname{Spec} R/f = \operatorname{Spec} R_2$ which is an open and closed subset of $\operatorname{Spec} R$. Therefore, the morphism i is an open immersion as well and thus is trivially sequence-regular. This shows that even dividing out a zero-divisor can be a sequence-regular immersion.

Lemma 1.2.12. *Let A be a ring and $f \in A$ a non zero-divisor. Then f is a non zero-divisor in $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} A$ and $\frac{f}{1} \cdot \frac{a}{b} = 0$ in $A_{\mathfrak{p}}$ with $b \notin \mathfrak{p}$. Then there is a $t \notin \mathfrak{p}$ with $f \cdot a \cdot t = 0$ in A. As f is a non zero-divisor, this implies $a \cdot t = 0$ and hence $\frac{a}{b} = 0$ in $A_{\mathfrak{p}}$. □

Proposition 1.2.13. *Let $f : X \to Y$ be a sequence-regular immersion. Then for all $x \in X$ the ideal $\ker(\mathcal{O}_{Y,f(y)} \to \mathcal{O}_{X,x})$ is sequence-regular.*

Proof. This follow by induction from the above Lemma 1.2.12. □

The following two statements show that in the noetherian case even the inverse of 1.2.13 holds: Sequence-regular sequences on the stalks can be lifted to a neighbourhood.

Lemma 1.2.14 (Lifting of non zero-divisors)**.** *Let A be a noetherian ring, $\mathfrak{p} \in \operatorname{Spec} A$ a prime ideal, and $f \in A$ such that the induced $f \in A_\mathfrak{p}$ is not zero. Let the annulator of f in $A_\mathfrak{p}$ be trivial, i.e., $\operatorname{Ann}_{A_\mathfrak{p}}(f) = 0$. Then there is an element $b \in A \setminus \mathfrak{p}$ such that $\operatorname{Ann}_{A_b}(f) = 0$.*

Proof. Let I be the ideal

$$I = \{g \in \mathfrak{p} \mid \exists x \notin \mathfrak{p} : xgf = 0\} .$$

This is in fact an ideal: Let $g_1, g_1 \in I$ with $x_1, x_2 \notin \mathfrak{p}$ and $x_1g_1f = x_2g_2f = 0$. Then $x_1x_2 \notin \mathfrak{p}$ and $x_1x_2(g_1 + g_2)f = 0$, thus $g_1 + g_2 \in I$. Let $g \in I$ with $xgf = 0$ and $\lambda \in A$. Then obviously $x\lambda gf = 0$, thus $\lambda g \in I$. As A is noetherian, $I = (g_1, \dots, g_n)$ is finitely generated. For all g_i one has $g_if = \frac{x_i}{x_i} g_if = 0$ in $A_\mathfrak{p}$. But as $\operatorname{Ann}_{A_\mathfrak{p}}(f) = 0$ it follows that $g_i = 0$ in $A_\mathfrak{p}$, therefore, there is an element $b_i \notin \mathfrak{p}$ with $b_ig_i = 0$ in A. Now let $b = b_1 \cdots b_n$. As $\mathfrak{p}$ is prime, obviously $b \notin \mathfrak{p}$. We have to show that $\operatorname{Ann}_{A_b}(f) = 0$, thus let $\frac{y}{b^m} f = 0$ in A_b, i.e., $b^{m'}yf = 0$ in A for some $m' \geq 0$. But then $y = a_1g_1 + \cdots + a_ng_n \in I$ and thus in A_b one has $y = 0$. Hence, $\operatorname{Ann}_{A_b}(f) = 0$. □

Proposition 1.2.15. *Let Y be a locally noetherian scheme and let $f : X \to Y$ be a closed immersion defined by a sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_Y$. Let $x \in X$. If $\mathcal{J}_{f(x)}$ is sequence-regular, then there is an affine open neighbourhood V of $f(x)$ such that $\mathcal{J}(V)$ is generated by a sequence-regular sequence, i.e., f is sequence-regular in x.*

Proof. Let $\mathcal{J}_{f(x)} \subset \mathcal{O}_{Y,f(x)}$ be sequence-regular. Then it is generated by a sequence-regular sequence $f_{1,x}, \dots, f_{n,x}$. By [GW10, 7.7.29] this sequence lifts to a generating sequence $f_1, \dots, f_n$ on some neighbourhood V' of $f(x)$. It remains to show that on some neighbourhood this sequence is even sequence-regular, what will be done by induction on n.

V' can be assumed to be affine. For $n = 1$, the statement follows from 1.2.14. Now let $f_{1,x}, \dots, f_{n+1,x}$ be sequence-regular. Then $f_{1,x}, \dots, f_{n,x}$ is sequence-regular, too, and can be lifted by induction hypothesis to some affine neighbourhood $V'' = \operatorname{Spec} A \subset V'$. Let $\mathfrak{p}_x \subset A$ denote the prime ideal corresponding to x and $J = (f_1, \dots, f_n) \subset A$. As $f_{n+1,x}$ is a non zero-divisor in $(A/J)_{\bar{\mathfrak{p}}_x}$, there is a $\bar{b} \in A/I$, $\bar{b} \notin \bar{\mathfrak{p}}_x$ such that f_{n+1} is a non zero-divisor in $(A/J)_{\bar{b}}$. Let $b \in A \setminus \mathfrak{p}_x$ be a lift of $\bar{b}$, then on $V = \operatorname{Spec} A_b$ the sequence is sequence-regular. □

Remark 1.2.16. The definition given in [Liu02, 6.3.4] for a regular immersion $f : X \to Y$ uses the property shown just before in 1.2.15: It requires for all $x \in X$ the ideal $\ker(\mathcal{O}_{Y,f(y)} \to \mathcal{O}_{X,x})$ to be regular. As the definition is given there only for locally noetherian schemes there, this is equivalent to the definition used here. If one would extend Liu's definition to non noetherian schemes, sequence-regular immersions would not necessarily be Koszul-regular immersions, too.

Lemma 1.2.17 ([Koe89]). *Let A be a noetherian ring, $I \subset A[X_1, \ldots, X_n]$ an ideal inducing a sequence-regular immersion* $\operatorname{Spec} B \to \operatorname{Spec} A[X_1, \ldots, X_n]$ *where* $B = A[X_1, \ldots, X_n]/I$. *Then the scheme* $\operatorname{Spec} B$ *is locally of the form* $\operatorname{Spec} B_0$ *with* $B_0 = A[X_0, \ldots, X_n]/(P_0, \ldots, P_d)$ *with a sequence-regular sequence* $P_0, \ldots, P_d$.

Proof. By the definition of sequence-regular immersion (1.2.7), there is an $f \in A[X_1, \ldots, X_n]$ and a sequence-regular sequence $P_1, \ldots, P_d \in A[X_1, \ldots, X_n]_f$ such that $I_f = (P_1, \ldots, P_d)$. By neglecting the (invertible) denominators, we can assume that $P_i \in A[X_1, \ldots, X_n]$. Now we define $P_0 := fX_0 - 1 \in A[X_0, \ldots, X_n]$ which is a non zero-divisor with $A[X_0, \ldots, X_n]/(P_0) \cong A[X_1, \ldots, X_n]_f$. Therefore, $P_0, \ldots, P_d$ form a sequence-regular sequence in $B_0 = A[X_0, \ldots, X_n]$ with $B_0/(P_0, \ldots, P_d) = A[X_0, \ldots, X_n]_f/(P_1, \ldots, P_d)$. □

Proposition 1.2.18 ([EGAIV.4, 19.8.2]). *Let (λ) be a projective filtered system with minimal element α. Let S_λ be a projective system of schemes and let X_λ, Y_λ be projective systems over S_λ-schemes flat and locally of finite presentation with X_α and Y_α quasi-compact. Let $S = \varprojlim S_\lambda$, $X = \varprojlim X_\lambda$, and $Y = \varprojlim Y_\lambda$. Let $j_\alpha : X_\alpha \to Y_\alpha$ be an S_α immersion locally of finite presentation. Then the induced immersion $j : X \to Y$ is sequence-regular if and only if there is an index $\lambda \geq \alpha$ such that $j_\lambda : X_\lambda \to Y_\lambda$ is sequence-regular.*

Proof. The condition is sufficient as sequence-regularity is stable under arbitrary base change under the conditions given here ([EGAIV.4, 19.2.4] and [EGAIV.4, 19.2.7 (ii)]).

Let $y \in j(X)$. Let s be the image of y in S and let s_λ be the image of s in S_λ. If one denotes by X_s, Y_s the fibers of s, then X_s, Y_s are the projective limits of $(X_\lambda)_{s_\lambda}$ and $(Y_\lambda)_{s_\lambda}$. As by 1.2.9 $j_s : X_s \to Y_s$ is sequence-regular and the induced transition maps $s_\lambda \to s_{\lambda'}$ on the images of s are faithfully flat, [EGAIV.4, 19.8.1 (ii)] shows that for every s there is a $\lambda(s)$ such that the immersion $(X_\lambda)_{s_\lambda} \to (Y_\lambda)_{s_\lambda}$ is sequence-regular and thus by 1.2.9 there is a neighbourhood $V_\lambda(y_\lambda)$ of y_λ such that $j_\lambda|_{j_\lambda^{-1}(V_\lambda(y_\lambda))} \to V_\lambda(y_\lambda)$ is regular. Denote by $V(y)$ the preimage of $V_\lambda(y_\lambda)$ in Y. As X is quasi-compact, $j(X)$ is quasi-compact, too, and only finitely many $y_1, \ldots, y_n$ are needed such that the $V(y_i)$ cover $j(X)$. For every λ, let $V_\lambda = \bigcup_{i=1}^n V_\lambda(y_{i,\lambda})$. Thus there is a largest λ such that $j_\lambda|_{j_\lambda^{-1}(V_\lambda)} \to V_\lambda$ is sequence-regular. As furthermore, the preimage of V_λ in Y equals Y, by [EGAIV.3, 8.3.4] there is a possibly larger λ with $V_\lambda = Y_\lambda$ and hence the proposition follows. □

Proposition 1.2.19 ([Kun86, B.23]). *Let R and P be noetherian local rings, and $R \to P$ a flat local homomorphism. Let $I \subset R$ be an ideal and $R = R/I$, $\bar{P} = P/IP$. Let $f_1, \ldots, f_d$ be a sequence of elements of P and let $\bar{f}_i$ denote the images of the f_i in $\bar{P}$. Then the following statements are equivalent:*

(a) The $f_1, \ldots, f_d$ form a P-sequence-regular sequence and $P/(f_1, \ldots, f_d)$ is a flat R-module.

(b) *The* $\bar{f}_1, \ldots, \bar{f}_d$ *for a* $\bar{P}$*-sequence-regular sequence and* $\bar{P}/(\bar{f}_1, \ldots, \bar{f}_d)$ *is a flat* $\bar{P}$*-module.*

1.3. Comparison of the Definitions

While Koszul-regularity provides the powerful tools of homological algebra, sequence-regularity is often easier to verify and in particular to falsify as it is more closely related to the concept of non zero-divisors (nevertheless, bearing in mind examples like 1.2.11). Therefore, it is of great interest to compare these two notions. This section aims at doing this.

Proposition 1.3.1 ([EGAIII.1, 1.1.4])**.** *Let* R *be a ring,* $f = (f_1, \ldots, f_r)$ *a sequence of elements of* R*,* M *an* R*-module. If* f *is* M*-sequence-regular, then* f *is* M*-Koszul-regular.*

Proof. The proof is done via induction on r: The case $r = 1$ is trivial by 1.1.8. Let f be M-sequence-regular and assume the statement has been proven up to $r-1$. Define $f' = (f_1, \ldots, f_{r-1})$, which is obviously again M-sequence-regular. Let $L_\bullet = K_\bullet(f', M)$ then by induction hypothesis $H_i(L_\bullet) = 0$ for all $i > 0$. Furthermore we have $H_0(L_\bullet) = M_{r-1}$ (1.1.3). Let $K_\bullet = K_\bullet(f_r) = 0 \to K_1 \to K_0 \to 0$ with $K_0 = K_1 = R$ and $d_1 : K_1 \to K_0$ the multiplication with f_r, then we have $K_\bullet(f, M) = K_\bullet \otimes_R L_\bullet$. Thus, for $k \geq 2$ one has $H_k(f, M) = 0$ by 1.1.5 and by the induction hypothesis. For $k = 1$ we have to show that $H_1(K_\bullet \otimes_R H_0(L_\bullet)) = 0$. But we have $H_1(K_\bullet \otimes_R H_0(L_\bullet)) = \ker(K_1 \otimes_R M_{r-1} \longrightarrow K_0 \otimes_R M_{r-1}) = \ker(M_{r-1} \longrightarrow M_{r-1}, z \mapsto f_r \cdot z) = 0$. □

Corollary 1.3.2. *Every sequence-regular immersion is Koszul-regular as well.*

The converse does not hold in general, as will be demonstrated in 1.3.8 and 1.3.9. However, given the right conditions, the definitions are in fact equivalent, as mentioned earlier. This will now be proved in the following theorem, which constitutes the main result of this section:

Theorem 1.3.3 ([EGAIV.4, 19.5.1])**.** *Let* A *be a ring,* $f = (f_1, \ldots, f_r)$ *a sequence of elements in* A*, and* M *an* A*-module. Let* I *be the ideal generated by* $f_1, \ldots, f_r$*. If every quotient module of a submodule of* M *is separated with respect to the* I*-adic topology, then the following statements are equivalent:*

(a) f *is* M*-sequence-regular.*

(b) $H_i(f, M) = 0$ *for all* $i \geq 0$ *(i.e.,* f *is Koszul-regular).*

(c) $H_1(f, M) = 0$*.*

Without the hypothesis of separateness, still (a) $\Rightarrow$ *(b)* $\Rightarrow$ *(c) holds.*

Proof. The implication (a) $\Rightarrow$ (b) has been shown in 1.3.1, the implication (b) $\Rightarrow$ (c) is trivial. Thus, it is sufficient to show (c) $\Rightarrow$ (a), given that every quotient module of a submodule of M is separated with respect to the I-adic topology. The proof will be done via induction on r. For $r = 1$, one has $0 = H_1(f, M) = \ker(M \to M, z \mapsto f_1 \cdot z)$, thus $f = (f_1)$ is M-sequence-regular. Let $r \geq 2$ and $f' = (f_1, \dots, f_{r-1})$, then one has, as above, $K_\bullet(f, M) = K_\bullet(f_r) \otimes_A K_\bullet(f', M)$. With 1.1.5 one obtains an exact sequence

$$0 \longrightarrow H_0(f_r, H_1(f', M)) \longrightarrow H_1(f, M) \longrightarrow H_1(f_r, H_0(f', M)) \longrightarrow 0.$$

Thus, from $H_1(f, M) = 0$ one gets $H_0(f_r, H_1(f', M)) = 0$ and $H_1(f_r, H_0(f', M)) = 0$. With 1.1.3 we obtain $H_0(f_r, H_1(f', M)) = H_1(f', M)/f_r H_1(f', M)$, thus $f_r H_1(f', M) = H_1(f', M)$. By definition, the module $H_1(f', M)$ is isomorphic to a quotient N/N', where N is a submodule M^{r-1}. Let M^{r-1} be equipped with the filtration given by M^j, $j \leq r-1$, let N be equipped with the induced filtration, and let $H_1(f', M)$ be equipped with the quotient filtration of N. Then $H_1(f', M)$ has a finite filtration whose quotients are isomorphic to quotients of submodules of M and hence $H_1(f', M)$ is separated with respect to the I-adic topology by hypothesis. Now, because of $f_r H_1(f', M) = H_1(f', M)$, we have already $I \cdot H_1(f', M) = H_1(f', M)$ and thus $H_1(f', M) = 0$ since it is I-adically separated. Therefore, by induction hypothesis, the sequence f' is M-sequence-regular. But $H_1(f_r, H_0(f', M)) = 0$ shows, also by induction hypothesis, that f_r is $H_0(f', M) = M/(f_1 M + \cdots + f_{r-1} M)$-sequence-regular and thus f is M-sequence-regular. $\square$

Corollary 1.3.4 ([EGAIV.4, 19.5.2])**.** *Let A be a noetherian ring, $f_1, \dots, f_r \in \operatorname{rad} A$, and M a finitely generated A-module (e.g., A). Then the statements (a), (b), and (c) of 1.3.3 are equivalent. In particular, the sequence $f_1, \dots, f_r$ is sequence-regular if and only if it is Koszul-regular.*

Proof. As A is noetherian, every submodule of the finitely generated A-module M is finitely generated as well. Quotient modules of these finitely generated submodules obviously are finitely generated as well. Being finitely generated modules, all these modules are $(f_1, \dots, f_r)$-adically separated ([EGAI, 0.7.3.5]). Therefore, theorem 1.3.3 can be applied. $\square$

Corollary 1.3.5. *Let $(A, \mathfrak{m})$ be a local noetherian ring, $f_1, \dots, f_r \in \mathfrak{m}$. Then $f_1, \dots, f_r$ is sequence-regular if and only if it is Koszul-regular.*

Corollary 1.3.6. *Let $(A, \mathfrak{m})$ be a local noetherian ring, $f_1, \dots, f_r \in \mathfrak{m}$. If $f_1, \dots, f_r$ is sequence-regular, then every reordering of the sequence is sequence-regular, too.*

Corollary 1.3.7. *Let $i : X \longrightarrow Y$ be an immersion of schemes and assume Y is locally noetherian. Then i is a Koszul-regular immersion (1.1.16) if and only if it is a sequence-regular immersion (1.2.7).*

Proof. Let i be Koszul-regular and $x \in X$. Then there is an affine open set $\operatorname{Spec} A = V \subset Y$ and an affine open neighbourhood $U \subset X$ of x such that the closed immersion $U \to V$ is induced by an ideal I generated by a Koszul-regular sequence $f_1, \dots, f_n \in A$. The point $i(x) \in Y$ corresponds to a prime ideal $\mathfrak{p} \subset A$ with $I \subset \mathfrak{p}$. Therefore, $I_\mathfrak{p} \neq A_\mathfrak{p}$ and thus 1.3.5 can be applied: The element $f_1, \dots, f_n$ form a sequence-regular sequence in $A_\mathfrak{p} = \mathcal{O}_{Y,i(x)}$. With 1.2.15 this shows that i is sequence-regular in x. The inverse statement has already been proven in 1.3.2. $\square$

However, the definitions of Koszul-regularity and sequence-regularity are not equivalent in all cases. In order to clarify their relation, in the following we will construct two examples, which do not fulfill both definitions.

Example 1.3.8 ([Eis99, 17.3])**.** Let k be a field and $R = k[X, Y, Z]/(X - 1)Z$. The elements X and $(X-1)Y$ generate the ideal $(X, (X-1)Y) = (X, Y) \neq R$. Furthermore, X is a non zero-divisor in R and in $R/(X) = k[Y, Z]/Z$ the element $(X - 1)Y = Y$ is a non zero-divisor. Therefore, the sequence $X, (X - 1)Y$ is sequence-regular and hence Koszul-regular. But we have $(X - 1)Y \cdot Z = 0$ in R and thus $(X - 1)Y$ is a zero-divisor in R. Therefore, the sequence $(X - 1)Y$, X cannot be sequence-regular. However, it is of course Koszul-regular as it is just a reordering of a Koszul-regular sequence.

As the sequence-regularity of an immersion is defined locally, it is of some interest to find an example even in local rings where Koszul-regularity and sequence-regularity do not coincide.

Example 1.3.9 ([EGAIV.4, 16.9.6])**.** Consider the sheaf of real-valued $\mathcal{C}^\infty$-functions on $\mathbb{R}$. Let F be the ring of germs of this sheaf in 0. This local ring contains the maximal ideal $\mathfrak{m} = \{f \mid f(0) = 0\}$. (To simplify notation, germs will be written by representatives). Furthermore F contains the function $t : x \mapsto x$. By A.1.4, the ideal $\mathfrak{m}$ is generated by t.

As the functions $x^{-k} \exp(-\frac{1}{x^2})$ can be extended to $\mathcal{C}^\infty$-functions in 0 for all k (by iterated application of de l'Hospital), one has $\exp(-\frac{1}{x^2}) \in \bigcap_k \mathfrak{m}^k =: \mathfrak{n} \neq 0$. This shows that $\mathfrak{m}$ is not separated.

Let now $A = F[T]/\mathfrak{n}TF[T]$ and let f_1, f_2 be the images of t and T in A. Then the sequence (f_1, f_2) is sequence-regular in A: First we show that f_1 is a non-zero divisor. Let $P(T) \in F[T]$ with $tP(T) \in \mathfrak{n}TF[T]$. For reasons of degree the coefficients of $P(T)$ have to be in $\mathfrak{n}$ and thus $P(T) \in \mathfrak{n}TF[T]$. It remains to show that f_2 is no zero-divisor in A/f_1A. As we have $B/tB \cong \mathbb{R}$, we have that $A/f_1A \cong \mathbb{R}[T]$ is an integral domain. Hence, the element $f_2 = T \neq 0$ is a non zero-divisor in $\mathbb{R}[T]$.

On the other hand, (f_2, f_1) is not sequence-regular in A: Let $x \in \mathfrak{n}$, $x \neq 0$ in A. Then one has $xT = 0$ in A and thus f_2 divides zero.

To show that this is in fact a counter example, only a few steps are left: As (f_1, f_2) is sequence-regular in A, we have $H_i((f_1, f_2), A) = 0$ for $i > 0$ with 1.3.1, i.e., (f_1, f_2) is Koszul-regular. But then we have also $H_i((f_2, f_1), A) = 0$: Whether $K_\bullet$ is acyclic does not depend on the ordering of the sequence (see 1.1.4). But as shown before, (f_2, f_1) is not sequence-regular in A and thus is Koszul-regular but not sequence-regular.

This counter example illustrates why the definition of sequence-regularity does not behave well in non-noetherian contexts: In general one is interested in the geometric properties of an immersion defined by an ideal. But this properties should not depend on the generating system of the ideal and in particular not on the ordering of the generating system. The Koszul complex gives rise to exactly this kind of behaviour. For more examples comparing different properties of sequence-regularity and Koszul-regularity see [Kab71].

Lemma 1.3.10 ([Koe89]). *Let A be a noetherian ring, $I \subset A[X_1, \dots, X_n]$ an ideal such that* $\operatorname{Spec} B \to \operatorname{Spec} A[X_1, \dots, X_n]$ *with* $B = A[X_1, \dots, X_n]/I$ *is a regular immersion. Then the scheme* $\operatorname{Spec} B$ *is (Zariski-)locally of the form* $\operatorname{Spec} B_0$ *with* $B_0 = A[X_0, \dots, X_n]/(P_0, \dots, P_d)$ *with a regular sequence $P_0, \dots, P_d$.*

Proof. Since A is noetherian, the notions of Koszul-regularity and sequence-regularity coincide. Thus, the statement follows from 1.2.17. □

Proposition 1.3.11 ([SGA6, Exp. VII, 1.10]). *Let X and Y be smooth S-schemes. Then every S-immersion $X \to Y$ is sequence-regular and hence Koszul-regular. In particular, every section of a smooth morphism is a sequence-regular and hence Koszul-regular immersion.*

Proof. By [EGAIV.4, 17.12.1], the immersion is a sequence-regular immersion and therefore is Koszul-regular by 1.3.2. □

2. Local Complete Intersections

This chapter introduces local complete intersections. They are a generalization of regular immersions, which turn out – in spite of exhibiting nice properties – to be too restrictive for our needs. As a main difference we will now allow composition with a smooth morphism. This enables us to handle not only subschemes but a much larger class of morphisms.

2.1. SGA Local Complete Intersections

Definition 2.1.1 ([SGA6, Exp. VIII, 1.1])**.** Let $f : X \longrightarrow Y$ be a morphism of schemes. Then f is called a *local complete intersection in* $x \in X$ if there is a neighbourhood U of x and a smooth Y-scheme V such that the restriction of f to U factorizes via V:

$$\begin{tikzcd} x \in U \arrow[rr, "f"] \arrow[dr, "i"'] & & Y, \\ & V \arrow[ur, "\text{smooth}"'] & \end{tikzcd}$$

where i is a Koszul-regular immersion. f is called a *local complete intersection* if it is a complete intersection in every point. In [SGA6] this is called simply *complete intersection*.

Remark 2.1.2. Smooth morphisms are trivially local complete intersections.

By replacing V by an open subset one can assume that i is a closed Koszul-regular immersion. One can even replace V by some affine space $\mathbb{A}^n_Y$ over Y: In fact, as the notion is local, on can assume that V is affine and that the image in Y is contained in an affine open $W \subset Y$. Then there is an $n \in \mathbb{N}$ and a closed W-immersion $V \longrightarrow \mathbb{A}^n_W$, respectively a Y-immersion $V \longrightarrow \mathbb{A}^n_Y$, which is Koszul-regular by 1.3.11. Since the composition of two Koszul-regular immersion is Koszul-regular, one gets a Koszul-regular Y-immersion $U \longrightarrow \mathbb{A}^n_Y$ which factorizes f.

Proposition 2.1.3 ([SGA6, Exp. VIII, 1.3])**.** *Suppose one has a commutative diagram*

$$\begin{tikzcd} & V' \arrow[d, "p"] \\ X \arrow[ur, "j"] \arrow[r, "i"] & V, \end{tikzcd}$$

where i and j are immersions and p is a smooth morphism. Then i is a Koszul-regular immersion if and only if j is a Koszul-regular immersion.

Proof. See [SGA6, Exp. VIII, 1.3]. □

Corollary 2.1.4. *Let $f : X \longrightarrow Y$ be a local complete intersection. Then every Y-immersion $i : X \longrightarrow Z$ from X to a smooth Y-scheme is Koszul-regular.*

Proof. Let $x \in X$. Then there is an open neighbourhood $U \subset X$ of x such that f factors as $U \to V \to Y$ where $U \to V$ is a Koszul-regular immersion and $V \to Y$ is a smooth morphism. The restriction $i : U \to Z$ is an immersion as well and there is a commutative diagram

$$\begin{tikzcd}
 & & V \arrow[dr] & \\
U \arrow[r] \arrow[urr] \arrow[drr] & V \times_Y Z \arrow[ur] \arrow[dr] & & Y, \\
 & & Z \arrow[ur] &
\end{tikzcd}$$

where $V \to Y$ and $Z \to Y$ are smooth, therefore the projections $V \times_Y Z \to V$ and $V \times_Y Z \to Z$ are smooth, too. Since $U \to Z$ and $U \to V$ are immersions, $U \to V \times_Y Z$ is an immersion, too (A.1.5). The morphism $U \to V$ is even Koszul-regular, and thus 2.1.3 shows that $U \to V \times_Y Z$ is Koszul-regular, too. By another application of this argument it follows that $U \to Z$ is Koszul-regular. □

Proposition 2.1.5 ([SGA6, Exp. VIII, Prop. 1.5])**.** *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two local complete intersections. Then $g \circ f$ is a local complete intersection, too.*

Proof. Let $x \in X$. There are neighbourhoods U of x and V of $f(x)$ and Koszul-regular immersions i, j such that

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[dr, "f"] & \mathbb{A}^n_V \arrow[r, "j'"] \arrow[d] & \mathbb{A}^{m+n}_Z \arrow[d] \\
 & V \arrow[r, "j"] \arrow[dr, "g"] & \mathbb{A}^m_Z \arrow[d] \\
 & & Z,
\end{tikzcd}$$

where j' is the base change of j by the flat projection $\mathbb{A}^{m+n}_Z \longrightarrow \mathbb{A}^m_Z$ and thus a Koszul-regular immersion. With 1.1.23, the morphism $j' \circ i$ is a Koszul-regular immersion from U to a smooth scheme over Z. □

Proposition 2.1.6 ([SGA6, Exp. VIII, 1.6]). *Let $f : X \longrightarrow Y$ be a morphism of schemes and $g : Y' \longrightarrow Y$ a flat base change. If f is a local complete intersection, then $f' : X \times_Y Y' \longrightarrow Y'$ is a local complete intersection, too. If g is quasi-compact and surjective, the inverse holds, too.*

Proof. If f is a local complete intersection, there is a covering $X = \bigcup_i U_i$ such that f factors locally as $U_i \longrightarrow V_i \longrightarrow Y$ where $U_i \longrightarrow V_i$ is a Koszul-regular immersion and $V_i \longrightarrow Y$ is smooth. The $U_i \times_Y Y'$ form a covering of $X \times_Y Y'$ and the $U_i \times_Y Y' \longrightarrow V_i \times_Y Y'$ are Koszul-regular by 1.1.19. As base changes of smooth morphisms the morphisms $V_i \times_Y Y' \longrightarrow Y'$ are smooth ([GW10, 6.15]). Thus, f' is a local complete intersection.

Let f' be a local complete intersection and assume that g is quasi-compact and faithfully flat. Then by faithfully flat descent ([GW10, 14.51 (1)]), f is locally of finite presentation. Therefore, locally, there is a factorization $f : X \to \mathbb{A}^n_Y \to Y$ via an immersion followed by a smooth morphism which induces a factorization $f' : X \times_Y Y' \to \mathbb{A}^n_{Y'} \to Y'$: Thus, we have

$$\begin{tikzcd}
X \arrow[r] & \mathbb{A}^n_Y \arrow[r] & Y \\
X \times_Y Y' \arrow[r] \arrow[u] & \mathbb{A}^n_{Y'} \arrow[r] \arrow[u] & Y'. \arrow[u]
\end{tikzcd}$$

Since immersions are stable under base change ([GW10, 4.32]), $X \times_Y Y' \to \mathbb{A}^n_{Y'}$ is an immersion, too, and therefore it is Koszul-regular by 2.1.4, as $\mathbb{A}'^n_Y \to Y'$ is smooth. In addition, $\mathbb{A}^n_{Y'} \to \mathbb{A}^n_Y$ is as base change of a faithfully flat and quasi-compact morphism faithfully flat and quasi-compact itself, thus 1.1.19 shows that $X \to \mathbb{A}^n_Y$ is a Koszul-regular immersion and therefore f is a local complete intersection. □

2.2. EGA Local Complete Intersections

In [EGAIV.4, 19.3] Grothendieck also gives definitions for local complete intersections which are more algebraic and thus often easier to handle. In the syntomic case which is interesting for us, the definitions will be equivalent to the definition given in 2.1.1.

Definition 2.2.1 ([EGAIV.4, 19.3.1]). Let A be a local noetherian ring. Then A is said to be an *EGA absolute complete intersection*, if its completion $\hat{A}$ is isomorphic to a quotient of a local noetherian complete regular ring B by a sequence-regular ideal.

A locally noetherian scheme X is an *EGA absolute complete intersection* in a point $x \in X$, if $\mathcal{O}_{X,x}$ is an EGA absolute complete intersection.

Proposition 2.2.2 ([EGAIV.4, 19.3.2]). *Let B be a local noetherian regular ring, $I \subset B$ an ideal. Then $A = B/I$ is an EGA absolute complete intersection if and only if the ideal I is sequence-regular.*

Thus, in particular, Definition 2.2.1 is independent of the chosen ring B. Furthermore, this shows that every regular local noetherian ring is an EGA absolute complete intersection.

Corollary 2.2.3 ([EGAIV.4, 19.3.4])**.** *Let k be a field, X a scheme locally of finite type over k, $k' \supseteq k$ a field extension, $X' = X \otimes_k k'$. Let $x' \in X'$, x the projection of x' to X. Then X is an absolute complete intersection in x if and only if X' is an absolute complete intersection in x'.*

Definition 2.2.4 ([EGAIV.4, 19.3.6])**.** Let $f : X \longrightarrow S$ be a flat morphism locally of finite presentation. Then X is called a *EGA local complete intersection in $x \in X$* (in [EGAIV.4] called *complete intersection relative to S in $x \in X$*), if the fiber $f^{-1}(f(x))$ is an EGA absolute complete intersection in x. The S-scheme X is called an *EGA local complete intersection* (in [EGAIV.4, 19.3.6] *complete intersection relative to S*) and f an *EGA local complete intersection*, if X is an EGA complete intersection in all of its points.

Remark 2.2.5. Note that $f^{-1}(f(x)) = X \times_S \operatorname{Spec} \kappa(f(x))$ is locally of finite presentation over $\operatorname{Spec} \kappa(f(x))$ and thereby locally noetherian. Thus the definition of EGA absolute complete intersections can be applied.

Proposition 2.2.6. *Let $f : X \longrightarrow S$ be a flat morphism locally of finite presentation. Then f is an EGA local complete intersection if and only if for all $x \in X$ the ring $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is an absolute complete intersection.*

Proof. The morphism $X \longrightarrow S$ is an EGA local complete intersection if for all $x \in X$ the fiber $f^{-1}(f(x))$ is an absolute local complete intersection. This is the case if and only if for all $x \in X$ the local ring of x in $f^{-1}(f(x)) = X \times_S \kappa(f(x))$ which is $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,f(x)}} (\mathcal{O}_{S,f(x)}/\mathfrak{m}_{f(x)}\mathcal{O}_{S,f(x)})$ is an absolute complete intersection. $\square$

Proposition 2.2.7 ([EGAIV.4, 19.3.7])**.** *Given the following commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & S, & \end{tikzcd}$$

where g and h are flat and locally of finite presentation, h smooth and f an immersion. Let $x \in X$ and $y = f(x)$, $s = g(x)$. Then g is an EGA local complete intersection in x if and only if f is sequence-regular in a neighbourhood of x.

Proof. The morphism g is an EGA local complete intersection in x if and only if $X_s = g^{-1}(s)$ is an EGA absolute complete intersection in x. Since h is smooth, $Y_s = h^{-1}(s)$ is smooth over $\kappa(s)$ and as such is regular in y by [GW10, 6.26]. As moreover, $\mathcal{O}_{X_s,x} = \mathcal{O}_{Y_s,y}/\ker f_x$, by 2.2.2 the fiber X_s is an EGA absolute complete intersection in x if and only if $f_s : X_s \to Y_s$ is sequence-regular in x, which is by 1.2.9 equivalent to f being sequence-regular in a neighbourhood of x. $\square$

Proposition 2.2.8 ([SGA6, Exp. VIII, 1.6]). *Let $f : X \longrightarrow Y$ be a morphism and $g : Y' \longrightarrow Y$ a base change. If f is an EGA local complete intersection, then $f' : X \times_Y Y' \longrightarrow Y'$ is an EGA local complete intersection, too.*

Proof. Let f be an EGA local complete intersection. It follows immediately that f' is flat and locally of finite presentation. Then, for every $y' \in Y'$, $y = g(y')$, $f'^{-1}(y') = X \times_Y Y' \times_{Y'} \kappa(y') = X \times_Y \kappa(y')$ is an absolute complete intersection if and only if $f^{-1}(y) = X \times_Y \kappa(y)$ is an absolute complete intersection by 2.2.3. $\square$

2.3. Comparison

The following proposition is of particular interest, as it states that local complete intersections and EGA local complete intersections are equivalent in the case considered in the following in this thesis, namely, the syntomic case.

Proposition 2.3.1 ([SGA6, Exp. VIII, 1.4]). *Let $f : X \longrightarrow Y$ be a flat morphism locally of finite presentation. Then f is a local complete intersection if and only if it is an EGA local complete intersection.*

Proof. Let f be a local complete intersection and let $x \in X$. Then there is a neighbourhood U of x and a factorization

$$\begin{tikzcd} x \in U \arrow[rr, "f"] \arrow[dr, "i"'] & & Y, \\ & V \arrow[ur, "g"'] & \end{tikzcd}$$

where i is a Koszul-regular immersion (without loss of generality closed) and g is a smooth morphism. Without loss of generality, we have $V = \operatorname{Spec} B$, $U = \operatorname{Spec} B/I = \operatorname{Spec} A$, and $Y = \operatorname{Spec} C$.

Let $y = f(x)$, $U_y := f^{-1}(y)$, $V_y := g^{-1}(y)$. The claim is that the induced immersion $i_y : U_y \longrightarrow V_y$ is Koszul-regular in x: The closed immersion $i : \operatorname{Spec} A \to \operatorname{Spec} B$ is defined by a Koszul-regular ideal $I \subset B$. Therefore, the Koszul complex of i (denoted by $K_\bullet(i)$ in the following) is (possibly after localization) a resolution of $B/I = A$. Since A and B are finite over C and the Koszul complex consists of free B-modules, the complex $K_\bullet(i)$ is even a C-flat resolution. Since $V_y = \operatorname{Spec} B \otimes_C \kappa(y)$ one has $K_\bullet(i_y) = K_\bullet(i) \otimes_C \kappa(y)$. Therefore, the homology groups of $K_\bullet(i_y)$ are equal to $\operatorname{Tor}_i^C(B/I, \kappa(y))$. As $X = \operatorname{Spec} B/I$ is assumed to be flat over Y, these groups vanish for $i > 0$ and therefore i_y is Koszul-regular. This shows that I_y a Koszul-regular ideal and hence by 1.3.7 sequence-regular.

Next, note that the local ring of y in V_y is regular, as V_y is smooth over $\operatorname{Spec} \kappa(y)$ ([GW10, 6.26]). Therefore, U_y is the quotient of a regular ring by a sequence-regular ideal and hence an EGA complete intersection by 2.2.2. This shows that f is an EGA local complete intersection.

Now let f be an EGA local complete intersection and $x \in X$. Let $U' = \operatorname{Spec} A$ be an affine open neighbourhood of $f(x)$ and U an affine open neighbourhood of x in X contained in the preimage of U'. With f being locally of finite presentation, there is – maybe after further localization – a factorization of $f : U \to U'$ via $V = \operatorname{Spec} A[T_1, \ldots, T_n]$, where $i : U \to V$ is a closed immersion and $g : V \to U'$ is flat. With 2.2.7 this shows that i is a sequence-regular immersion and hence Koszul-regular. Therefore, f is a local complete intersection. □

Remark 2.3.2. A famous example for the use of local complete intersections is Andrew Wiles' proof of Fermat's Last Theorem: He proves that a certain Hecke algebra is a complete intersection (see [TW95] and [Wil95]). For an introduction to complete intersections in the context of Fermat's Last Theorem, see [SRS97].

3. Syntomic Morphisms

This chapter concludes the first part of the thesis by introducing syntomic morphisms. They will form the constitutive element of the topologies introduced later in this thesis. We present some of their basic properties and establish some important criteria that will allow us to show that some morphisms are syntomic coverings. n particular, this enables us to show that on the syntomic site the Kummer sequence is exact in characteristic $p > 0$, too, which is one of the most important properties of the syntomic topology.

Definition 3.1.1. A morphism $f : X \longrightarrow S$ of schemes is called *syntomic*, if it is locally of finite presentation, flat and a local complete intersection.

Remark 3.1.2. The name "syntomic" was created by Barry Mazur. In a mail to Thanos D. Papaïoannou he explains its meaning ([Maz]):

> Thanks for your question. I'm thinking of "local complete intersection" as being a way of cutting out a (sub-) space from an ambient surrounding space; the fact that it is flat over the parameter space means that each such "cutting" as you move along the parameter space, is—more or less—cut out similarly. I'm also thinking of the word "syntomic" as built from the verb temnein (i.e., to cut) and the prefix "syn" which I take in the sense of "same" or "together". So I think it fits.

Lemma 3.1.3 ([Bau92, 1.2])**.**

(i) Open immersions are syntomic.

(ii) The composition of syntomic morphisms is syntomic.

(iii) Syntomic morphisms are stable under arbitrary base change.

Proof. (i): Since an open immersions is smooth ([GW10, 6.15]), it is trivially a local complete intersection. Also open immersions are locally of finite presentation ([GW10, 10.35]) and flat and thus syntomic. (ii): Flat morphisms and morphisms locally of finite presentation are stable under composition. Local complete intersections are stable under composition by 2.1.5. (iii) Flat morphisms and morphisms locally of finite presentation are stable under base change. Local complete intersections which are flat and locally of finite presentation are stable under base change by 2.2.8. □

Lemma 3.1.4. *Let $f : X \longrightarrow Y$ be a flat morphism of smooth S-schemes. Then f is syntomic.*

Proof. The morphism f is of finite presentation as it is a morphism of S-schemes locally of finite presentation ([GW10, 10.35]). It remains to show that f factors locally through a Koszul-regular immersion followed by a smooth morphism. Consider the factoring through the graph Γ_f

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Gamma_f"] \arrow[drr, "f"] \arrow[ddr, "\mathrm{id}"'] & & \\
& X \times_S Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d] \\
& X \arrow[r] & S,
\end{tikzcd}$$

where π_Y is smooth as it is a base change of a smooth morphism. The graph Γ_f is an immersion [GW10, 9.5] between smooth S-schemes and as such a Koszul-regular immersion by 1.3.11. Since f is flat by hypothesis, it is syntomic. $\square$

Proposition 3.1.5 ([Koe89])**.** *Let $A = \varinjlim A_\lambda$ be a filtered limit of rings and A' a finitely generated syntomic A-algebra. Then there is an index ν and a finitely generated syntomic A_ν-algebra A'_ν such that $A' = A \otimes_{A_\nu} A'_\nu$.*

Proof. A' is locally of finite presentation. Since all the statements are local and affine schemes are quasi compact, we can assume without restriction that A' is of finite presentation. Then by [GW10, 10.65] there is an index μ and an A_μ-algebra A'_μ of finite presentation with $A' = A'_\mu \otimes_{A_\mu} A$. Therefore, we have $A'_\mu = A_\mu[X_1, \dots, X_n]/(f_1, \dots, f_d)$ with certain $f_i \in A_\mu[X_1, \dots, X_n]$. By defining $A'_\lambda = A'_\mu \otimes_{A_\mu} A_\lambda$ for $\lambda \geq \mu$ we have $A' = \varinjlim A'_\lambda = A'_\mu \otimes_{A_\mu} A$. Since A' is flat as A-module, by [EGAIV.3, 11.2.6.1] there is a $\tilde{\mu} \geq \mu$ such that A'_λ is flat as A_λ-algebra for all $\lambda \geq \tilde{\mu}$. So by now we can assume without restriction that for all λ the algebras A'_λ are flat and locally of finite presentation as A_λ-algebras and therefore the definitions of SGA-local complete intersections and EGA-local complete intersections coincide (2.3.1). Thus, it is sufficient to show that there is a ν such that A'_ν is an EGA-local complete intersection as A_ν-algebra. Since A' is an EGA local complete intersection as well, the immersion $\operatorname{Spec} A[T_1, \dots, T_n] \to \operatorname{Spec} A[T_1, \dots, T_n]/(f_1, \dots, f_d) = A'$ is sequence-regular by 2.2.7. Therefore, Proposition 1.2.18 shows that there is a ν such that the immersion $A_\nu[T_1, \dots, T_n] \to A_\nu[T_1, \dots, T_n]/(f_1, \dots, f_d) = A'_\nu$ is sequence-regular and hence $A_\nu \to A'_\nu$ is an EGA local complete intersection. $\square$

Proposition 3.1.6. *Let A be a noetherian ring and X be a syntomic scheme over* $\operatorname{Spec} A$*. Then locally X is of the form*

$$U = \operatorname{Spec} A[T_1, \dots, T_n]/(P_1, \dots, P_d),$$

where $P_1, \dots, P_d \in A[T_1, \dots, T_n]$ form a sequence-regular (and therefore Koszul-regular) sequence.

Proof. By definition of local complete intersections, $X \to \operatorname{Spec} A$ factors locally as $X \to \mathbb{A}^n_A \to \operatorname{Spec} A$, with $X \to \mathbb{A}^n_A$ being a Koszul-regular immersion. Therefore, the statement follows by 1.3.10. □

Nil immersions play an important role in the construction of the crystalline site (cf. 5.2.1). In 5.3.6 we will construct a morphism from the crystalline-syntomic topos to the syntomic topos. Crucial for this result will be the following theorem stating that syntomic coverings of schemes locally can be lifted to syntomic coverings of nil immersions.

Proposition 3.1.7 ([Koe89]). *(Lifting of syntomic morphisms) Consider the following diagram*

$$\begin{tikzcd} U' \arrow[d, "u"] & \\ U \arrow[r, hook] & T, \end{tikzcd}$$

where u is syntomic and $U \hookrightarrow T$ is a closed nil immersion. Then there is a Zariski covering $\{U_i \longrightarrow U'\}$ of U' and T_i such that the diagram

$$\begin{tikzcd} U_i \arrow[r, hook] \arrow[d, hook] & T_i \arrow[dd, "v_i"] \\ U' \arrow[d, "u"] & \\ U \arrow[r, hook] & T \end{tikzcd}$$

is cartesian with v_i syntomic.

Proof. Without loss of generality, let $U = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$ be affine and $U' = \operatorname{Spec} A'$ with A' a finitely generated syntomic A-algebra. We denote by I the kernel of the nil immersion $B \to A$, i.e., $A = B/I$. First assume that A and B are noetherian. By 1.3.10 we can even assume without loss of generality $A' = A[X_0, \dots, X_n]/(P_0, \dots, P_d)$ with $(P_0, \dots, P_d)$ sequence-regular. Choosing lifts $S_i \in B[X_0, \dots, X_n]$ for the $P_i \in A[X_0, \dots, X_n]$, one defines $B' = B[X_0, \dots, X_n]/(S_0, \dots, S_d)$. It remains to show that the S_i generate a Koszul-regular ideal in $B[X_0, \dots X_n]$ and that B' is flat over B.

This will be done by application of 1.2.19. For this, let $\mathfrak{p} \in \operatorname{Spec}(B[X_0, \ldots, X_n])$. Define $P = B[X_0, \ldots, X_n]_{\mathfrak{p}}$, $R = B_{\mathfrak{p} \cap B}$ and $J = I_{\mathfrak{p} \cap B} \subset R$. Since I is a nil ideal, $J \neq R$. One has $\bar{R} = R/J = A_{\overline{\mathfrak{p} \cap B}}$ and $\bar{P} = P/I_{\mathfrak{p}} = A[X_0, \ldots, X_n]_{\bar{\mathfrak{p}}}$. Let $f_i := S_{i\mathfrak{p}} \in P$ be the germs of the S_i in $\mathfrak{p}$. Obviously the $\bar{f}_i \in \bar{P}$ form a sequence-regular sequence in $\bar{P}$ and $\bar{P}/(\bar{f}_1, \ldots, \bar{f}_d) = A'_{\mathfrak{p} \cap A}$ is flat over $\bar{R}$. Therefore, by 1.2.19, the f_i form a sequence-regular sequence in P and $P/(f_1, \ldots, f_d)$ is flat over R. This shows that B' is a syntomic B-algebra.

Now let A and B be arbitrary. One has $B = \varinjlim B_\lambda$ where B_λ runs through all finitely generated $\mathbb{Z}$-subalgebras of B. Recalling that $A = B/I$, one defines $A_\lambda = B_\lambda/(I \cap B_\lambda)$ and therefore has $A = \varinjlim A_\lambda$. By 3.1.5 there is a ν and a finitely generated syntomic A_ν-algebra A'_ν with $A' = A \otimes_{A_\nu} A'_\nu$. Since A_ν and B_ν are noetherian, by the first part of the proof there is a syntomic lift B'_ν of B_ν, thus by defining $B' = B \otimes_{B_\nu} B'_\nu$ one gets a syntomic lift of B. $\square$

The following proposition is crucial for the proof of the conjecture of Birch and Swinnerton-Dyer as it shows that adjoining p-th roots to a noetherian ring is syntomic. In particular, the relative Frobenius over a Witt ring is syntomic.

Proposition 3.1.8. *Let A be a noetherian ring. Then the homomorphism*

$$\begin{aligned} A[T_1, \ldots, T_m] &\longrightarrow A[T_1, \ldots, T_m] \\ T_i &\longmapsto T_i^p \end{aligned}$$

is syntomic and faithfully flat. Therefore, it is a syntomic covering.

Proof. $A[T_1]$ is a free $A[T_1]$-module via the map $T_1 \mapsto T_1^p$ (a base is given, e.g., by $T_0^0, \ldots, T_0^{p-1}$) and thus it is flat. Since flatness is stable under products, the morphism is flat for all m. As $A[T_1, \ldots, T_m]$ is smooth over A, the morphism is syntomic by 3.1.4.

The morphism is obviously even faithfully flat, as it is topologically the identity. $\square$

The proposition above is a generalized version from the one given in [Koe89], which now follows as a corollary:

Corollary 3.1.9 ([Koe89])**.** *Let k be a perfect field of characteristic $p > 0$, $s \geq 1$ and $W_s := W_s(k)$ the ring of Witt vectors of length s. Then the relative Frobenius*

$$\begin{aligned} W_s[T_1, \ldots, T_m] &\longrightarrow W_s[T_1, \ldots, T_m] \\ T_i &\longmapsto T_i^p \end{aligned}$$

is syntomic and faithfully flat. Therefore, it is a syntomic covering.

4. Some Facts about Grothendieck Topologies

This section aims to collect some statements about Grothendieck topologies that will be used later to characterize covering families on the syntomic sites and to show that the Kummer sequences on the syntomic sites are exact. As there are quite a number of slightly different definitions for a Grothendieck topology (e.g.,, in [SGA3.1], [Tam94], and [Mil80]), the one used here shall be introduced in the following.

Definition 4.1.1. Let $\mathcal{C}$ be a category (with fiber products). A topology on $\mathcal{C}$ is the datum for every $S \in \operatorname{ob}\mathcal{C}$ of a set of families of morphisms in $\mathcal{C}$ with target S called *covering families* or *coverings* such that the following axioms hold:

(C 1) For every covering family $\{S_i \longrightarrow S\}$ and every morphism $T \longrightarrow S$ the family $\{S_i \times_S T \longrightarrow G\}$ is a covering (*stable under base change*).

(C 2) Let $\{S_i \longrightarrow S\}$ be a covering and for every i let $\{S_{ij} \longrightarrow S_i\}$ be a covering. Then the composite family $\{S_{ij} \longrightarrow S\}$ is a covering (*stable under composition*).

(C 3) Let $\{T_j \longrightarrow S\}$ be a covering and $\{S_i \longrightarrow S\}$ be a family such that for every j there is an i such that

$$\begin{tikzcd} T_j \arrow[rr] \arrow[dr] & & S \\ & S_i \arrow[ur] & \end{tikzcd}$$

commutes, then $\{S_i \longrightarrow S\}$ is a covering (*Saturation*).

(C 4) Every family consisting of just an isomorphism is a covering.

It is often useful to extend the notion of coverings to families of arbitrary morphisms in the category of presheaves $\hat{\mathcal{C}} = \operatorname{Hom}(\mathcal{C}^{op}, (\text{set}))$. This can be done uniquely by the following definition/axiom:

(C 0) Let $\{F_i \longrightarrow F\}$ be a family of morphisms in $\hat{\mathcal{C}}$. If for every representable base change $S \longrightarrow F$ (i.e., S, $F_i \times_F S \in \mathcal{C}$) the family $\{F_i \times_F S \to S\}$ is a covering, then the initial family a covering.

This is compatible with the Yoneda embedding $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$, $S \mapsto \mathrm{Hom}_{\mathcal{C}}(-, S)$: A family $\{F_i \to F\}$ in $\mathcal{C}$ is a covering if and only if the induced family in $\hat{\mathcal{C}}$ is a covering. Thus, one can regard all objects of $\mathcal{C}$ as objects of $\hat{\mathcal{C}}$, too.

A category $\mathcal{C}$ together with a topology on $\mathcal{C}$ is called a *site*.

Remark 4.1.2. Note that **(C 2)** and **(C 3)** already imply the following statement: Let $\{S_i \longrightarrow S\}$ be a family of morphisms and $\{T_j \longrightarrow S\}$ a covering such that for all j the family $\{S_i \times_S T_j \longrightarrow T_j\}$ is a covering. Then $\{S_i \longrightarrow S\}$ is already a covering (*local coverings are coverings*).

Remark 4.1.3. In [SGA3.1, Exp. IV] instead of the notion of covering families of morphisms, the notion of sieves, that is subobjects in $\hat{\mathcal{C}}$, is used mostly. These two notions are equivalent for categories possessing fiber products ([SGA3.1, Exp. IV, 4.2.4]). Today, the notion of families of morphisms is used almost exclusively. Therefore, all statements are presented here using the language of families of morphisms.

Remark 4.1.4. Today many authors use the weaker notion of a pretopology [SGA3.1, Exp. IV, 4.2.5]. A pretopology does not have to be saturated which makes handling it a lot easier. For most sheaf related statements it is sufficient to hold at the level of a pretopology ([SGA4.1, Exp. II, 2.4]) because a pretopology is cofinal in the generated topology ([SGA4.1, Exp. II, 1.4]).

Definition 4.1.5. Let $\mathcal{C}$ be a category with fiber products. Given for any $S \in \mathrm{ob}\,\mathcal{C}$ a set of families of morphisms in $\mathcal{C}$ one can consider the coarsest topology in which all the given families are covering families. This topology is well-defined by [SGA3.1, Exp. IV, 4.2.2] and is called the *topology generated by the given families.*

Definition 4.1.6. Let $\mathcal{C}$ be a category. A presheaf on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to (set) and thus an object of $\hat{\mathcal{C}} = \mathrm{Hom}(\mathcal{C}^{op}, (\mathrm{set}))$. A presheaf P is called *representable* if it is isomorphic to a presheaf in the image of the Yoneda embedding, i.e., there is an $S \in \mathcal{C}$ with $P \cong \mathrm{Hom}_{\mathcal{C}}(-, S)$. Assume $\mathcal{C}$ to be equipped with a topology. A presheaf P is called a *sheaf*, if for all $S \in \mathrm{ob}\,\mathcal{C}$ and all coverings $\{S_i \longrightarrow S\}$ the diagram

$$P(S) \longrightarrow \prod_i P(S_i) \rightrightarrows \prod_{i,j} P(S_i \times_S S_j)$$

is exact in the usual sense in (set). In general, one can assign to each presheaf P a sheaf $P^\dagger$ with the universal property $\mathrm{Hom}_{\hat{\mathcal{C}}}(P, G) = \mathrm{Hom}_{\tilde{\mathcal{C}}}(P^\dagger, G)$ for any sheaf G. The sheaf $P^\dagger$ is called the *sheaf associated to* P.

The category of sheaves on a site $\mathcal{C}$ is a full subcategory of the category of presheaves $\hat{\mathcal{C}}$ and often is denoted by $\tilde{\mathcal{C}}$. A category which is equivalent to the category of sheaves for some site is called a *topos*.

Definition 4.1.7. Let $\{F_i \xrightarrow{f_i} F\}$ be a family of morphisms in $\hat{\mathcal{C}}$. One defines the image of this family to be the presheaf $\mathrm{im}\{f_i\} : S \mapsto \bigcup_i f_i(S)(F_i(S)) \subset F(S)$.

Proposition 4.1.8. *Let F_i, G be sheaves, $\{F_i \xrightarrow{\varphi_i} G\}$ a covering family, and* $\mathrm{im}\{\varphi_i\} \subset G$ *the image presheaf. Then one has* $\mathrm{im}\{\varphi_i\}^\dagger = G$.

Proof. Let $T \in \mathcal{C}$ and $g : T \longrightarrow G \in \mathrm{Hom}_{\hat{\mathcal{C}}}(T, G) = G(T)$. From the diagram

$$\begin{tikzcd}
F_i \times_G T \arrow[r] \arrow[d, "f_i"] \arrow[dr, dashed, "g_i"] & T \arrow[d, "g"] \\
F_i \arrow[r, "\varphi_i"] & G
\end{tikzcd}$$

one gets morphisms $f_i : F_i \times_G T \longrightarrow F_i \in F_i(F_i \times_G T)$ which induce morphisms $g_i = \varphi_i \circ f_i : F_i \times_G T \longrightarrow G$ in $\mathrm{im}\{\varphi_i\}(F_i \times T)$. Furthermore, $\{F_i \times_G T \longrightarrow T\}$ is a covering by 4.1.1 and one has clearly $g_i = g_{|F_i \times_G T}$. Thus, the morphisms g_i form a gluing datum on $\{F_i \times_G T\}$ in $\mathrm{im}\{\varphi_i\}$ which glues to g in $\mathrm{im}\{\varphi_i\}^\dagger$. □

Remark 4.1.9. This proposition shows that a covering family which is usually thought of as something surjective becomes in fact surjective if one considers the image sheaf. This analogy is even more accurate as the following proposition shows the inverse statement:

Proposition 4.1.10. *Let $f : F \to G$ be a morphism of sheaves that is surjective, i.e.,* $\mathrm{im}\{f\}^\dagger = G$*. Then f is a covering morphism.*

Proof. Let $F, G \in \mathcal{C}$ be representable sheaves. Then for all $S \in \mathcal{C}$ the morphism $F(S) = \mathrm{Hom}_C(S, F) \to \mathrm{Hom}_C(S, G) = G(S)$ is surjective. In particular for the case $S = G$ one has $\mathrm{id} \in \mathrm{Hom}_{\mathcal{C}}(G, G)$. Since f is surjective as a morphism of sheaves, there is a covering $\{G_i \xrightarrow{\psi_i} G\}$ and there are morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(G_i, F) = F(G_i)$ such that $f \circ \varphi_i = \mathrm{id}|_{G_i} = \psi_i$ as in the following diagram:

$$\begin{tikzcd}
F \arrow[r, "f"] & G \\
G_i \arrow[u, "\varphi_i"] \arrow[r, equal] & G_i \arrow[u, "\psi_i"']
\end{tikzcd}$$

As the morphisms $f \circ \varphi_i = \psi_i$ form a covering of G, the saturation axiom (C3) shows that f is a covering as well.

Now let $F, G \in \hat{\mathcal{C}}$ be arbitrary sheaves. If all representable base changes of $F \to G$ are surjective and therefore coverings as shown before, $F \to G$ is a covering by (C0). Therefore, the proof is finished by the following lemma. □

Lemma 4.1.11. *Let $f : F \to G$ be a surjective morphism of sheaves. Then every base change of f is surjective, too.*

Proof. Let $g : H \to G$ be a morphism of sheaves. Let $X \in \mathcal{C}$ and $\psi \in H(X) = \mathrm{Hom}_{\hat{\mathcal{C}}}(X, H)$. Then $g \circ \psi \in \mathrm{Hom}_{\hat{\mathcal{C}}}(X, G)$. Therefore, there is a covering $\{X_i \xrightarrow{t_i} X\}$ and there are morphisms $s_i : X_i \to F$ with $f \circ s_i = (g \circ \psi)|_{X_i} = g \circ \psi \circ h_i$. The universal property of the fiber product shows that the morphisms s_i factor through $F \times_G H$, making the following diagram commute:

$$\begin{tikzcd}
F \arrow[r, two heads, "f"] & G \\
F \times_G H \arrow[u] \arrow[r, "f'"] & H \arrow[u, "g"'] \\
 & X \arrow[u, "\psi"'] \\
X_i \arrow[uu, dotted, "\varphi_i"] \arrow[uuu, bend left, "s_i"] \arrow[r, equal] & X_i \arrow[u, "t_i"']
\end{tikzcd}$$

Since $f' \circ \varphi_i = \psi|_{X_i}$, this shows that $F \times_G H \to H$ is a surjective morphism of sheaves. $\square$

Corollary 4.1.12. *Let $f : F \longrightarrow G$ be a morphism of sheaves. Then this morphism is surjective, i.e.,* $\mathrm{im}\{f\}^{\dagger} = G$, *if and only if it is covering.*

A lot of common topologies are based on the Zariski topology together with some refinement (e.g., the étale topology or the flat topology). The covering families in this topologies could be some quite complicated combinations of the morphisms used to construct the topology. The following proposition, however, shows that in special cases (which will be fulfilled in all our applications) the covering families can be easily characterized.

Proposition 4.1.13 ([SGA3.1, Exp. IV, 6.2.1])**.** *Let $\mathcal{C}$ be a category and $\mathcal{C}'$ a full subcategory. Let P be a set of families $\{S_i \to S\}$ of morphisms in $\mathcal{C}$, stable under base change and composition (i.e., they fulfill the axioms (P1) and (P2) in [SGA3.1, Exp. IV, 4.2.5]). Let P' be a set of families of morphisms $\{S_i \to S\}$ in $\mathcal{C}'$. Suppose P' contains the families consisting of just an identity-isomorphism (P3) and fulfills the following properties:*

(i) If $\{S_i \longrightarrow S\} \in P'$ (and thus $S_i, S \in \mathrm{ob}\,\mathcal{C}'$) and if $T \longrightarrow S$ is a morphism in $\mathcal{C}'$, then the fiber products $S_i \times_S T \in \mathrm{ob}\,\mathcal{C}$ exist and the family $\{S_i \times_S T \longrightarrow T\}$

is an element of P' (thus $S_i \times_S T \in \operatorname{ob}\mathcal{C}'$, too). This condition implies that P' is stable under base change, but is not equivalent as it implies further that the inclusion functor $\mathcal{C}' \longrightarrow \mathcal{C}$ commutes with certain fiber products.

(ii) For every $S \in \operatorname{ob}\mathcal{C}$ there is a family $\{S_i \longrightarrow S\} \in P$ with $S_i \in \operatorname{ob}\mathcal{C}'$ for all i.

(iii) In the situation

$$\begin{tikzcd}
 & S_{ijk} \arrow[d, "P'"] \\
S_i \arrow[d, "P'"] & S_{ij} \arrow[l, "P"] \\
S,
\end{tikzcd}$$

where $S, S_i, S_{ij}, S_{ijk} \in \operatorname{ob}\mathcal{C}'$, $\{S_i \longrightarrow S\} \in P'$, $\{S_{ij} \longrightarrow S_i\} \in P$ for all i and $\{S_{ijk} \longrightarrow S_{ij}\} \in P'$ for all i, j (like indicated in the diagram in a somewhat imprecise manner), there is a family $\{T_n \longrightarrow S\} \in P'$ and for every n there is a multi-index ijk such that the following diagram commutes:

$$\begin{tikzcd}
T_n \arrow[rr] \arrow[dr] & & S \\
 & S_{ijk} \arrow[ur] &
\end{tikzcd}$$

Let $\mathcal{C}$ be equipped with the topology generated by P and P'. Let $S \in \operatorname{ob}\mathcal{C}$ and $\{R_k \longrightarrow S\}$ a family of morphisms (in [SGA3.1] a sieve $R \longrightarrow S$ is used). Then this family is a covering if and only if there is a composite morphism $\{S_{ij} \longrightarrow S_i \longrightarrow S\}$, where $S_i, S_{ij} \in \operatorname{ob}\mathcal{C}'$, $\{S_i \longrightarrow S\} \in P$, $\{S_{ij} \longrightarrow S_i\} \in P'$ for all i such that the family $\{S_{ij} \longrightarrow S\}$ is a refinement of $\{R_k \longrightarrow S\}$, i.e., it factors through $\{R_k \longrightarrow S\}$: For every i, j there is a k such that the morphism $S_{ij} \longrightarrow S$ factors through $R_k \longrightarrow S$.

Proof. The families of P and P' are coverings and thus are also composite families of those by (C2). Therefore, families of which the composite families are refinements are coverings as well. This proves one direction of the claimed equivalence.

In order to show the converse, it suffices to show that these families already form a topology, i.e., they satisfy (C1) to (C4).

(C4) Let $S \in \operatorname{ob}\mathcal{C}$. According to (ii) there is a family $\{S_i \longrightarrow S\} \in P$ with $S_i \in \operatorname{ob}\mathcal{C}'$. The families $\{\mathrm{id}_{S_i} : S_i \longrightarrow S_i\}$ are elements of P' by assumption. Therefore the identity morphism of S is of the desired form:

$$\begin{tikzcd}
S_i \arrow[d, "\wr"] \arrow[dr] & \\
S_i \arrow[d] & S \arrow[dl] \\
S &
\end{tikzcd}$$

(C3) Clear.

(C1) Let $\{R_k \longrightarrow S\}$ be a covering of the desired form and $T \longrightarrow S$ a morphism in $\mathrm{ob}\,\mathcal{C}$. Then we consider the following diagram:

$$\begin{tikzcd}
& S_{ij} \arrow[d, "P'"] \arrow[ddl] & & U_{ilj} \arrow[lll] \arrow[d] \\
& S_i \arrow[d, "P"] & T_i \arrow[l] \arrow[d, "P"] & U_{il} \arrow[l, "P"'] \\
R_k \arrow[r] & S & T \arrow[l] &
\end{tikzcd}$$

Here, T_i is given by $T_i = S_i \times_S T$. Then $\{T_i \longrightarrow T\} \in P$ as P is stable under base change. By (ii) one obtains a family $\{U_{il} \longrightarrow T_i\} \in P$ with $U_{il} \in \mathrm{ob}\,\mathcal{C}'$. With P being stable under composition, one has $\{U_{il} \longrightarrow T\} \in P$ as well. By (i), $U_{ilj} = U_{il} \times_{S_i} S_{ij} \in \mathrm{ob}\,\mathcal{C}'$ and $\{U_{ilj} \longrightarrow U_{il}\} \in P'$. Since we have a diagram

$$\begin{tikzcd}
U_{ilj} \arrow[r] \arrow[d] & R_k \arrow[d] \\
T \arrow[r] & S,
\end{tikzcd}$$

the morphisms $\{U_{ilj} \longrightarrow R_k\}$ factor through $R_k \times_S T$, which shows that $\{R_k \times_S T \longrightarrow T\}$ is indeed of the desired form.

(C2) Let $\{R_k \longrightarrow S\}$ be of the desired form and for all i let $\{T_{kn} \longrightarrow R_k\}$ be of this form as well. Then, one has a commutative diagrams

$$
\begin{tikzcd}
S_{ij} \arrow[d, "P'"] \arrow[ddr] & R_{klm} \arrow[d, "P'"] \arrow[ddr] & \\
S_i \arrow[d, "P"] & R_{kl} \arrow[d, "P"] & \\
S & R_k \arrow[l] & T_{kn}. \arrow[l]
\end{tikzcd}
$$

From this one can construct a diagram

$$
\begin{tikzcd}
V_{ijkltm} \arrow[rr] \arrow[d, "P'"] \arrow[drr, phantom, "\square"] & & R_{klm} \arrow[d, "P'"] \arrow[ddddr] & \\
V_{ijklt} \arrow[r, "P"] \arrow[dr, "P"] & S_{ijkl} \arrow[r] \arrow[d, "P"] \arrow[dr, phantom, "\square"] & R_{kl} \arrow[d, "P"] & \\
& S_{ij} \arrow[r] \arrow[d, "P'"] & R_k \arrow[dd, equal] & \\
& S_i \arrow[d, "P"] & & \\
& S & R_k \arrow[l] & T_{kn}, \arrow[l]
\end{tikzcd}
$$

where $S_{ijkl} = S_{ij} \times_{R_k} R_{kl}$ and thus $\{S_{ijkl} \longrightarrow S_{ij}\} \in P$. Because of (ii) one gets a family $\{V_{ijklt} \longrightarrow S_{ijkl}\} \in P$ with $V_{ijklt} \in \operatorname{ob} \mathcal{C}'$ for all t. As P' is stable under base change, one gets a family of morphisms $\{V_{ijkltm} = V_{ijklt} \times_{R_{kl}} R_{klm} \longrightarrow V_{ijklt}\} \in P'$. From this, one obtains a composite family $\{V_{ijkltm} \longrightarrow V_{ijklt} \longrightarrow S_{ij} \longrightarrow S_i \longrightarrow S\}$ composed as $P \circ P' \circ P \circ P'$, which factors through $\{T_{kn} \longrightarrow S\}$ and in which all objects but S are in $\mathcal{C}'$. Now (iii) gives for every family $\{V_{ijkltm} \longrightarrow S_i\}$ a family $\{T_{iq} \longrightarrow S_i\} \in P'$ factoring through V_{ijkltm} and thus $\{T_{iq} \longrightarrow S_i \longrightarrow S\}$ factors through T_{kn}. This shows that $\{T_{kn} \longrightarrow S\}$ is indeed of the desired form.

$\square$

Corollary 4.1.14 ([SGA3.1, Exp. IV, 6.2.2])**.** *Let $S \in \operatorname{ob} \mathcal{C}'$ and $\{R_k \longrightarrow S\}$ be a family of morphisms. Then $\{R_i \longrightarrow S\}$ is covering if and only if there is a family $\{T_l \longrightarrow S\} \in P'$ factoring through $\{R_k \longrightarrow S\}$.*

Proof. Such a family is a covering by (C3). On the other hand, given a covering $\{R_k \longrightarrow S\}$, there exists a composite family $\{S_{ij} \longrightarrow S_i \longrightarrow S\}$ factoring through $\{R_k \longrightarrow S\}$. By composing with the identity isomorphism of S, one gets by (iii) the desired family $\{T_l \longrightarrow S\} \in P'$. $\square$

Corollary 4.1.15 ([SGA3.1, Exp. IV, 6.2.3])**.** *A presheaf F on $\mathcal{C}$ is a sheaf if and only if the diagram*

$$F(S) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

is exact for all families $\{S_i \longrightarrow S\}$ from the following two cases:

(i) $\{S_i \longrightarrow S\} \in P$.

(ii) S, $S_i \in \operatorname{ob} \mathcal{C}'$ *and* $\{S_i \longrightarrow S\} \in P'$.

Remark 4.1.16 ([SGA3.1, Exp. IV, 6.2.5])**.** Condition (iii) of 4.1.13 is satisfied in particular whenever

(i) P' is stable under composition and

(ii) if $\{S_i \longrightarrow S\}$ is a family of morphisms in $\mathcal{C}'$ and an element of P, then there is a subfamily which is in P'.

The following proposition is generalized from [SGA3.1, Exp. VI, Prop. 6.3.1 (i)].

Proposition 4.1.17. *Let $\mathcal{C}, \mathcal{C}'$ have finite direct sums and have an initial object I such that $S \times_{S \amalg T} T = I$. Suppose, for all S_i, $i = 1, \ldots, n$ one has $\{S_i \longrightarrow \coprod_i S_i\} \in P$. Suppose further that for all $\{S_i \longrightarrow S\} \in P'$ one has $\{\coprod_i S_i \longrightarrow S\} \in P'$. Then a presheaf F on $\mathcal{C}$ is a sheaf if and only if the sheaf sequence (4.1.6) is exact for all families from the following two cases:*

(i) $\{S_i \longrightarrow S\} \in P$.

(ii) $S', S \in \operatorname{ob} C'$ *and* $\{S' \longrightarrow S\} \in P'$.

Proof. Since we have $\{S_i \longrightarrow \coprod_i S_i\} \in P$, one has an exact sequence

$$F(\textstyle\coprod_i S_i) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_{\coprod_k S_k} S_j) = F(I) = \emptyset,$$

and thus $F(\coprod_i S_i) \simeq \prod_i F(S_i)$. Now let $S_i, S \in \operatorname{ob} C'$ and $\{S_i \longrightarrow S\} \in P'$. Then by assumption one has $\{\coprod_i S_i \longrightarrow S\} \in P'$ and a commutative diagram

$$\begin{tikzcd}
F(S) \arrow[r] \arrow[d, equal] & \prod_i F(S_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"] & \prod_{i,j} F(S_i \times_S S_j) \arrow[d, "\wr"] \\
F(S) \arrow[r] & F(\coprod_i S_i) \arrow[r, shift left] \arrow[r, shift right] & F(\coprod_{i,j} S_i \times_S S_j = \coprod_i S_i \times_S \coprod_j S_j).
\end{tikzcd}$$

Since the lower sequence is exact by assumption, the upper sequence is exact, too, and the conditions of 4.1.15 are fulfilled. This proves the proposition. $\square$

In order to do cohomology calculations on a site, we need to define a global section functor. In general, this is a little bit more tricky than on the usual sites like the Zariski site or the étale site, as there is not necessarily a final object in the site. However, the topos always has a final object. This is what will be used:

Definition 4.1.18. Let $\mathcal{C}$ be a site with topos $\mathcal{T}$ and let $T \in \mathcal{C}$ be an object. The section functor $\Gamma(T, -) : \mathcal{T} \longrightarrow (\mathrm{Set})$ is defined as $\Gamma(T, F) = F(T)$ for all sheaves $F \in \mathcal{T}$. Since $F(T) = \mathrm{Hom}_{\hat{\mathcal{C}}}(T, F)$ this is generalized by $\Gamma(G, F) = F(G) := \mathrm{Hom}_{\hat{\mathcal{C}}}(G, F)$ for all $G \in \mathcal{T}$.

Definition 4.1.19. Let $\mathcal{C}$ be a site. Then $\hat{e} : T \mapsto \{0\}$ is the final object in the category $\hat{\mathcal{C}}$ of presheaves on $\mathcal{C}$. Its associated sheaf e is the final object of the topos $\tilde{\mathcal{C}}$. The global section functor $\Gamma : \tilde{\mathcal{C}} \longrightarrow (\mathrm{Set})$ is defined as $\Gamma(\mathcal{F}) := \Gamma(e, \mathcal{F})$.

Remark 4.1.20. If the site $\mathcal{C}$ has a final object X (e.g., X in $X_{\text{ét}}$ or X_{ZAR}), then e is represented by X, thus $\Gamma(\mathcal{F}) = \mathcal{F}(X)$. In general one has, by the universal property of the associated sheaf, $\mathrm{Hom}_{\tilde{\mathcal{C}}}(e, \mathcal{F}) = \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{e}, \mathcal{F})$, where a morphism φ in $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{e}, \mathcal{F})$ is given by a compatible system of maps $\varphi(T) : \hat{e}(T) = \{0\} \longrightarrow \mathcal{F}(T)$ for all $T \in \mathcal{C}$. Such a system of maps is just a compatible system of elements $\varphi(T)(0) \in \mathcal{F}(T)$ for all $T \in \mathcal{C}$. Thus, one has $\Gamma(\mathcal{F}) = \varprojlim_{T \in \mathcal{C}} \mathcal{F}(T)$.

With these definitions, cohomology for, e.g., abelian sheaves can be defined in the usual way as the derived functor of the global section functor (see, e.g., [Tam94, 3.3]). If a site has a canonical structure sheaf, its notation is often suppressed, e.g., $H^q_{\mathrm{cris}}(X/S) = H^q((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S})$ denotes the q-th cohomology group of the sheaf $\mathcal{O}_{X/S} \in (X/S)_{\mathrm{cris}}$.

5. The Syntomic Sites

In this chapter, the techniques developed the previous chapter will be used to construct the different syntomic sites. First, the syntomic site of a scheme will be constructed. Secondly, the crystalline syntomic site will be constructed. Finally some comparison theorems will be proved. They will allow to construct comparison morphisms between the different topoi and thus give rise to the possibility to calculate some crystalline cohomology groups on the syntomic site – a technique fundamental for the proof of the conjecture of Birch and Swinnerton-Dyer as given in [Bau92] that will be presented later in this thesis.

5.1. The Syntomic Site of a Scheme

Definition 5.1.1. Let Y be a scheme. Let $\mathcal{C}$ be the category of Y-schemes and $\mathcal{C}' \subset \mathcal{C}$ the full subcategory of absolute affine Y-schemes, i.e., Y-schemes that are affine as a scheme (not as a scheme over Y). Let P be the set of surjective families of open immersions in $\mathcal{C}$ and P' the set of finite surjective families of syntomic morphisms in $\mathcal{C}'$. Then the site of $\mathcal{C}$ together with the topology generated by P and P' (cf. 4.1.5) is called *(big) syntomic site* $\mathrm{SYN}(Y)$ *of* Y. The full subcategory $\mathrm{syn}(Y) \subset \mathrm{SYN}(Y)$ of syntomic Y-schemes with the induced topology is called *small syntomic site of* Y. The associated topoi are denoted by Y_{SYN} and Y_{syn}.

Proposition 5.1.2. *A family of morphisms* $\{Z_k \longrightarrow X\}$ *is a covering in* $\mathrm{SYN}(Y)$ *if and only if there is a composite family* $\{X_{ij} \longrightarrow X_i \longrightarrow X\}$ *with* X_{ij}, X_i *affine for all* i, j, $\{X_i \longrightarrow X\} \in P$ *and* $\{X_{ij} \longrightarrow X_i\} \in P'$ *such that* $\{X_{ij} \longrightarrow X_i \longrightarrow X\}$ *factors through* $\{Z_k \longrightarrow X\}$. *In particular, these composite families form a pretopology.*

Proof. It suffices to show that P and P' fulfill the conditions in 4.1.13. Since syntomic morphisms are stable under base change (3.1.3), (i) is fulfilled. Condition (ii) is obviously fulfilled. For (iii) it is with 4.1.16 sufficient to show that syntomic families are stable under composition (3.1.3) and that for all families $\{X_i \longrightarrow X\} \in P$ with $X_i, X \in \mathrm{ob}\,\mathcal{C}'$ there is a subfamily of $\{X_i \longrightarrow X\}$ in P'. But as affine schemes are quasi-compact, there is a finite subfamily of $\{X_i \longrightarrow X\}$ which is surjective and thus in P', as open immersions are syntomic (3.1.3). □

Corollary 5.1.3. *A presheaf on* $\mathrm{SYN}(Y)$ *is a sheaf if and only if the sheaf sequence is exact for all* $\{X_i \to X\} \in P$ *and for all* $\{Y \to X\} \in P'$.

Proof. This follows from 4.1.17. □

Remark 5.1.4. The syntomic topology is obviously finer than the étale or the smooth topology. On the other hand it is coarser than the flat or $fppf$-topology. The next propositions shall make the position of the syntomic topology relatively to the other topologies clearer.

One of the big advantages of the étale topology over the Zariski topology is that it is fine enough to have an exact Kummer sequence in characteristic $p = 0$ ([SGA4 1/2, II 2.5]). In the flat or the $fppf$-topology the Kummer sequence is always exact. The syntomic topology is already fine enough to show the same behaviour:

Proposition 5.1.5 ([Bau92, 1.4])**.** *Let S be a scheme of characteristic $p > 0$, G/S a smooth group scheme for which multiplication by p is a faithfully flat morphism. Let $\mathcal{G}$ be the sheaf represented by G on $\mathrm{SYN}(S)$. Then $\mathcal{G} \xrightarrow{p^n} \mathcal{G}$ is an epimorphism of syntomic sheaves for any $n \in \mathbb{N}$. In particular for $\mathbb{G}_m$ resp. an abelian scheme A/S one gets exact sequences in $\mathrm{SYN}(S)$:*

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \longrightarrow 0$$

$$0 \longrightarrow {}_{p^n}A \longrightarrow A \xrightarrow{p^n} A \longrightarrow 0$$

Proof. The morphism $p^n : G \longrightarrow G$ is a flat morphism of smooth schemes and thus syntomic by 3.1.4. Since it is faithfully flat, it is surjective and thus a syntomic covering. By definition, the induced morphism $\mathcal{G} \longrightarrow \mathcal{G}$ is a covering as well. In 4.1.8 it has been shown that then this morphism is surjective. The first part of the sequence is of course exact by definition of ${}_{p^n}A$ resp. μ_{p^n}. □

Lemma 5.1.6. *Let G/S be a smooth group scheme. Then for all $j \geq 0$ the canonical maps*

$$H^j(S_{\mathrm{SYN}}, G) \to H^j(S_{\mathrm{fppf}}, G)$$

induced by the Leray spectral sequence for the morphism $S_{\mathrm{fppf}} \longrightarrow S_{\mathrm{SYN}}$ obtained by coarsing the topology are isomorphisms.

Proof. In [Mil80, III 3.9] it is shown that $H^j(S_{\mathrm{fl}}, G) \cong H^j(S_{\mathrm{\acute{e}t}}, G)$. The proof, however, shows already that $H^j(S_{\mathrm{\acute{e}t}}, G)$ is isomorphic to any cohomology of a topology between the étale and the flat topology. □

Corollary 5.1.7 ([Bau92, 1.5])**.** *Let A/S be an abelian scheme. Then for any $j \geq 0$ one has*

(i) $H^j(S_{\mathrm{fppf}}, {}_{p^n}A) \simeq H^j(S_{\mathrm{SYN}}, {}_{p^n}A)$.

(ii) $H^j(S_{\mathrm{fppf}}, \mu_{p^n}) \simeq H^j(S_{\mathrm{SYN}}, \mu_{p^n})$,

Proof. For all sheaves F on S_{fppf} there are canonical homomorphisms $H^j(S_{\text{SYN}}, \varphi_* F) \to H^j(S_{\text{fppf}}, F)$ from the Leray spectral sequence for the morphism $\varphi : S_{\text{fppf}} \longrightarrow S_{\text{SYN}}$ (note that $\varphi_* F = F$, as the syntomic topology is coarser than the fppf topology). By the above lemma 5.1.6 these morphisms are isomorphisms for a sheaf represented by a smooth group scheme. Thus, the statement follows from the five-lemma in the diagram

$$\begin{array}{ccccccccc} H^{j-1}(S_{\text{SYN}}, A) & \longrightarrow & H^{j-1}(S_{\text{SYN}}, A) & \longrightarrow & H^j(S_{\text{SYN}}, {}_{p^n}A) & \longrightarrow & H^j(S_{\text{SYN}}, A) & \longrightarrow & H^j(S_{\text{SYN}}, A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{j-1}(S_{\text{fppf}}, A) & \longrightarrow & H^{j-1}(S_{\text{fppf}}, A) & \longrightarrow & H^j(S_{\text{fppf}}, {}_{p^n}A) & \longrightarrow & H^j(S_{\text{fppf}}, A) & \longrightarrow & H^j(S_{\text{fppf}}, A). \end{array}$$

The statement for μ_{p^n} of course follows by setting $A = \mathbb{G}_m$. □

Proposition 5.1.8. *The syntomic site* $\mathrm{SYN}(Y)$ *has enough points: A morphism of syntomic sheaves* $f : F' \longrightarrow F$ *in* Y_{SYN} *is an isomorphism if and only if it is an isomorphism in all points*

Proof. See [Sta, 38.30.1]. □

Remark 5.1.9. Although by the above proposition we know that the syntomic site has enough points, so far there are no results on what these points actually are. This is not the case for, e.g., the étale topology, where we know that the points are the spectra of strict hensilian rings or the Nisnevich topology, where the points are the spectra of hensilian rings.

In the following an exactness criterion on some syntomic sites will be proven which will be crucial for the techniques used in the proof of the conjecture of Birch and Swinnerton-Dyer [Bau92] discussed later. It is generalized from [Koe89], where it is only given for the special case $R = W_s(k)$, $Y = \operatorname{Spec} R$ for k an perfect field of characteristic p. The version given here is able to handle a much wider range of base schemes.

Proposition 5.1.10. *Let R be a noetherian ring and Y a syntomic R-scheme. The set*

$$\begin{aligned} \mathcal{M}_Y := \{ X_0 = \operatorname{Spec} A_0 \in \operatorname{syn}(Y) \ &\textit{with } A_0 \cong R[T_1, \ldots, T_m]/(P_1, \ldots, P_d) \\ &\textit{with a sequence-regular sequence } P_1, \ldots, P_d \in R[T_1, \ldots, T_m] \} \end{aligned}$$

is cofinal in $\operatorname{syn}(Y)$.

Proof. By 3.1.6, $\mathcal{M}_Y$ is cofinal in $\operatorname{syn}(Y)$: We have $\operatorname{syn}(Y) \subset \operatorname{syn}(\operatorname{Spec} R)$ as $Y \to \operatorname{Spec} R$ is syntomic. □

For an scheme $X_0 \in \mathcal{M}_Y$ we define a family $X_i := \operatorname{Spec} A_i \in \operatorname{syn}(Y)$ by base change with the morphism $\varphi : \mathbb{A}^m_R \longrightarrow \mathbb{A}^m_R$ given on $A[T_1, \ldots, T_m]$ by $T_i \mapsto T_i^p$ (this is the relative Frobenius in those cases where a relative Frobenius exists):

$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[d] & \mathbb{A}^m_A \arrow[d, "\varphi^i"] \\ X_0 \arrow[r, hook] & \mathbb{A}^m_A \end{tikzcd}$$

By 3.1.8 all morphisms $X_i \to X_0$ are syntomic coverings. Having established these prerequisites, we can formulate the exactness criterion:

Proposition 5.1.11 ([Koe89])**.** *Let R be a noetherian ring and Y a syntomic R-scheme. Let*

$$\mathcal{F}_\bullet := \mathcal{F}_1 \xrightarrow{f} \mathcal{F}_2 \xrightarrow{g} \mathcal{F}_3$$

be a sequence of abelian presheaves on $\mathrm{syn}(Y)$ *such that* $\varinjlim_i \mathcal{F}_\bullet(X_i)$ *is exact in the category of abelian groups (Ab) for all $X_0 \in \mathcal{M}_Y$. Then the sheafified sequence $\tilde{\mathcal{F}}_\bullet$ is an exact sequence of abelian sheaves.*

Proof. First we will show that $\tilde{g} \circ \tilde{f} = 0$: For this let $U \in \mathrm{syn}(Y)$ and let $s \in \tilde{\mathcal{F}}_1(U)$ be a section. By passing to a covering we can assume that $s \in \mathcal{F}_1(U)$ by the definition of the sheafification. As mentioned above, this U can be covered with objects from $\mathcal{M}_Y$, thus it can be assumed that $U = X_0 \in \mathcal{M}_Y$. But then by hypothesis $(g \circ f)(s) = 0 \in \varinjlim_i \mathcal{F}_3(X_i)$ which shows that there is an i such that $(g \circ f)(s)|_{X_i} = 0$. Now, because $X_i \longrightarrow X_0$ is a syntomic covering, this shows that $(\tilde{g} \circ \tilde{f})(s) = 0 \in \tilde{\mathcal{F}}_3(X_0)$.

It remains to show that $\ker(\tilde{g}) \subseteq \mathrm{im}(\tilde{f})$. For this, let again be $U \in \mathrm{syn}(Y)$ and $s \in \ker(\tilde{g})(U) \subseteq \tilde{\mathcal{F}}_2(U)$. By passing to coverings it can be assumed first that $s \in \mathcal{F}_2(U)$ and then even $g(s) = 0$. With another covering one gets $U = X_0 \in \mathcal{M}_Y$. Now by hypothesis there is a $t \in \varinjlim_i \mathcal{F}_1(X_i)$ with $f(t) = s \in \varinjlim_i \mathcal{F}_2(X_i)$ and thus there is an i with $s|_{X_i} \in \mathrm{im}\tilde{f}$. Exactly as above, $X_i \longrightarrow X_0$ is a syntomic covering and thus $s \in \mathrm{im}\tilde{f}(U)$, hence the statement. ☐

Now let k be a perfect field of characteristic p, $R = W_s(k)$ and Y an syntomic R-scheme. The following proposition underlines the importance of the exactness criterion:

Lemma 5.1.12 ([Koe89])**.** *Let $A = \varinjlim_i A_i$ for $A_0 \cong W_s[T_1, \ldots, T_m]/(P_1, \ldots, P_d)$ as in the definition of $\mathcal{M}_Y$. Then one has*

$$A \cong W_s[T_1^{p^{-\infty}}, \ldots, T_m^{p^{-\infty}}]/(P_1, \ldots, P_d)$$

and the absolute Frobenius of $A/(p)$ is surjective.

Proof. By definition one has

$$W_s[T_1^{p^{-\infty}}, \ldots, T_m^{p^{\infty}}] = \varinjlim(W_s[T_1, \ldots, T_m] \xrightarrow{\text{rel. Frob.}} W_s[T_1, \ldots, T_m] \longrightarrow \ldots).$$

and by definition of $X_i = \operatorname{Spec} A_i$ one has

$$A_i = W_s[T_1^{p^{-i}}, \dots, T_m^{p^{-i}}]/(P_1, \dots, P_d),$$

thus, the identity follows.

Since $W_s/(p) = k$ is perfect, the absolute Frobenius on $k[T_1^{p^{-\infty}}, \dots, T_m^{p^{-\infty}}]$ is surjective and therefore on A, too. □

Corollary 5.1.13. *Let Y be a syntomic k-scheme and $\mathcal{O}_Y$ the structure sheaf on* $\operatorname{syn}(Y)$*. Then*

$$\mathcal{O}_Y \xrightarrow{Frob} \mathcal{O}_Y$$

is surjective. In particular, this gives a more hands-on proof for the exactness of the Kummer sequence on the syntomic site of a perfect field in characteristic $p > 0$ than the one given in 5.1.5.

Proof. The structure sheaf $\mathcal{O}_Y$ is defined by $\mathcal{O}_Y(Z) = \Gamma(Z, \mathcal{O}_Z)$. Let $X_0 = \operatorname{Spec} A_0 \in M_Y$ with $A_0 \cong k[T_1, \dots, T_m]/(P_1, \dots, P_d)$ for a sequence-regular sequence $P_1, \dots, P_d$. Then one has $\mathcal{O}_Y(X_0) = k[T_1, \dots, T_m]/(P_1, \dots, P_d)$ and therefore $\varinjlim_i \mathcal{O}_Y(X_i) = k[T_1^{p^{-\infty}}, \dots, T_m^{p^{\infty}}]/(P_1, \dots, P_d) =: A$. Hence, by 5.1.12 the absolute Frobenius on $A/(p) = A$ is surjective. By 5.1.11, this shows that $(-)^p : \mathcal{O}_Y \to \mathcal{O}_Y$ is surjective. □

5.2. The Crystalline-Syntomic Site

The site underlying the crystalline topology can also be equipped with a syntomic topology which will make way for interesting comparison theorems. The following part will make some use of the concepts of crystalline topology and divided power structures. For a comprehensive introduction into these topics, see [BO78]. The definition given here is based on [Koe89].

In the following p is a prime, (S, I, γ) a PD-scheme. Let X be an S-scheme such that p is locally nilpotent on X and that γ extends to X.[1]

In the important case of the conjecture of Birch and Swinnerton-Dyer discussed later, we will have $S = \operatorname{Spec} W(\mathbb{F}_p) = \operatorname{Spec} \mathbb{Z}_p$ or $S = \operatorname{Spec} W_n(\mathbb{F}_p) = \operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z}$.

Definition 5.2.1. Let $\mathrm{CRIS}(X/S, I, \gamma)$ or short $\mathrm{CRIS}(X/S)$ be the category whose objects are quadruples (U, T, i, δ) (or short (U, T, δ)) with U an X-scheme, T an S-scheme where p is locally nilpotent, $i : U \hookrightarrow T$ a closed S-immersion and δ divided powers on the ideal defining i in $\mathcal{O}_T$ compatible with γ. In particular, the immersion i is a nil immersion. A morphism $(U', T', \delta') \longrightarrow (U, T, \delta)$ is a commutative diagram

[1][Bau92] actually demands that γ extends to all X-schemes. However, [BO78, 5] shows that this is not necessary: The PD structure γ already extends to all Zariski open subschemes of X if it extends to X.

$$
\begin{tikzcd}
U' \arrow[r, hook, "i'"] \arrow[d, "u"] & T' \arrow[d, "v"] \\
U \arrow[r, hook, "i"] & T,
\end{tikzcd}
$$

where u is an X-morphism and v an S-PD-morphism. A short notation is just (u, v) or, when there is no confusion possible, even just $T' \longrightarrow T$.

Definition 5.2.2.

(i) An object (U, T, δ) is called *affine* if T is affine.

(ii) A morphism (u, v) is called *cartesian*, if the commutative diagram is cartesian.

(iii) A morphism (u, v) is called open immersion, syntomic, surjective, etc., if it is cartesian and v is an open immersion, syntomic, surjective, etc.

Definition 5.2.3. Let $\mathrm{CRIS}'(X/S) \subset \mathrm{CRIS}(X/S)$ the full subcategory consisting of all affine (U, T, δ). Let P be the set of surjective families of open immersions in $\mathrm{CRIS}(X/S)$ and P' the set of finite surjective families of syntomic morphisms in $\mathrm{CRIS}'(X/S)$. The category $\mathrm{CRIS}(X/S)$ with the topology generated by P and P' is called *big crystalline-syntomic site* $\mathrm{CRIS}(X/S)_{\mathrm{SYN}}$ of X/S. The full subcategory $\mathrm{Cris}(X/S)_{\mathrm{syn}}$ of objects $i : U \longrightarrow T$ where $U \longrightarrow X$ is syntomic together with the induced topology is called *small crystalline-syntomic site* of X/S. The associated topoi are denoted by $(X/S)_{\mathrm{CRIS,SYN}}$ and $(X/S)_{\mathrm{cris,syn}}$.

The usual crystalline site and topos as introduced in [BO78], i.e., only with P as coverings, will be denoted as $\mathrm{CRIS}(X/S)_{\mathrm{ZAR}}$ and $(X/S)_{\mathrm{CRIS,ZAR}}$.

Remark 5.2.4. This is the definition as given in [Koe89]. Berthelot actually defines in [BO78] the crystalline site slightly different. He defines only the small site: He requires for the objects (U, T) that U is Zariski-open in X and he requires for morphisms (u, v) that u is an open immersion.

Furthermore, he does not require the coverings to be cartesian, explicitly. However, this is equivalent on the small site:. Let an arbitrary morphisms $(u, v) : (U', T') \to (U, T)$ be given such that the morphism $v : T' \to T$ is an open immersion. As we are working on the small site, the morphism $u : U' \to U$ is an open immersion, too. Furthermore the morphisms $U' \to T'$ and $f : U \to T$ are nilpotent closed immersions, hence topological homeomorphisms. Therefore, one has an identification $U' = f^{-1}(T') = U \times_T T'$.

On the big site, the definition given here behaves better than the one given by Berthelot in [BO78]: For an open immersion (u, v), the morphisms u and v are guaranteed to be open immersions, like for any property stable under base change. The definition given by Berthelot ensures only that v has the requested property.

Note that for a morphism $(U', T', \delta') \to (U, T, \delta)$ surjectivity of $T' \to T$ and $U' \to U$ is equivalent, as they are topologically identical.

Later, in 5.3.10 we will show that the crystalline sheaves $\mathcal{O}_{X/S}$, $\mathcal{O}$ and $\mathcal{I}_{X/S}$ defined in [BO78, 5.2] are sheaves on the syntomic crystalline site as well. A sheaf $\mathcal{F}$ on $\mathrm{CRIS}(X/S)$ is often characterized by the Zariski sheaves $\mathcal{F}_{(U,T)}$ on T for all $(U,T) \in \mathrm{CRIS}(X/S)$ given by $\mathcal{F}_{(U,T)}(T') = \mathcal{F}(U \cap T', T')$ (cf. [BO78, 5.1]).

Lemma 5.2.5. *In* $\mathrm{CRIS}(X/S)$*, fiber products exist and syntomic morphisms are stable under base change.*

Proof. Consider the situation

$$\begin{tikzcd} & (U_1, T_1, \delta_1) \arrow[d] \\ (U', T', \delta') \arrow[r, "\text{synt.}"] & (U, T, \delta) \end{tikzcd}$$

in $\mathrm{CRIS}(X/S)$. The fiber product $(U_1 \times_U U', T_1 \times_T T')$ can be constructed via the following commutative diagram:

$$\begin{tikzcd} & U_1 \times_U U' \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & T_1 \times_T T' \arrow[dl] \arrow[dd] \\ U_1 \arrow[rr] \arrow[dd] & & T_1 \arrow[dd] & \\ & U' \arrow[rr, dotted] \arrow[dl, dotted] & & T' \arrow[dl, "\text{synt.}"] \\ U \arrow[rr] & & T & \end{tikzcd}$$

The bottom square is cartesian because $(U', T', \delta') \longrightarrow (U, T, \delta)$ is syntomic. The left and the right squares are cartesian by construction. Thus, the universal property of the fiber product shows that the upper square is cartesian as well.

Since $T' \longrightarrow T$ is syntomic, so is $T_1 \times_T T' \longrightarrow T_1$. Hence $(U_1 \times_U U', T_1 \times_T T', \delta'') \longrightarrow (U_1, T_1, \delta_1)$ is syntomic. $\square$

With these properties, the coverings in $\mathrm{CRIS}(X/S)_{\mathrm{SYN}}$ again can be characterized by 4.1.13:

Corollary 5.2.6. *A family of morphisms* $\{(U'_k, T'_k) \longrightarrow (U,T)\}$ *is a covering in* $\mathrm{CRIS}(X/S)_{\mathrm{SYN}}$ *if and only if there is a composite family* $\{(V_{ij}, T_{ij}) \longrightarrow (V_i, T_i) \longrightarrow (U,T)\}$ *with* (V_{ij}, T_{ij}) *and* (V_i, T_i) *affine for all* i, j*,* $\{(V_i, T_i) \longrightarrow (U,T)\} \in P$ *and*

$\{(V_{ij}, T_{ij} \longrightarrow (V_i, T_i)\} \in P'$ *such that* $\{(V_{ij}, T_{ij}) \longrightarrow (V_i, T_i) \longrightarrow (U, T)\}$ *factors through* $\{(U'_k, T'_k) \longrightarrow (U, T)\}$. *In particular, these composite families form a pretopology.*

Proof. It suffices to show that P and P' fulfill the conditions in 4.1.13. Syntomic crystalline morphisms are stable under base change by 5.2.5. Condition (ii) is fulfilled as all schemes allow an affine covering by Zariski open subschemes. This property lifts to the crystalline site by base change. For (iii) the argument is the same as in 5.1.2. $\square$

Corollary 5.2.7. *A presheaf on* $\mathrm{CRIS}(Y)_{\mathrm{SYN}}$ *is a sheaf if and only if the sheaf sequence is exact for all* $\{(U_i, T_i) \to (U, T)\} \in P$ *and for all* $\{(U', T') \to (U, T)\} \in P'$.

Proof. This follows from 4.1.17. $\square$

In general, the category $\mathrm{CRIS}(X/S)$ has no final object. Because of this, the global section functor is only defined via the final presheaf (4.1.19). However, here we will show that in some cases final objects in $\mathrm{CRIS}(X/S)$ do exist, which will be very important later on. This follows the argument which Köck gives in [Koe89] based on a talk of Günther Tamme in Oberwolfach ([Tam89]), but is generalized in order to allow the case of $\mathrm{syn}(S)$ for S a syntomic $W_n(k)$-scheme which will be a smooth curve over $\mathbb{F}_p$ in the conjecture of Birch and Swinnerton-Dyer later on in this thesis.

In the following let k be a perfect field of characteristic $p > 0$. Let $W_n = W_n(k)$ be the ring of Witt vectors of length n. Unless otherwise noted, k is considered a $W_n(k)$-algebra via

$$W_n(k) \xrightarrow{\mathrm{Frob}^{-n}} W_n(k) \xrightarrow{\pi_0} k,$$

where π_0 is the projection onto the first component, i.e., reduction modulo p.

Furthermore, let A be a k-algebra and (B, I) an affine PD-thickening of A/W_n: (B, I) is a PD-algebra over (W_n, pW_n) together with a W_n-isomorphism $B/I \to A$, where A is considered a W_n-algebra via the W_n-algebra structure of k (see above). First we need a technical lemma:

Lemma 5.2.8. *Let* $0 \leq r \leq n$, $b \in B$, *and* $\delta \in I$. *Then*

$$(b + \delta)^{p^r} \equiv b^{p^r} \mod p^r I.$$

Proof. This is proven using induction on r. For $r = 0$ the statement is obvious. Assume it is known for r. Then by induction hypothesis $(b+\delta)^{p^{r+1}} = (b^{p^r} + p^r\delta')^p$. Modulo $p^{r+1}I$ we get

$$(b^{p^r} + p^r\delta')^p \equiv b^{p^{r+1}} + p^{pr}\delta'^p \equiv b^{p^{r+1}} + p^{pr}p!\gamma_p(\delta'),$$

because $\delta' \in I$ and I is a PD-ideal (γ_p denotes the p-th divided powers on I). The last term is of course equal to $b^{p^{r+1}} \mod p^{r+1}$ $\square$

Corollary 5.2.9. *There is a well-defined homomorphism of W_n-algebras given by*

$$\begin{array}{ccccc} \theta_n : W_n(A) & \dashrightarrow & W_n(B) & \xrightarrow{\text{ghost comp. } W_n} & B \\ (a_0, \dots, a_{n-1}) & \mapsto & (\hat{a}_0, \dots, \hat{a}_{n-1}) & \longmapsto & \hat{a}_0^{p^n} + \dots + p^{n-1}\hat{a}_{n-1}^p, \end{array}$$

where $\hat{a}_i$ denotes a lift of $a_i \in A$ under the projection $B \longrightarrow A$.

Proof. The morphism is well-defined by the above lemma. It is a ring homomorphism because the ghost component $W_n(B) \longrightarrow B$ is so. The W_n-linearity needs more work:

Let $s : W_n \longrightarrow k$ be the W_n-structure morphism of k (see above), $\varphi : k \longrightarrow A$ be the structure morphism of A, $\varphi' = \varphi \circ s : W_n \longrightarrow A$ the induced W_n-structure morphism and $\psi : W_n \longrightarrow B$ the structure morphism of B. The induced structure morphism $W_n(k) \longrightarrow W_n(A)$ is denoted by $\tilde{\varphi}$.

Now let $x = (x_0, \dots, x_{n-1}) \in W_n(k)$. We have to show that $\theta_n(\tilde{\varphi}(x)) = \psi(x)$. First note that $\varphi(x_i) = \varphi'(\hat{x}_i)$ for arbitrary lifts of x_i in $W_n(k)$ by construction of φ'. Thus, $\tilde{\varphi}((x_0, \dots, x_{n-1})) = (\varphi(x_0), \dots, \varphi(x_{n-1})) = (\varphi'(\hat{x}_0), \dots, \varphi'(\hat{x}_{n-1})) \in W_n(A)$. As $B \longrightarrow A$ is a morphism of W_n algebras, $(\psi(\hat{x}_0), \dots, \psi(\hat{x}_{n-1}))$ is a lift in $W_n(B)$ and thus $\theta_n(\tilde{\varphi}(x)) = \psi(\hat{x}_0)^{p^n} + \dots + p^{n-1}\psi(\hat{x}_{n-1})^p$.

On the other hand we have in $x = (x_0, \dots, x_{n-1}) = \hat{x}_0^{p^n} + \dots + p^{n-1}\hat{x}_{n-1}^p$ in $W_n(k)$ (A.1.2) and this shows that $\psi(x) = \theta_n(\tilde{\varphi}(x)) \in B$. □

This definition can be extended to global sections:

Proposition 5.2.10. *Let $(X, T) \in \mathrm{CRIS}(S/W_n)$. There is a homomorphism of W_n-algebras which is functorial in (X, T)*

$$\theta_n : W_n(\Gamma(X_1, \mathcal{O}_{X_1})) \longrightarrow \Gamma(T, \mathcal{O}_T),$$

where $X_1 = X \times_{W_n} k$.

Proof. First assume $(U, T) \in \mathrm{CRIS}(S/W_n)$ be affine, i.e., T is affine. Let $B = \Gamma(T, \mathcal{O}_T)$ and $A = \Gamma(U, \mathcal{O}_U)/(p)$. The ideal $I = (p) + \ker(B \twoheadrightarrow \Gamma(U, \mathcal{O}_U)) = \ker(B \to A)$ is equipped with a PD-structure. That makes (B, I) an affine PD-thickening of A. By the above proposition there is a homomorphism of W_n-algebras which is functorial in (U, T):

$$\theta_n : W_n(\Gamma(U, \mathcal{O}_U)/(p)) \longrightarrow \Gamma(T, \mathcal{O}_T)$$

Note that $A = \Gamma(U, \mathcal{O}_U)/(p) = \Gamma(U, \mathcal{O}_U) \otimes_{W_n} k$. Therefore, by gluing we get the statement. □

In the next step, this morphism will be extended to PD-structure. For this define for an S-scheme X the ideals

$$\begin{aligned} \mathcal{I}_n(X) &:= \ker(\theta_n : W_n(\Gamma(X_1, \mathcal{O}_{X_1}) \to \Gamma(X_1, \mathcal{O}_{X_1})) \\ &= \left\{ (a_0, \dots, a_{n-1}) \in W_n(\Gamma(X_1, \mathcal{O}_{X_1})) \;\middle|\; a_0^{p^n} = 0 \right\}, \\ \mathcal{J}_n(X) &:= \ker(\theta_n : W_n(\Gamma(X_1, \mathcal{O}_{X_1}) \to \Gamma(X, \mathcal{O}_X)), \end{aligned}$$

corresponding to the homomorphisms θ_n for the trivial PD-thickenings (X_1, X_1) and (X, X).

Lemma 5.2.11. *One has*

$$\mathcal{J}_n(X) + (p) \subseteq \mathcal{I}_n(X).$$

If the Frobenius is surjective on X_1, this is an equality.

Proof. Obviously one has $\mathcal{J}_n(X) + (p) \subset \mathcal{I}_n(X)$. Let the Frobenius on X_1 be surjective and $A = \Gamma(X, \mathcal{O}_X)$. Now let $a = (a_0, \ldots, a_{n-1}) \in \mathcal{I}_n(X)$, i.e., $(a_0, \ldots, a_{n-1}) \in W_n(A/p)$ with $a_0^{p^n} = 0 \in A/p)$, that is $\hat{a}_0^{p^n} = py$ in A for some $y \in A$. Define $b = (0, a_1, \ldots, a_{n-1})$. Since the Frobenius on A/p is surjective, one has $b \in (p)$. Furthermore let $\bar{y}$ be the projection of y in A/p and $c_1^{p^{n-1}} = \bar{y}$. By defining $c = (0, c_1, 0, \ldots, 0) \in (p)$ we have $\theta_n(a - y - c) = 0 \in A$. This shows $a - y - c \in \mathcal{J}_n(X)$ and thereby $a \in \mathcal{J}_n(X) + (p)$. $\square$

Remark 5.2.12. If the Frobenius on X_1 is not surjective, the inclusion is in general proper: Let $n = 2$, $k = \mathbb{F}_p$ and $A = \mathbb{Z}/p^2[X]$. Then $(0, T) \in W_2(A/p)$ is obviously in $\mathcal{I}_n(\operatorname{Spec} A)$, but as there is no p-th root of T, there is no change of $(0, T)$ lying within $\mathcal{J}_n(\operatorname{Spec} A) + (p)$. Köck claims in [Koe89] actually that the equality in 5.2.11 holds always. But the statement is used only in cases of surjective Frobenius, thus it does not matter.

With this let

$$W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1})) := D_{W_n(\Gamma(X_1, \mathcal{O}_{X_1})), (p)}(\mathcal{J}_n(X))$$

be the PD-envelope of $W_n(\Gamma(X_1, \mathcal{O}_{X_1}))$ with respect to the ideal $\mathcal{J}_n(X)$ over the ring $(W_n, (p))$ (For the notation cf. [BO78, 3.19]).

If the Frobenius on X_1 is surjective, one has

$$D_{W_n(\Gamma(X_1, \mathcal{O}_{X_1})), (p)}(\mathcal{I}_n(X)) = D_{W_n(\Gamma(X_1, \mathcal{O}_{X_1})), (p)}(\mathcal{J}_n(X))$$

because of the equality $\mathcal{J}_n(X) + (p) = \mathcal{I}_n(X)$ (5.2.11). By abuse of notation let $\mathcal{J}_n(X) \subset W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1}))$ denote the PD-ideal.

With this notation we can finally extend θ_n like we wish:

Proposition 5.2.13. *Let $(X, T) \in \mathrm{CRIS}(S/W_n)$. There is a PD-homomorphism of W_n-algebras functorial in (X, T)*

$$\theta_n : W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1})) \longrightarrow \Gamma(T, \mathcal{O}_T),$$

satisfying the commutative diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{O}_X) & & \Gamma(T, \mathcal{O}_T). \arrow[ll, two heads] \\
& W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1})) \arrow[ul, "\theta_n"] \arrow[ur, "\theta_n"'] &
\end{tikzcd}$$

Proof. This follows directly from the universal property of the PD-envelope and the functoriality of θ_n as defined above by using the morphism $(X, X) \to (X, T)$ in $\mathrm{CRIS}(S/W_n)$. $\square$

Corollary 5.2.14. *Let X be a W_n-scheme. There is a homomorphism of W_n-algebras functorial in X:*

$$W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1})) \longrightarrow H^0_{\mathrm{cris}}(X/W_n)$$

Proof. Recall that by 4.1.20 we have $H^0_{\mathrm{cris}}(X/W_n) = \varprojlim_{(U,T)} \Gamma(T, \mathcal{O}_T)$ where (U, T) runs through $\mathrm{CRIS}(X/W_n)$. Let $(U, T) \in \mathrm{CRIS}(X/W_n)$. By the above one has a morphism

$$\theta_n : W_n^{DP}(\Gamma(U_1, \mathcal{O}_{U_1})) \longrightarrow \Gamma(T, \mathcal{O}_T).$$

By functoriality using the morphism $(U, U) \to (X, X)$ one has a morphism

$$W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1})) \to W_n^{DP}(\Gamma(U_1, \mathcal{O}_{U_1})).$$

Concatenating these morphisms one gets compatible morphisms $W_n^{DP}(\Gamma(X_1, \mathcal{O}_{X_1})) \to \Gamma(T, \mathcal{O}_T)$ which factorize through the limit by the universal property of the projective limit. $\square$

Proposition 5.2.15. *Let $X = \operatorname{Spec} A$ be an affine scheme, flat over W_n with the absolute Frobenius on A_1 surjective. Then*

$$(X, \operatorname{Spec} W_n^{DP}(A_1))$$

is a final object in $\mathrm{CRIS}(X, W_n)$.

Proof. First we have to prove that $(X, W_n^{DP}(A_1)) \in \mathrm{CRIS}(X, W_n)$. Using the results from 5.2.13, it remains to show that it is a PD-thickening of X. Thus, it is sufficient to show $W_n^{DP}(A_1)/\mathcal{J}_n(A_1) \cong A$. By the properties of the PD-envelope, we have

$$W_n^{DP}(A_1)/\mathcal{J}_n(A_1) = W_n(A_1)/\mathcal{J}_n(A_1),$$

thus it is sufficient to show $W_n(A_1)/\mathcal{J}_n(A_1) \cong A$, i.e., that $\theta_n : W_n(A_1) \longrightarrow A$ is surjective.

This proof is done via induction on n. First let $n = 1$. Then one has $A_1 = A$ and θ_1 induced by $\theta_1 : W_1(A_1) \longrightarrow A_1$, $(a_0) \mapsto a_0^p$ which is clearly surjective as the Frobenius on A_1 is surjective.

Now let the statement be proven up to n and let A be a flat W_{n+1}-algebra with surjective Frobenius on A_1. Then $A_n := A \otimes_{W_{n+1}} W_n$ is a flat W_n-algebra with surjective Frobenius on $A_{n,1} = A_1$ and thus by induction hypothesis $\theta_n : W_n(A_1) \longrightarrow A_n$ is surjective, just like $\theta_1 : W_1(A_1) \longrightarrow A_1$.

Because k is perfect, there is an exact sequence

$$0 \longrightarrow W_1(k) \xrightarrow{\nu^n} W_{n+1}(k) \xrightarrow{\rho_1} W_n(k) \longrightarrow 0,$$

where ν^n denotes the shift map $(x_0) \mapsto (0, \ldots, 0, x_0)$ and $\rho_k : W_n(k) \to W_{n-k}(k)$ is given by $(x_0, \ldots, x_{n-1}) \mapsto (x_0^{p^k}, \ldots, x_{n-k-1}^{p^k})$. This is a sequence of W_{n+1}-modules (A.1.3), where W_{n-k} is a W_{n+1}-module via ρ_k. Since A is flat over W_{n+1}, this induces an exact sequence

$$0 \longrightarrow A_1 \xrightarrow{\nu^n} A \xrightarrow{\rho_1} A_n \longrightarrow 0$$

with $A_1 := A \otimes_{W_{n+1}} W_1(k)$, $A = A \otimes_{W_{n+1}} W_{n+1}(k)$, and $A_n := A \otimes_{W_{n+1}(k)} W_n(k)$. As we will prove now, this sequence gives rise to a commutative diagram

$$\begin{tikzcd}
& W_1(A_1) \arrow[r, "\nu^n"] \arrow[d, "\theta_1"] & W_{n+1}(A_1) \arrow[r, "\rho_1"] \arrow[d, "\theta_{n+1}"] & W_n(A_1) \arrow[r] \arrow[d, "\theta_n"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "\mathrm{id} \otimes \nu^n"] & A \arrow[r, "\mathrm{id} \otimes \rho_1"] & A_n \arrow[r] & 0.
\end{tikzcd}$$

First we show the commutativity of the left square: Let $(a) \in W_1(A_1)$. Then one has $(a) \overset{\nu^n}{\longmapsto} (0, \ldots, 0, a) \overset{\theta_{n+1}}{\longmapsto} p^n \hat{a}^p$. On the other hand we have $(a) \overset{\theta_1}{\longmapsto} a^p = \hat{a}^p \otimes 1 \overset{\mathrm{id} \otimes \nu^n}{\longmapsto} \hat{a}^p \otimes p^n = p^n \hat{a}^p$. Thus the left square commutes.

For the right square, let $(a_0, \ldots, a_n) \in W_{n+1}(A_1)$. Then one has

$$\begin{aligned}
(a_0, \ldots, a_n) &\overset{\rho}{\longmapsto} (a_0^p, \ldots, a_{n-1}^p) \\
&\overset{\theta_n}{\longmapsto} \widehat{a_0^p}^{p^n} + \cdots + p^{n-1} \widehat{a_{n-1}^p}^{\,p} \\
&= \hat{a}_0^{p^{n+1}} + \cdots + p^{n-1} \hat{a}_{n-1}^{p^2}.
\end{aligned}$$

For the last step note that $\hat{a}^p - \widehat{a^p} \in \ker(A_n \to A_1) = pA$, with pA being a PD-ideal. Therefore, by 5.2.8 in A_n one has $p^k \widehat{a^p}^{p^{n-k}} = p^k \hat{a}^{p^{n+1-k}}$. On the other hand:

$$\begin{aligned}
(a_0, \ldots, a_n) &\overset{\theta_{n+1}}{\longmapsto} \hat{a}_0^{p^{n+1}} + \cdots + p^n \hat{a}_n^p \\
&\overset{\mathrm{id} \otimes \rho_1}{\longmapsto} \hat{a}_0^{p^{n+1}} + \cdots + p^{n-1} \hat{a}_{n-1}^{p^2},
\end{aligned}$$

as $p^n = 0$ in A_n. Furthermore one has

$$\begin{aligned}
&\ker(\rho_1 : W_{n+1}(A_1) \to W_n(A_1)) \\
&\qquad = \left\{ (a_0, \ldots, a_n) \in W_{n_1}(A_1) \mid a_i^p = 0 \forall i = 0, \ldots, n-1 \right\}.
\end{aligned}$$

and there is a commutative triangle

$$\begin{tikzcd}
\ker(\rho_1) \arrow[r, hook, "\subseteq"] \arrow[d, "\pi_n"] & W_{n+1}(A_1), \\
W_1(A_1) \arrow[ur, "\nu_n"] &
\end{tikzcd}$$

where $\pi_n : (a_0, \dots, a_n) \mapsto (a_n)$ denotes the projection onto the last component. All this together allows the construction of a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\rho_1) \arrow[r] \arrow[d, "\theta_1 \circ \pi_n"] & W_{n+1}(A_1) \arrow[r, "\rho_1"] \arrow[d, "\theta_{n+1}"] & W_n(A_1) \arrow[r] \arrow[d, "\theta_n"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \operatorname{coker}(\theta_{n+1}) \arrow[r] & 0, &
\end{tikzcd}$$

where the first two rows are exact. By the snake lemma, we get $\operatorname{coker}(\theta_{n+1}) = 0$ and thus $W_{n+1}(A_1)/\mathcal{J}_{n+1}(A_1) \cong A$. Therefore, $(X, W_n^{DP}(A_1)) \in \mathrm{CRIS}(X/W_n)$. In 5.2.14 it has already been shown that this object is final. $\square$

Corollary 5.2.16. *Let $X = \operatorname{Spec} A$ be an affine scheme, flat over W_n with the absolute Frobenius on A_1 surjective. Then the morphism*

$$\theta_n : W_n^{DP}(A_1) \longrightarrow H^0_{\mathrm{cris}}(X/W_n)$$

is an isomorphism.

Remark 5.2.17. Fontaine constructs in [Fon94] universal PD-thickenings as well. However, his construction is more complicated, as he constructs the limit for $n \to \infty$. Therefore, he needs additionally the ring $R = \varprojlim A_1$ formed over the morphisms $A_1 \to A_1$, $x \mapsto x^p$ and gets a ring $W^{DP}(R)$ for the universal PD-thickening. The ring R appears here, as $W^{DP}(R)$ corresponds to $\varprojlim_n W_n^{DP}(A_1)$ where the morphisms are given by $\rho_1 : W_{n+1}^{DP}(A_1) \to W_n^{DP}(A_1)$.

5.3. Comparison Morphisms

Crucial for the applications of the syntomic topology and cohomology will be the possibility to compare it with other topologies. In the following some morphisms of topologies and of topoi will be constructed in order to make this possible. Recall:

Definition 5.3.1 ([SGA4.1, Exp. IV, 3.1]). A morphism of topoi $f : T' \longrightarrow T$ is a pair (f_*, f^*) of functors, where $f_* : T' \longrightarrow T$ has a left adjoint $f^* : T \longrightarrow T'$ which commutes with finite inverse limits.[2]

[2] [BO78, 5.4] defines a morphism of topoi slightly different to be a functor f_* admitting a left adjoint like f^*. Since the left adjoint is unique only up to isomorphism, this definitions allows less morphisms of topoi.

While a morphism of topologies gives rise in a canonical way to a morphism of the associated topoi, there are cases where it is not possible to construct a useful morphism of the sites and nevertheless morphisms of the topoi can be defined. This is often done using the following statements which show that it is sufficient to define the morphism for representable sheaves. This statements can be found in [SGA4.1, Exp. IV, 4.9.4]. Here they are given more detailed and without introducing that much additional notation.

Proposition 5.3.2 ([BO78, 5.7])**.** *Let $\mathcal{C}'$ and $\mathcal{C}$ be categories. Suppose $\varphi : \mathcal{C} \longrightarrow \hat{\mathcal{C}}'$ is a functor. Then there is a unique pair of functors $\varphi^* : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}'$ and $\varphi_* : \hat{\mathcal{C}}' \longrightarrow \hat{\mathcal{C}}$ such that $\varphi^*|_{\mathcal{C}} = \varphi$ and such that φ^* is left adjoint to φ_*.*

These functors are defined by

$$\begin{aligned} \varphi_*(G') &: T \mapsto \operatorname{Hom}_{\hat{\mathcal{C}}'}(\varphi(T), G') && T \in \mathcal{C}, G' \in \hat{\mathcal{C}}' \\ \varphi^*(G) &: T' \mapsto \lim_{\{T' \to \varphi(T)\}} G(T) && T' \in \mathcal{C}', G \in \hat{\mathcal{C}}. \end{aligned}$$

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "\varphi"'] \arrow[d, hook, "i"] \\ \hat{\mathcal{C}}' \arrow[r, shift right, "\varphi_*"'] & \hat{\mathcal{C}} \arrow[l, shift right, "\varphi^*"'] \end{tikzcd}$$

If $\mathcal{C}$ and $\mathcal{C}'$ are sites, given some more properties, these morphisms even introduce a morphism of the topoi $\tilde{\mathcal{C}} \to \tilde{\mathcal{C}}'$:

Theorem 5.3.3. *Let the notation be as in 5.3.2. Let finite inverse limits be representable in $\mathcal{C}'$. Let the following properties be given:*

(i) $\varphi(T)$ *is a sheaf for all objects* $T \in \mathcal{C}$.

(ii) $\varphi_*(G')$ *is a sheaf for all sheaves* G' *on* $\mathcal{C}'$.

(iii) $g^* : \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{C}}' = (\varphi^*(-))^\dagger$ *commutes with finite inverse limits (for example if it commutes with finite products and finite fiber products).*

Then $g_* : \tilde{\mathcal{C}}' \longrightarrow \tilde{\mathcal{C}} = \varphi_*|_{\tilde{\mathcal{C}}'}$ *is a morphism of topoi.*

Proof. Let $g_* : \tilde{\mathcal{C}}' \longrightarrow \tilde{\mathcal{C}} = \varphi_*|_{\tilde{\mathcal{C}}'}$ be defined by $g_*(G') = \varphi_*(G')$. Let $g^* : \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{C}}'$ be defined by $g^*(G) = (\varphi^*(G))^\dagger$. Then g^* is left adjoint to g_*:

$$\begin{aligned} \operatorname{Hom}(g^*(G), G') &= \operatorname{Hom}((\varphi^*(G))^\dagger, G') = \operatorname{Hom}(\varphi^*(G), G') \\ &= \operatorname{Hom}(G, \varphi_*(G')) = \operatorname{Hom}(G, g_*(G')) \end{aligned}$$

Since g^* commutes with finite inverse limits, these functors constitute a morphism of topoi. $\square$

With the following proposition, hypothesis (iii) can be checked quite easily in a lot of cases:

Proposition 5.3.4 ([SGA4.1, Exp IV, 4.9.2]). *With the notions of 5.3.3: Let (i) and (ii) be given. If $\mathcal{C}$ is a small category where finite inverse limits are representable, then g^* commutes already with finite inverse limits, if φ does.*

Proof. Let φ commute with finite inverse limits. Since $\mathcal{C}$ is small with representable finite inverse limits, φ^* commutes with them as well ([SGA4.1, Exp I, Prop. 5.4 4]). Now, from 5.3.3 (i) and (ii) follows as $\mathcal{C}$ is small that g^* commutes also with finite inverse limits ([SGA4.1, Exp. III, Prop. 1.3 5]). □

Remark 5.3.5. In terms of [SGA4.1, Exp. IV, 4.9], the theorem 5.3.3 can be reformulated: The hypotheses (i) and (ii) say that φ is a continuous functor $\mathcal{C} \longrightarrow \tilde{\mathcal{C}'}$. Together with (iii) this means that φ induces a morphism of sites from $\tilde{\mathcal{C}'}$ to $\mathcal{C}$. With this notation the statement of this proposition can be extended to an equivalence of categories

$$\mathrm{Homtop}(E, \tilde{\mathcal{C}}) \xrightarrow{\sim} \mathrm{Morsite}(E, \mathcal{C}),$$

where $\mathcal{C}$ is a site and E is a topos which can be considered as a site, too, using the notion of covering families of presheaves.

With these propositions we are now able to construct a morphism of topoi $\nu : (X/S)_{\mathrm{CRIS,SYN}} \longrightarrow X_{\mathrm{SYN}}$, which is done here according to the construction given in [Koe89].

Theorem 5.3.6. *There is a morphism of topoi $\nu : (X/S)_{\mathrm{CRIS,SYN}} \longrightarrow X_{\mathrm{SYN}}$.*

Proof. We would like to apply theorem 5.3.3. Therefore, the setting will be as following:

$$\begin{tikzcd}
\mathrm{CRIS}(X/S) = \mathcal{C}' \arrow[d, hook] & \mathcal{C} = \mathrm{SYN}(X) \arrow[dl, "\nu"] \arrow[d, hook] \\
(X/S)_{\mathrm{CRIS,SYN}} = \tilde{\mathcal{C}'} \arrow[r, shift right, "\nu_*"'] & \tilde{\mathcal{C}} = X_{\mathrm{SYN}} \arrow[l, shift right, "\nu^*"']
\end{tikzcd}$$

Thus, we need a morphism $\nu : \mathrm{SYN}(X) \longrightarrow (X/S)_{\mathrm{CRIS,SYN}}$. Let $Z \in \mathrm{SYN}(X)$. We define $\nu(Z)$ to be the presheaf

$$\nu(Z)(U, T) = \mathrm{Hom}_X(U, Z).$$

Now we have to check the conditions of 5.3.3:

(i): $\nu(Z)$ is a sheaf: Let $(U_i, T_i) \longrightarrow (U, T)$ be a Zariski-covering, i.e., the diagrams

$$\begin{tikzcd}
U_i \arrow[r, hook] \arrow[d, "u_i"] & T_i \arrow[d, "v_j"] \\
U \arrow[r, hook] & T
\end{tikzcd}$$

are cartesian and the morphisms $\{T_i \longrightarrow T\}$ form a surjective family of open immersions. Since open immersions are stable under base change, $\{U_i \longrightarrow U\}$ is also a family of open immersions which is even surjective because nil-immersions are topological homeomorphisms. Now the u_i are X-morphisms and thus $\{U_i \longrightarrow U\}$ is a covering of U in the Zariski topology of X. Because $\mathrm{Hom}_X(-, Z)$ is a Zariski sheaf on X, we have an exact sequence $\mathrm{Hom}_X(U, Z) \to \prod_i \mathrm{Hom}_X(U_i, Z) \rightrightarrows \prod_{i,j} \mathrm{Hom}_X(U_i \times_U U_j, Z)$.

Now let $(U', T') \longrightarrow (U, T)$ be a surjective syntomic morphism of affine objects. As above this means that $U' \longrightarrow U$ is surjective and syntomic and thus fppf. Since $\mathrm{Hom}_X(-, Z)$ is also an fppf sheaf on X, we have that $\mathrm{Hom}_X(U, Z) \to \mathrm{Hom}_X(U', Z) \rightrightarrows \mathrm{Hom}_X(U' \times_U U', Z)$ is exact.

This shows with 4.1.17 that $\nu^* Z$ is a sheaf on $\mathrm{CRIS}(X/S)_{\mathrm{SYN}}$.

(ii): $\nu_* F$ is a sheaf on $\mathrm{SYN}(X)$ for all $F \in (X/S)_{\mathrm{CRIS,SYN}}$: According to 5.3.2, $\nu_* F$ is defined by $\nu_* F(Z) = \mathrm{Hom}(\nu^* Z, F)$. Showing the exactness of the sheaf sequence for a Zariski covering, that is, a surjective family $\{Z_i \longrightarrow Z\}$ of open immersions, involves quite some calculations and is shown in [BO78, 5.8.2]. Thus, is remains to show the exactness for a surjective and syntomic $\varphi : Z' \longrightarrow Z$ in $\mathrm{SYN}(X)$. Then again $\nu_* F$ is already a sheaf by 4.1.17.

Let $u' \in \ker(\mathrm{Hom}(\nu^* Z', F) \rightrightarrows \mathrm{Hom}(\nu^*(Z' \times_Z Z'), F))$. We have to show that there is a unique $u \in \mathrm{Hom}(\nu^* Z, F)$ such that $u|_{\nu^* Z'} = u'$. In the following this sheaf morphism is constructed. Let $(U, T, i, \delta) \in \mathrm{CRIS}(X/S)$ and $f \in \mathrm{Hom}_X(U, Z) = \nu^* Z(U, T)$. One has to give $u(U, T)(f) \in F(U, T)$.

By base change f comes locally in $\mathrm{SYN}(X)$ from an $f' : Z' \times_Z U \to Z'$. However, the syntomic covering $Z' \times_Z U \to U$ does not have to induce an object in $\mathrm{CRIS}(X/S)_{\mathrm{SYN}}$ or even a covering of (U, T). Thus, it is no section of $\nu^* Z'$. But, the lifting proposition 3.1.7 states that such a crystalline covering exists locally: Let U_i be a Zariski covering of $Z' \times_Z U$ such that there is a cartesian diagram

$$\begin{tikzcd}
 & U_i \arrow[dl, "f'_i"'] \arrow[r, hook] \arrow[d, hook] & T_i \arrow[dd, "v"] \\
Z' \arrow[d, "\varphi"] & Z' \times_Z U \arrow[l, "f'"] \arrow[d] & \\
Z & U \arrow[l, "f"] \arrow[r, hook] & T
\end{tikzcd}$$

with v syntomic. Such a covering exists by 3.1.7. Note that $f_i' : U_i \to Z' \in \mathrm{Hom}_X(U_i, Z') = \nu^*(Z')(U_i, T_i)$. By defining $f_i := \varphi \circ f_i'$, we have to define $u(U_i, T_i)(f_i) = u'|_{\nu^* Z'}(U_i, T_i)(f_i') = u'(U_i, T_i)(f_i') \in F(U_i, T_i)$. Because $f_i = f|_{U_i}$, we have to show that these sections glue in $F(U, T)$.

For this let $u(f_i) = u'(f_i') := u'(U_i, T_i)(f_i')$, $U_{ij} = U_i \times_U U_j$, and $T_{ij} = T_i \times_T T_j$ for the sake of readability. Then we have

$$\begin{aligned} u'(f_i)|_{(U_{ij},T_{ij})} &= u'(U_{ij}, T_{ij})(f_i|_{(U_{ij},T_{ij})}) && \text{As } u' \text{ is a sheaf hom} \\ &= u'|_{1,Z'\times_Z Z'}(U_{ij}, T_{ij})(U_{ij} \stackrel{f_i\times f_j}{\longrightarrow} Z' \times_Z Z') && \text{Definition of restriction} \\ &= u'|_{2,Z'\times_Z Z'}(U_{ij}, T_{ij})(U_{ij} \stackrel{f_i\times f_j}{\longrightarrow} Z' \times_Z Z') && \text{As } u'|_{1,Z'\times_Z Z'} = u'|_{2,Z'\times_Z Z'} \\ &= u'(f_j)|_{(U_{ij},T_{ij})}. \end{aligned}$$

This shows that the $u(f_i) \in F(U_i, T_i)$ fulfill the gluing axioms and give rise to a section in $F(U, T)$, which we define to be $u(f)$. Now this defines a sheaf homomorphism $u : \nu^* Z \longrightarrow F$ with $u|_{\nu^* Z'} = u'$ by construction, as can be seen in the above diagram. It is also unique with this property as noted in the construction, thus this shows the exactness of the sheaf sequence for $Z' \longrightarrow Z$. Hence, $\nu^* Z$ is a sheaf.

(iii): By 5.3.4 is suffices to check that ν commutes with finite inverse limits or even with finite products and fiber products:

$$\begin{aligned} \nu(Z_1 \times_Z Z_2)(U, T) &= \mathrm{Hom}_X(U, Z_1 \times_Z Z_2) \\ &= \mathrm{Hom}_X(U, Z_1) \times_{\mathrm{Hom}_X(U,Z)} \mathrm{Hom}_X(U, Z_2) \\ &= \nu(Z_1)(U, T) \times_{\nu(Z)(U,T)} \nu(Z_2)(U, T) \\ &= (\nu(Z_1) \times_{\nu(Z)} \nu(Z_2))(U, T) \end{aligned}$$

This shows that ν commutes with fiber products. With $Z = X$ one gets the statement for products. This concludes the proof. □

Theorem 5.3.7 ([Koe89])**.** *There is a commutative diagram of morphisms of topoi*

$$\begin{tikzcd} (X/S)_{\mathrm{CRIS,ZAR}} \arrow[r, "u_{X/S}"] & X_{\mathrm{ZAR}} \\ (X/S)_{\mathrm{CRIS,SYN}} \arrow[u, "\beta"] \arrow[r, "\nu_{X/S}"] & X_{\mathrm{SYN}}, \arrow[u, "\alpha"'] \end{tikzcd}$$

where α and β are the morphisms of topoi created by coarsing the topology. The morphism $u_{X/S}$ is defined in [BO78, 5.18].

Proof. It suffices to show $\nu_{X/S}^* \circ \alpha^* = \beta^* \circ u_{X/S}^*$ for $Z \in \mathrm{ZAR}(X)$ as this determines all the morphisms uniquely by 5.3.3. So let $Z \in \mathrm{ZAR}(X)$. Then one has $(\nu_{X/S}^* \circ \alpha^*)(Z) = \nu(Z)$, as representable functors are already syntomic sheaves. On the other hand, $(\beta^* \circ u_{X/S}^*)(Z)$ is, by definition, the associated sheaf to $u_{X/S}^* Z$ in the syntomic topology. But $u_{X/S}^* Z$ and $\nu(Z)$ are by construction the same sheaf, the latter one being already a sheaf in $\mathrm{CRIS}(X/S)_{\mathrm{CRIS,SYN}}$. Thus $(\beta^* \circ u_{X/S}^*)(Z) = u_{X/S}^* Z = \nu(Z)$. □

Lemma 5.3.8 ([Koe89]). *Let $\mathcal{F} \in (X/S)_{\text{CRIS,SYN}}$, $Z \in \text{SYN}(X)$, and $\nu = \nu_{X/S}$. Then one has*

$$\nu_*\mathcal{F}(Z) = \text{Hom}_{Z/S}(e, \mathcal{F}|_{(Z/S)_{\text{CRIS,SYN}}}) = H^0((Z/S)_{\text{CRIS,SYN}}, \mathcal{F}|_{(Z/S)_{\text{CRIS,SYN}}}).$$

Proof. By definition in 5.3.2 we have

$$\nu_*\mathcal{F}(Z) = \text{Hom}_{X/S}(\nu(Z), \mathcal{F}),$$

where $\nu(Z) : (U,T) \mapsto \text{Hom}_X(U,Z)$. Thus, it is sufficient to show $\text{Hom}_{X/S}(\nu(Z), \mathcal{F}) \cong \text{Hom}_{Z/S}(e, \mathcal{F}|_{(Z/S)})$. Therefore, let $u : \nu(Z) \to \mathcal{F}$. We define an morphism $u' : e \to \mathcal{F}|_{(Z/S)}$ on (Z/S): Let $(U \xrightarrow{f} Z, T) \in \text{CRIS}(Z/S)$. Then u' is defined by

$$\begin{aligned} u'(U \to Z, T) : e(U,T) = \{e\} &\to \mathcal{F}|_{(Z/S)}(U,T) = \mathcal{F}(U,T) \\ e \ &\mapsto u(U,T)(f). \end{aligned}$$

On the other hand let $u' : e \to \mathcal{F}_{(Z/S)}$. Then a morphism $u : \nu(Z) \to \mathcal{F}$ on (X/S) is defined for $(U,T) \in \text{CRIS}(X/S)$ by

$$\begin{aligned} u(U,T) : \nu(Z)(U,T) = \text{Hom}_X(U,Z) &\to \mathcal{F}(U,T) \\ f &\mapsto u'(U \xrightarrow{f} Z, T)(e). \end{aligned}$$

The assignments $u \leftrightarrow u'$ are mutually inverse which shows the identity. The identity $\text{Hom}_{Z/S}(e, \mathcal{F}|_{(Z/S)_{\text{CRIS,SYN}}}) = H^0((Z/S)_{\text{CRIS,SYN}}, \mathcal{F}|_{(Z/S)_{\text{CRIS,SYN}}})$ is just by definition of H^0 (4.1.19). $\square$

In the following the cohomology groups of this different topoi will be compared. Of course, they can not be expected to be isomorphic in general, but there are special cases which will be sufficiently general for our purposes. In order to formulate this conditions, we need the notion of crystals, which are the objects giving rise to the name of crystalline cohomology. For the usual crystalline-Zariski site they are defined as follows:

Definition 5.3.9 ([Bau92, 1.13]). A *crystal* of $\mathcal{O}_{X/S}$-modules is a sheaf $\mathcal{F}$ of $\mathcal{O}_{X/S}$-modules in $(X/S)_{\text{CRIS,ZAR}}$ such that for every morphism $(u,t) : (U',T') \longrightarrow (U,T)$ in $\text{CRIS}(X/S)_{\text{ZAR}}$ the canonical morphism of $\mathcal{O}_{T'}$-modules

$$\rho_{(u,v)} \otimes 1 : v^*(\mathcal{F}_{(U,T)}) \longrightarrow \mathcal{F}_{(U',T')}$$

is an isomorphism. *Morphisms* of crystals are morphisms of $\mathcal{O}_{X/S}$-modules. A crystal is called *quasi-coherent* resp. *locally free of finite rank* if for any object (U,T) the $\mathcal{O}_T$-module $\mathcal{F}_{(U,T)}$ has this property.

The following proposition states that a quasi-coherent crystal is already a sheaf in the syntomic topology (and in fact even for *fppf* and *fpqc*, see [BBM82, 1.1.19]) and compares their cohomology.

Proposition 5.3.10 ([Bau92, 1.14]). *Let $\mathcal{F} \in (X/S)_{\mathrm{CRIS,ZAR}}$ be a quasi-coherent crystal of $\mathcal{O}_{X/S}$-modules. Then $\mathcal{F}$ is also a sheaf for the syntomic cohomology on* $\mathrm{CRIS}(X/S)$ *and for any $j > 0$*

$$\mathbf{R}^j \beta_* \mathcal{F} = 0,$$

where β is the canonical morphism $(X/S)_{\mathrm{CRIS,SYN}} \longrightarrow (X/S)_{\mathrm{CRIS,ZAR}}$. In particular, there is no need to distinguish between crystals on $(X/S)_{\mathrm{CRIS,ZAR}}$ and $(X/S)_{\mathrm{CRIS,SYN}}$.

Proof. See [BBM82, 1.1.18, 1.1.19] or [Bar]. □

Corollary 5.3.11. *If $\mathcal{F}$ is a quasi-coherent crystal of $\mathcal{O}_{X/S}$-modules, one has for any $j \geq 0$:*

$$H^j((X/S)_{\mathrm{CRIS,SYN}}, \mathcal{F}) \cong H^j((X/S)_{\mathrm{CRIS,ZAR}}).$$

Remark 5.3.12. This tells us that the exact sequence

$$0 \longrightarrow \mathcal{J}_{X/S} \longrightarrow \mathcal{O}_{X/S} \longrightarrow \mathcal{O} \longrightarrow 0$$

([BO78, 5.2]) exists on the crystalline syntomic site, too.

It would be helpful to have a similar statement for the morphism $u = u_{X/S}$: $(X/S)_{\mathrm{CRIS,SYN}} \longrightarrow X_{\mathrm{SYN}}$. However, much more restrictive hypotheses on the underlying schemes are needed in this case and some syntomic properties will enter in the proof. First we note the following lemma which allows us to calculate certain crystalline cohomology groups using Zariski cohomology. It is a combined version of the slightly more special ones in [Bau92, 1.16] and [Koe89].

Lemma 5.3.13 ([Bau92, 1.16], [Koe89]). *Let k be a perfect ring of characteristic p, $W_n := W_n(k)$ the ring of Witt vectors of length n endowed with the canonical divided power structure on the ideal $(p) \subset W_n(k)$. Let R be a W_n-algebra such that the Frobenius morphism is surjective on the k-algebra $R_0 = R/(p)$ and let $S = \operatorname{Spec} R$. Then there exists an affine PD-thickening $D = \operatorname{Spec} W_n^{DP}(R_0)$ of S, such that for all $j \geq 0$ and all abelian sheaves $\mathcal{F}$ in $(S/W_n(k))_{\mathrm{CRIS,ZAR}}$ one has*

$$H^j((S/W_n)_{\mathrm{CRIS,ZAR}}, \mathcal{F}) \cong H^j_{\mathrm{ZAR}}(D, \mathcal{F}_D).$$

Proof. By 5.2.15, (S, D) is a final object in $\mathrm{CRIS}(S/W_n)$ and for any $(U, T) \in \mathrm{CRIS}(S/W_n)$ there is a morphism

$$\theta_n : (U, T) \longrightarrow (S, D)$$

compatible with morphisms in $\mathrm{CRIS}(S/W_n)$. This allows to define a morphism of topoi

$$\begin{aligned} \Theta : (S/W_n)_{\mathrm{CRIS,ZAR}} &\longrightarrow D_{\mathrm{ZAR}} \\ \Theta_* \mathcal{F} &:= \mathcal{F}_D \\ (\Theta^* \mathcal{G})_{(U,T)} &:= \theta_n^{-1} \mathcal{G}, \end{aligned}$$

where $\mathcal{F} \in (S/W_n)_{\mathrm{CRIS,ZAR}}$, $\mathcal{G} \in D_{\mathrm{ZAR}}$ and $(U, T) \in \mathrm{CRIS}(S/W_n)$. Since Θ_* is exact ([BO78, 5.26]), the proposition follows from the Leray spectral sequence. □

With this tool, it is possible to show that in this context the crystalline-syntomic and the syntomic cohomology groups are isomorphic:

Proposition 5.3.14 ([Bau92, 1.17]). *Let k be a perfect ring of characteristic p, $W_n := W_n(k)$ the ring of Witt vectors of length n, endowed with the canonical divided power structure on the ideal $(p) \subset W_n(k)$. Let further S be a k-scheme and $\mathcal{F}$ a quasi-coherent crystal on $\mathrm{CRIS}(S/\Sigma)_{\mathrm{SYN}}$. Then one has*

$$\mathbf{R}^j u_* \mathcal{F} = 0 \textit{ for } j > 0 \textit{ and}$$

$$H^j((S/\Sigma)_{\mathrm{CRIS,SYN}}, \mathcal{F}) \cong H^j(S_{\mathrm{SYN}}, u_*\mathcal{F}) \textit{ for } j \geq 0.$$

Proof. We have to calculate the higher direct images $\mathbf{R}^j u_* \mathcal{F}$ of $\mathcal{F}$. Such a $\mathbf{R}^j u_* \mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto F^j(U) := H^j((U/\Sigma)_{\mathrm{CRIS,SYN}}, \mathcal{F})$. Therefore, it is sufficient to show that this presheaf becomes locally trivial, or even the following: For any affine S-scheme $U = \operatorname{Spec} A$ there is a family of faithfully flat syntomic morphisms $\{V_i \longrightarrow U\}_{i \in I}$ satisfying $\varinjlim_{i \in I} F^j(V_i) = 0$ for $j > 0$.

Now, the crucial point is that for any $a \in A$ the morphism $\operatorname{Spec}(A[T]/(T^p - a)) \longrightarrow \operatorname{Spec} A$ is a syntomic covering: It is of finite presentation, it is flat as $A[T]/(T^p - a)$ is free as A-module and it factors via $\operatorname{Spec} A[T]/(T^p - a) \longrightarrow \operatorname{Spec} A[T] \longrightarrow \operatorname{Spec} A$ clearly via a regular immersion ($T^p - a$ is a non zero-divisor in $A[T]$) followed by a smooth morphism. Finally as $A \longrightarrow A[T]/(T^p - a)$ is an integral ring extension, it is surjective by going up.

Thus, we define a ring $\hat{A}$ by

$$\hat{A} := \varinjlim_{n \in \mathbb{N}} A_n, \quad A_0 := A, \quad A_{n+1} := A_n[(T_a)_{a \in A_n}]/((T_a^p - a)_{a \in A_n}).$$

This ring $\hat{A}$ can be written as $\hat{A} = \varinjlim_{i \in I} B_i$ where B_i are faithfully flat syntomic A-algebras (Let $I = \hat{A}$ and $a \leq a'$ if $A(a) \subset A(a')$. For every $a \in \hat{A}$ there is $n \in \mathbb{N}$ such that $a^{p^n} \in A$, thus a and $B_a = A(a) \subset \hat{A}$ can be generated by n syntomic ring extensions like above and thus B_a is a syntomic A-algebra). One has

$$\begin{aligned}
\varinjlim_{i \in I} F^j(\operatorname{Spec} B_i) &= \varinjlim_{i \in I} H^j((\operatorname{Spec} B_i/\Sigma)_{\mathrm{CRIS,SYN}}, \mathcal{F}) \\
&\cong \varinjlim_{i \in I} H^j((\operatorname{Spec} B_i/\Sigma)_{\mathrm{CRIS,ZAR}}, \mathcal{F}) && 5.3.11 \\
&\cong H^j((\operatorname{Spec} \hat{A}/\Sigma)_{\mathrm{CRIS,ZAR}}, \mathcal{F}). && \text{[Kat94, 2.4.3]}
\end{aligned}$$

In addition, the Frobenius morphism on $\hat{A}$ is clearly surjective by the definition of $\hat{A}$ and thus we can use 5.3.13: Let $D = \operatorname{Spec} W_n^{DP}(\hat{A})$. Then one gets

$$H^j((\operatorname{Spec} \hat{A}/\Sigma)_{\mathrm{CRIS,ZAR}}, \mathcal{F}) \cong H^j_{\mathrm{ZAR}}(D, \mathcal{F}_D) = 0,$$

where the second step is due to Serre's criterion [Sha96, 2.1.2], as $\mathcal{F}_D$ is a quasi-coherent on D.

□

Remark 5.3.15. The fact that locally the Frobenius becomes surjective is a special property of the syntomic topology. This would not work e.g., in étale topology: The polynom $T^p - a$ is inseparable in characteristic p and thus the induced morphism $\operatorname{Spec} A[T]/(T^p - a) \longrightarrow \operatorname{Spec} A$ is not étale in general. For an explicit example consider $A = \mathbb{F}_p(X)$ as $\mathbb{F}_p$-scheme. Then the morphism is not étale at $0 \in A[T]/(T^p - X)$ as the induced residue field extension $\mathbb{F}_p(X)(X^{1/p})|\mathbb{F}_p(X)$ is inseparable.

6. Sheaves on the Syntomic Site

This section will introduce some very important syntomic sheaves on sites of schemes over perfect fields of characteristic p. These sheaves will play a major part in the proof of BSD presented later. The material in this part is based on the manuscript *Die syntomische Kohomologie* by Köck [Koe89]. It was first formulated by Fontaine and Messing in [FM87]. Here it is generalized in order to allow the case of $\operatorname{syn}(S)$ for S a syntomic $W_n(k)$-scheme which will be the smooth curve from BSD later.

Let k be a perfect field of characteristic $p > 0$. Let $W_n = W_n(k)$ be the ring of Witt vectors of length n. Unless otherwise noted, k is considered a $W_n(k)$-algebra via

$$W_n(k) \xrightarrow{\mathrm{Frob}^{-n}} W_n(k) \xrightarrow{\pi_0} k,$$

where π_0 is the projection onto the first component, which corresponds to reduction modulo p.

Let $s \in \mathbb{N}$ and X be a W_s-scheme. For $i \leq s$ define $X_i = X \otimes_{W_s} W_i$. By 5.3.6 this induces for $n \geq i$ a morphism of topoi

$$\nu_i := \nu_{X_i/W_n} : (X_i/W_n)_{\mathrm{CRIS,SYN}} \longrightarrow (X_i)_{\mathrm{SYN}}.$$

Definition 6.1.1. Let $\mathcal{O}^{\mathrm{cris}}_{X,n,i}$ denote the sheaf $(\nu_i)_*(\mathcal{O}_{X_i/W_n})$ on $\mathrm{SYN}(X_i)$. The induced sheaf on $\operatorname{syn}(X_i)$ is also denoted by $\mathcal{O}^{\mathrm{cris}}_{X,n,i}$. If there is no ambiguity possible, often just $\mathcal{O}^{\mathrm{cris}}_{n,i}$ will be used.

Proposition 6.1.2. *The sections of the sheaf $\mathcal{O}^{\mathrm{cris}}_{X,n,i}$ for $Z \in \mathrm{SYN}(X_i)$ are*

$$\mathcal{O}^{\mathrm{cris}}_{X,n,i}(Z) = H^0((Z/W_n)_{\mathrm{CRIS,SYN}}, \mathcal{O}_{Z/W_n}).$$

They are commutative W_n-algebras with a Frobenius endomorphism $\mathcal{F}$.

Proof. By 5.3.8 and 4.1.20 one has

$$\begin{aligned}(\nu_i)_*(\mathcal{O}_{X_i/W_n})(Z) &= H^0((Z/W_n)_{\mathrm{CRIS,SYN}}, \mathcal{O}_{Z/W_n}) = \Gamma(\mathcal{O}_{Z/W_n}) \\ &= \varprojlim_{(U,T)} \mathcal{O}_{Z/W_n}((U,T)) = \varprojlim \mathcal{O}_T(T),\end{aligned}$$

where the limit runs through $\mathrm{CRIS}(Z/W_n)$. Since the T are W_n-schemes, their global sections are W_n-algebras and hence is the limit. The Frobenius is induced by the Frobenius on $\mathcal{O}_{Z/W_n}$. $\square$

Lemma 6.1.3. *Let $i \leq j$ and $\mathrm{inc} : X_j \hookrightarrow X_i$ the canonical closed immersion. Then one has*

$$(\mathrm{inc})_*(\mathcal{O}^{\mathrm{cris}}_{X,n,j}) = \mathcal{O}^{\mathrm{cris}}_{X,n,i}.$$

Proof. Let $Z \in \mathrm{SYN}(X_i)$. Then

$$\begin{aligned}(\mathrm{inc})_*(\mathcal{O}^{\mathrm{cris}}_{X,n,j})(Z) &= \mathcal{O}^{\mathrm{cris}}_{X,n,j}(Z \otimes_{W_i} W_j) \\ &= H^0_{\mathrm{cris}}((Z \otimes_{W_i} W_j)/W_n) && \text{by 5.3.8} \\ &= H^0_{\mathrm{cris}}(Z/W_n) && \text{rigidity theorem [BO78, 5.17]} \\ &= \mathcal{O}^{\mathrm{cris}}_{X,n,i}(Z).\end{aligned}$$

□

Remark 6.1.4. This allows us to suppress the index i from now on: $\mathcal{O}^{\mathrm{cris}}_{X,n} = \mathcal{O}^{\mathrm{cris}}_{X,i,n}$. Since $(\mathrm{inc})_*$ is exact ([BO78, 6.2]), this does not change the cohomology.

The following lemma shows that we can use the sheaves $\mathcal{O}^{\mathrm{cris}}_{X,n}$ to calculate crystalline cohomology groups on the syntomic site.

Lemma 6.1.5. *We have isomorphisms for $Z \in \mathrm{SYN}(X)$ resp. $Z \in \mathrm{syn}(X)$*

$$H^i_{\mathrm{SYN}}(Z, \mathcal{O}^{\mathrm{cris}}_{X,n}) = H^i_{\mathrm{CRIS,SYN}}(Z/W_n)$$
$$H^i_{\mathrm{syn}}(Z, \mathcal{O}^{\mathrm{cris}}_{X,n}) = H^i_{\mathrm{cris,syn}}(Z/W_n).$$

Proof. This follows from 5.3.14. □

Next, a more explicit description of $\mathcal{O}^{\mathrm{cris}}_{X,n}$ will be given for a syntomic W_s-scheme X, using the results from 5.2.15. From now on we work on the small syntomic site, as we are going to apply the exactness criterion 5.1.11. Let Z be in $\mathrm{syn}(X)$, then Z is a W_s-scheme and by 5.2.14 there is a canonical homomorphism

$$W^{DP}_n(\Gamma(Z_1, \mathcal{O}_{Z_1})) \longrightarrow H^0_{\mathrm{cris}}(Z_i/W_n)$$

for i sufficiently small ($i \leq s, n$). Let W^{DP}_n denote the presheaf $Z \mapsto W^{DP}_n(\Gamma(Z_1, \mathcal{O}_{Z_1}))$ on $\mathrm{syn}(X)$: This allows us to define a morphism of presheaves $W^{DP}_n \longrightarrow \mathcal{O}^{\mathrm{cris}}_{X,n}$ which can be extended to the associated sheaf $\widetilde{W}^{DP}_n$ on $\mathrm{syn}(X)$:

$$\widetilde{W}^{DP}_n \longrightarrow \mathcal{O}^{\mathrm{cris}}_{X,n}.$$

Proposition 6.1.6. *Let X be a syntomic W_s-scheme. Then the homomorphism*

$$\widetilde{W}^{DP}_n \longrightarrow \mathcal{O}^{\mathrm{cris}}_{X,n}$$

is an isomorphism.

Proof. This will be proved using the exactness criterion 5.1.11: Let $X_0 = \operatorname{Spec} A_0 \in \mathcal{M}_X$. We have to prove that $\varinjlim_i W_n^{DP}(A_i/p) \cong \varinjlim_i H^0_{\text{cris}}(A_i/W_n)$. Since the PD-envelope is a left-adjoint functor (A.1.6), one has $\varinjlim_i W_n^{DP}(A_i/p) = W_n^{DP}(\varinjlim_i A_i/p)$ which equals $H^0_{\text{cris}}(A/W_n)$ (5.2.15) as the Frobenius is surjective on $A = \varinjlim_i A_i$ by 5.1.12. As $H^0_{\text{cris}}(\varinjlim_i A_i/W_n) = \varinjlim_i H^0_{\text{cris}}(A_i/W_n)$ ([SGA4.2, Exp. VI, 5]), the statement follows. □

Proposition 6.1.7. *Let $n \geq m$. The canonical morphism*

$$\mathcal{O}^{\text{cris}}_{X,n} \xrightarrow{\nu} \mathcal{O}^{\text{cris}}_{X,m}$$

given for $Z \in \operatorname{syn}(X)$ as the morphism $H^0_{\text{cris}}(Z_i/W_n) \to H^0_{\text{cris}}(Z_i/W_m)$ induced by the canonical homomorphism $W_n \to W_m, (x_0, \ldots, x_{n-1}) \mapsto (x_0^{p^{n-m}}, \ldots, x_{m-1}^{p^{n-m}})$ is

$$\begin{aligned} \widetilde{W}_n^{DP} &\xrightarrow{\nu} \widetilde{W}_m^{DP} \\ (a_0, \ldots, a_{n-1}) &\longmapsto (a_0^{p^{n-m}}, \ldots, a_{m-1}^{p^{n-m}}). \end{aligned}$$

Proof. We have $\mathcal{O}^{\text{cris}}_n(Z) = H^0_{\text{cris}}(Z_i/W_n) = \varprojlim_{(U,T)} \mathcal{O}_T(T)$ where the limit runs trough $(U,T) \in \operatorname{CRIS}(Z_i/W_n)$. The morphism $\operatorname{Spec} W_m \to \operatorname{Spec} W_n$ induces an inclusion $\operatorname{CRIS}(Z_i/W_m) \subset \operatorname{CRIS}(Z_i/W_n)$. The morphism ν is given on the limit by projection:

$$\varprojlim_{\substack{(U,T)\in \\ \operatorname{CRIS}(Z_i/W_n)}} \mathcal{O}_T(T) \longrightarrow \varprojlim_{\substack{(U,T)\in \\ \operatorname{CRIS}(Z_i/W_m)}} \mathcal{O}_T(T)$$

The morphism $\widetilde{W}_n^{DP}(Z) \to \mathcal{O}^{\text{cris}}_n(Z)$ is given via the universal property of the projective limit by

$$\begin{aligned} \widetilde{W}_n^{DP}(Z) &\xrightarrow{\nu} \mathcal{O}_T(T) \\ (a_0, \ldots, a_{n-1}) &\longmapsto \sum_i p^i \hat{a}_i^{p^{n-i}} \end{aligned}$$

for $(Z,T) \in \operatorname{CRIS}(Z_i/W_n)$ and is an isomorphism, if the Frobenius on Z_1 is surjective. Let $Z_i = \operatorname{Spec} A$ affine and $(Z_i, T) \in \operatorname{CRIS}(Z_i/W_m) \subset \operatorname{CRIS}(Z_i/W_n)$. Then we have $\sum_{i=0}^{n-1} p^i \hat{a}_i^{p^{n-i}} = \theta_n(a_0, \ldots, a_{n-1}) = \theta_m(\nu(a_0, \ldots, a_n)) = \sum_{i=0}^{m-1} p^i \hat{b}_i^{p^{m-i}}$ with $\nu(a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{m-1})$. Hence the statement. □

Proposition 6.1.8. *The Frobenius homomorphism on $\mathcal{O}^{\text{cris}}_{X,n}$ reads under the isomorphism $\widetilde{W}_n^{DP} \cong \mathcal{O}^{\text{cris}}_{X,n}$ as*

$$\begin{aligned} \widetilde{W}_n^{DP} &\xrightarrow{\mathcal{F}} \widetilde{W}_n^{DP} \\ (a_0, \ldots, a_{n-1}) &\longmapsto (a_0^p, \ldots, a_{n-1}^p). \end{aligned}$$

Lemma 6.1.9. *Let B be a smooth W_n-algebra of finite type, $P_1, \dots, P_d \in B$ a sequence-regular sequence, and $J = (P_1, \dots, P_d)$ such that $A = B/J$ is flat over W_n. Then for all $r \geq 0$ the ideal $\bar{J}^{[r]} \subseteq D_{B,(p)}(J)$ ist flat over W_n. Here, $\bar{J}^{[r]}$ denotes the r-th divided power of the PD-ideal $\bar{J} \subseteq D_{B,(p)}(J)$.*

Proof. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d, "\alpha"] \\ \operatorname{Spec} W_n \arrow[r, "z"] & \mathbb{A}^d_{W_n}, \end{tikzcd}$$

where z is the zero section $T_i \mapsto 0$ and α is induced by $T_i \mapsto P_i$. This diagram is in fact cartesian: Let C be a ring with homomorphisms $c_1 : B \to C$ and $c_2 : W_n \to C$ commuting over $W_n[T_1, \dots, T_d]$. Because $\alpha((T_1, \dots, T_d)) \subseteq B$ generates J and $(T_1, \dots, T_d) = \ker(z)$ one has $J \subseteq \ker c_1$ and therefore c_1 factors uniquely via $B/J = A$. On the other hand, c_2 factors via the structure morphism $W_n \to B$ and therefore via B/J, too.

Let $x \in X$. The elements $T_1, \dots, T_d \in \mathcal{O}_{\mathbb{A}^d_{W_n},\alpha(x)}$ form a sequence-regular sequence. As these are mapped by α to the B-sequence-regular sequence $P_1, \dots, P_d$ in B and as $B/(P_1, \dots, P_d)$ is flat over $W_n[T_1, \dots, T_d]/(T_1, \dots, T_d) = W_n$, the stalk $\mathcal{O}_{Y,x}$ is flat over $\mathcal{O}_{\mathbb{A}^d_{W_n},\alpha(x)}$ by [EGAIV.1, 0.15.1.21]. Since the flat locus is open ([GW10, 14.42]), there exist an open neighbourhood U of X in Y such that $U \to \mathbb{A}^d_{W_n}$ is flat.

By A.1.7, one has $D_{X,(p)}(Y) = D_{X,(p)}(U)$. Furthermore, as $U \to \mathbb{A}^d_{W_n}$ is flat, by [BO78, 3.21] we get

$$D_{X,(p)}(U) = D_{\operatorname{Spec} W_n,(p)}(\mathbb{A}^d_{W_n}) \times_{\mathbb{A}^d_{W_n}} U.$$

Therefore, by flat base change we see that $D_{X,(p)}(Y) = D_{\operatorname{Spec} W_n,(p)}(\mathbb{A}^d_{W_n}) \times_{\mathbb{A}^d_{W_n}} U$ is flat over $D_{\operatorname{Spec} W_n,(p)}(\mathbb{A}^d_{W_n})$ and even that $\bar{J}^{[r]} \subseteq D_{B,(p)}(J)$ flat over $\bar{I}^{[r]} \subseteq D_{W_n[T_1,\dots,T_d],(p)}(T_1, \dots, T_d)$. Since $\bar{I}^{[r]} = \bigoplus_{|i| \geq r} W_n T_1^{[i_1]} \cdots T_d^{[i_d]}$ is a free W_n-module and as such flat over W_n, this implies the statement. $\square$

Lemma 6.1.10. *Let A be a k-algebra with surjective Frobenius. Then for all $n, m \geq 1$ there is an exact sequence*

$$W^{DP}_{n+m}(A) \xrightarrow{p^n} W^{DP}_{n+m}(A) \xrightarrow{\nu} W^{DP}_n(A) \longrightarrow 0,$$

where ν is given by $\nu : W_{n+m}(A) \to W_n(A), (x_0, \dots, x_{n+m-1}) \mapsto (x_0^{p^m}, \dots, x_{n-1}^{p^m})$.

Proof. Since the Frobenius on A is surjective, so is the map $\nu : W_{n+m}(A) \to W_n(A)$. Furthermore, one has $\nu(\mathcal{I}_{n+m}(A)) = \mathcal{I}_n(A)$ and as by 5.2.11 we have $\mathcal{I}_n(A) = \mathcal{J}_n(A)$, this induces a surjective map $W^{DP}_{n+m}(A) \to W^{DP}_n(A)$. Obviously, one has $\nu \circ p^n = 0$. Thus, it remains to show that $\ker(\nu) \subseteq \operatorname{im}(p^n)$.

There is a PD-thickening $\theta_{n+m} : W_{n+m}^{DP}(A) \to A$, which is a morphism of W_{n+m}-algebras. By factoring out p^n, we get a PD-thickening $W_{n+m}^{DP}(A)/(p^n) \to A/(p^n) = A$, which is a morphism of $W_{n+m}/(p^n) = W_n$-algebras. By 5.2.13, we get a morphism $\theta_n : W_n^{DP}(A) \to W_{n+m}^{DP}(A)$. We claim that it satisfies the following commutative diagram:

$$\begin{tikzcd} & W_n^{DP}(A) \arrow[dr, "\theta_n"] & \\ W_{n+m}^{DP}(A) \arrow[ur, "\nu"] \arrow[rr, "\pi"] & & W_{n+m}^{DP}(A)/p^n \end{tikzcd}$$

This would show that ν is injective up to p^n and therefore $\ker(\nu) \subseteq \operatorname{im}(p^n)$.

To show the commutativity, let $a = (a_0, \dots, a_{n+m-1}) \in W_{n+m}(A)$. Then $\nu(a) = (a_0^{p^m}, \dots, a_{n-1}^{p^m})$. To calculate $\theta_n((a_0^{p^m}, \dots, a_{n-1}^{p^m}))$, we need to choose lifts $\hat{a}_i$ for the $a_i^{p^m}$ in $W_{n+m}^{DP}(A)/p$ under the homomorphism $\theta_{n+m} : W_{n+m}^{DP}(A)/p^n \to A, (a_0, \dots, a_{n+m-1}) \mapsto a_0^{p^{m+n}} + \dots + p^{m+n-1}a_{m+n-1}^p = a_0^{p^{m+n}}$. Let $\hat{a}_i := [\tilde{a}_i] = (\tilde{a}_i, 0, \dots, 0) \in W_{n+m}^{DP}(A)/p^n$ with $\tilde{a}_i^{p^n} = a_i$ be such a lift. Then we have

$$\begin{aligned} \theta_n(\nu(a)) &= \theta_n(a_0^{p^m}, \dots, a_{n-1}^{p^m}) \\ &= \sum_{k=0}^{n-1} p^k \hat{a}_i^{n-k} = \sum_{k=0}^{n-1} p^k [\tilde{a}_i^{n-k}] \\ &= (\tilde{a}_0^{p^n}, \dots, \tilde{a}_{n-1}^{p^n}, 0, \dots, 0) \\ &= (a_0, \dots, a_{n-1}, 0, \dots, 0) = \pi(a). \end{aligned}$$

□

Remark 6.1.11. It is worth noting that this sequence becomes exact not before taking the PD-envelopes: For example, let $\varepsilon \in A$ with $\varepsilon^2 = 0$. Then obviously $(\varepsilon, 0, \dots, 0) \in \ker(\nu)$, but how should $(\varepsilon, 0, \dots, 0) \in (p^n)$ hold? In fact it doesn't have to be true in $W_{n+m}(A)$, but it is true in $W_{n+m}^{DP}(A)$: Let $\tilde{\varepsilon} \in A$ such that $\tilde{\varepsilon}^{p^n} = \varepsilon$. Then $\theta_{n+m}([\tilde{\varepsilon}]) = \tilde{\varepsilon}^{p^{m+n}} = 0$, thus $[\tilde{\varepsilon}] \in \mathcal{I}_{n+m}(A)$. As $[\varepsilon]$ is in the PD-ideal, one has $[\varepsilon] = [\tilde{\varepsilon}]^{p^n} = (p^n)!\gamma_{p^n}([\tilde{\varepsilon}]) \in (p^n)$.

This allows us to construct a fundamental exact sequence on the syntomic topos:

Proposition 6.1.12 ([Koe89]). *Let X be a syntomic W_s-scheme. Then there is an exact sequence on* $\operatorname{syn}(X)$*:*

$$\mathcal{O}_{X,n+m}^{\text{cris}} \xrightarrow{p^m} \mathcal{O}_{X,n+m}^{\text{cris}} \xrightarrow{p^n} \mathcal{O}_{X,n+m}^{\text{cris}} \xrightarrow{\nu} \mathcal{O}_{X,n}^{\text{cris}} \longrightarrow 0$$

Proof. First the exactness on the left side will be shown. Of course $p^n \circ p^m = 0$, thus it has to be shown that $\ker p^n \subseteq \operatorname{im} p^m$. Let $U \in \operatorname{syn}(X)$ and $s \in \mathcal{O}_{X,n+m}^{\text{cris}}(U)$ with $p^n s = 0$. It is sufficient to show that there is a covering $\{U_i \to U\}$ in $\operatorname{syn}(X)$ and $t_i \in \mathcal{O}_{X,n+m}^{\text{cris}}(U_i)$ with $p^m t_i = s|_{U_i}$.

By 5.1.10, U can be assumed to be of the form $U = \mathrm{Spec}(B/J)$ with $B = W_s[T_1, \dots, T_l]$ and $J = (P_1, \dots, P_d)$ with $P_1, \dots, P_d$ a sequence-regular sequence in B. Then we have

$$\begin{aligned}\mathcal{O}^{\mathrm{cris}}_{X,n+m}(U) &= H^0_{\mathrm{cris}}(U/W_{n+m}, \mathcal{O}_{U/W_{n+m}}) \\ &= \ker\left(\bar{J}^{[0]} \xrightarrow{d} \bar{J}^{[0]} \otimes \Omega^1_{\mathbb{A}^l_{W_{n+m}/W_{n+m}}}\right) \subseteq \bar{J}^{[0]} = D_{U,(p)}(\mathbb{A}^l_{W_{n+m}}).\end{aligned}$$

by [BO78, 7.2] as $\mathbb{A}^l_{W_{n+m}}$ is flat over W_{n+m}. Furthermore, $\bar{J}^{[0]}$ is flat over W_{n+m} by 6.1.9. Thus, the exact sequence

$$W_{n+m} \xrightarrow{p^m} W_{n+m} \xrightarrow{p^n} W_{n+m}$$

induces an exact sequence $\bar{J}^{[0]} \to \bar{J}^{[0]} \to \bar{J}^{[0]}$ which shows that there is a $t \in \bar{J}^{[0]}$ with $p^m t = s$. Let $dt = \sum_{i=1}^l a_i \otimes dT_i \in J^{[0]} \otimes \Omega^1_{\mathbb{A}^l_{W_{n+m}/W_{n+m}}}$ denote the image of t under d with $a_i \in \bar{J}^{[0]}$. Of course, $dt \neq 0$ in general. But by defining $V = U_{n+m}$ by base change with the relative Frobenius as in 5.1.11, one has in V: $T_i = (T_i^{p^{-n-m}})^{p^{n+m}}$ and thus $d(T_i|_V) = d(T_i^{p^{-n-m}})^{p^{n+m}} = p^{n+m} \cdot d(T_i^{p^{-n-m}})^{p^{n+m}-1} = 0$. Therefore, $d(t|_V) = 0$ and $t|_V \in \ker(d) = \mathcal{O}^{\mathrm{cris}}_{X,n+m}(V)$ which shows the exactness in the first part.

The exactness in the other parts is a consequence from 6.1.10 and the exactness criterion: Let $X_0 = \mathrm{Spec}\, A_0 \in \mathcal{M}_X$. Then

$$W^{DP}_{n+m}(\varinjlim A_i/p) \longrightarrow W^{DP}_{n+m}(\varinjlim A_i/p) \longrightarrow W^{DP}_{n}(\varinjlim A_i/p) \longrightarrow 0$$

is exact by 6.1.10. By A.1.6 this sequence equals

$$\varinjlim W^{DP}_{n+m}(A_i/p) \longrightarrow \varinjlim W^{DP}_{n+m}(A_i/p) \longrightarrow \varinjlim W^{DP}_{n}(A_i/p) \longrightarrow 0,$$

which shows the statement by the exactness criterion 5.1.11. ◻

Corollary 6.1.13. *Let X be a syntomic W_s-scheme. Then there is a short exact sequence on* $\mathrm{syn}(X)$

$$0 \longrightarrow \mathcal{O}^{\mathrm{cris}}_{X,m} \xrightarrow{\underline{p^n}} \mathcal{O}^{\mathrm{cris}}_{X,n+m} \xrightarrow{\nu} \mathcal{O}^{\mathrm{cris}}_{X,n} \longrightarrow 0.$$

Proof. From 6.1.12 one obviously gets a short exact sequence

$$0 \longrightarrow \mathcal{O}^{\mathrm{cris}}_{X,m+n}/p^m \xrightarrow{\underline{p^n}} \mathcal{O}^{\mathrm{cris}}_{X,n+m} \xrightarrow{\nu} \mathcal{O}^{\mathrm{cris}}_{X,n} \longrightarrow 0,$$

thus it remains to show that there is an isomorphism $\mathcal{O}^{\mathrm{cris}}_{X,m+n}/p^m \cong \mathcal{O}^{\mathrm{cris}}_{X,m}$. But this isomorphism is induced by the epimorphism $\nu : \mathcal{O}^{\mathrm{cris}}_{X,m+n} \twoheadrightarrow \mathcal{O}^{\mathrm{cris}}_{X,m}$ gained by exchanging n and m after factoring out the kernel $p^m\mathcal{O}^{\mathrm{cris}}_{X,m+n}$. ◻

Remark 6.1.14. Let A be a k-algebra with surjective Frobenius. Then $\mathcal{O}^{\mathrm{cris}}_{X,n}(A) = W_n^{DP}(A)$. In this case the morphism $\underline{p}^m : \mathcal{O}^{\mathrm{cris}}_{X,n} \to \mathcal{O}^{\mathrm{cris}}_{X,n+m}$ is given by $(a_0, \ldots, a_{n-1}) \mapsto (0, \ldots, 0, a_0, \ldots, a_{n-1})$.

Definition 6.1.15. Let X be a syntomic W_s-scheme. Like for $\mathcal{O}^{\mathrm{cris}}_{X,n,i}$ we define $\mathcal{O}_{X,n,i}$ to be the push-forward of $\mathcal{O}_{X_i}$ under the morphism ν_i. It again turns out to be independent of i. Therefore, we can omit the index and write $\mathcal{O}_{X,n} \in \mathrm{syn}(X)$.

Proposition 6.1.16. *It can be easily shown that $\mathcal{O}_n = \mathcal{O}_X/p^n$, with $\mathcal{O}_X$ the structure sheaf on* $\mathrm{syn}(X)$.

Proposition 6.1.17. *Let X be a syntomic W_s-scheme. Then there is an epimorphism of sheaves on* $\mathrm{syn}(X)$*:*

$$\mathcal{O}^{\mathrm{cris}}_n \to \mathcal{O}_n.$$

This epimorphism is induced by the exact sequence 5.3.12 on $(X_i/W_n)_{\mathrm{cris,syn}}$ and application of $\nu_{i,*}$.

Proposition 6.1.18. *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{cris}}_{n+m} \arrow[r] \arrow[d, "\nu"] & \mathcal{O}_{n+m} \arrow[d, "\nu'"] \\
\mathcal{O}^{\mathrm{cris}}_{n} \arrow[r] & \mathcal{O}_{n},
\end{tikzcd}$$

where ν' is induced by the morphism $W_{n+m} \to W_n$.

We define the sheaf of ideals $\mathcal{J}_n$ on $\mathrm{syn}(X)$ by the short exact sequence

$$0 \longrightarrow \mathcal{J}_n \longrightarrow \mathcal{O}^{\mathrm{cris}}_n \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

Note that for $s = 1$ we have $\mathcal{O}_n = \mathcal{O}_X = \mathbb{G}_a$ and thus

$$0 \longrightarrow \mathcal{J}_n \longrightarrow \mathcal{O}^{\mathrm{cris}}_n \longrightarrow \mathbb{G}_a \longrightarrow 0.$$

From 6.1.12 we note that

$$\ker(\mathcal{O}^{\mathrm{cris}}_{n+1} \to \mathcal{O}^{\mathrm{cris}}_n) = p^n \mathcal{O}^{\mathrm{cris}}_{n+1}.$$

Multiplication by p induces an injection

$$\underline{p} : \mathcal{O}^{\mathrm{cris}}_n \hookrightarrow \mathcal{O}^{\mathrm{cris}}_{n+1}$$

by 6.1.13. The commutative diagram 6.1.18 shows that the projection $\mathcal{O}^{\mathrm{cris}}_{n+1} \twoheadrightarrow \mathcal{O}^{\mathrm{cris}}_n$ induces a surjection $\mathcal{J}_{n+1} \twoheadrightarrow \mathcal{J}_n$.

From now on let X be a syntomic k-scheme, i.e., $s = 1$. Then $\mathcal{O}_n = \mathbb{G}_a = \mathcal{O}_X$. In this case the surjection $\mathcal{J}_{n+1} \twoheadrightarrow \mathcal{J}_n$ has kernel $p^n\mathcal{O}^{\mathrm{cris}}_{n+1}$ as well: Obviously, one has $\ker(\mathcal{J}_{n+1} \to \mathcal{J}_n) \subseteq \ker(\mathcal{O}^{\mathrm{cris}}_{n+1} \to \mathcal{O}^{\mathrm{cris}}_n) = p^n\mathcal{O}^{\mathrm{cris}}_{n+1}$. But as X is a k-scheme, we have $\mathrm{char}\ \mathcal{O}_{n+1} = \mathrm{char}\ \mathbb{G}_a = p$ and therefore $p^n\mathcal{O}^{\mathrm{cris}}_{n+1} \subset \mathcal{J}_{n+1}$.

Proposition 6.1.19. *On $\mathcal{O}_n^{\mathrm{cris}}$ there is a Frobenius $\mathcal{F}$. On $\mathcal{O}_1^{\mathrm{cris}}$ the kernel of $\mathcal{F}$ is $\mathcal{J}_1$.*

Proof. The proof will use the exactness criterion (5.1.11). Let $Z_0 = \operatorname{Spec} A_0 \in \mathcal{M}_X$ (see 5.1.10). By the exactness criterion, it is sufficient to show the equality for the ideals of $W_1^{DP}(A_i/p)$ corresponding to $\mathcal{J}_1$ and $\ker \mathcal{F}$. The first one is generated as a PD-ideal by the ideal $J_1 := \{(a_0) \in W_1(A_i/p) \mid \hat{a}_0^p = 0\}$, while the second one is generated as a PD-ideal by the ideal $I_1 := \{(a_0) \in W_1(A_i/p) \mid a_0^p = 0\}$. By 5.2.11, these ideals are equal, as $W_1(A_i/p)$ is of characteristic p. Therefore, the statement follows. □

Lemma 6.1.20. *By writing $\mathcal{J}_1 \subset \mathcal{O}_{n+1}^{\mathrm{cris}}$ for the image of $\mathcal{J}_1$ in $\mathcal{O}_{n+1}^{\mathrm{cris}}$ under the injection $\underline{p^n} : \mathcal{O}_1^{\mathrm{cris}} \hookrightarrow \mathcal{O}_{n+1}^{\mathrm{cris}}$ one defines*

$$\mathcal{I}_n := \mathcal{J}_{n+1}/J_1$$

and gets an exact sequence

$$0 \longrightarrow \mathcal{I}_n \longrightarrow \mathcal{O}_{n+1}^{\mathrm{cris}}/\mathcal{J}_1 \longrightarrow \mathbb{G}_{\mathbb{A}} \longrightarrow 0.$$

Proof. This is done by factoring out $\mathcal{J}_1$ from the short exact sequence $0 \to \mathcal{J}_{n+1} \to \mathcal{O}_{n+1}^{\mathrm{cris}} \to \mathcal{O}_{n+1} \to 0$. □

Since by definition, $\mathcal{J}_1 \subset p^n \mathcal{O}_{n+1}^{\mathrm{cris}}$, the surjection $\mathcal{J}_{n+1} \to \mathcal{J}_n$ factors via $\mathcal{I}_n$ giving rise to the exact sequence

$$0 \longrightarrow p^n \mathcal{O}_{n+1}^{\mathrm{cris}}/\mathcal{J}_1 \longrightarrow \mathcal{I}_n \longrightarrow \mathcal{J}_n \longrightarrow 0.$$

Proposition 6.1.21 ([Bau92, 3.4 (i)])**.** *The Frobenius $\mathcal{F}$ on $\mathcal{O}_{n+1}^{\mathrm{cris}}$ induces a morphism $p^{-1}F : \mathcal{I}_n \to \mathcal{O}_n^{\mathrm{cris}}$, which makes the following diagram commutative:*

$$\begin{tikzcd}
\mathcal{J}_{n+1} \arrow[r, two heads] \arrow[d, "\mathcal{F}"] & \mathcal{I}_n \arrow[d, "p^{-1}F"] \\
p\mathcal{O}_{n+1}^{\mathrm{cris}} & \mathcal{O}_n^{\mathrm{cris}}. \arrow[l, "\underline{p}"]
\end{tikzcd}$$

Proof. First we show that the Frobenius maps $\mathcal{J}_{n+1}$ to $p\mathcal{O}_{n+1}^{\mathrm{cris}}$: Using the exactness criterion we can work locally on a k-algebra A with surjective Frobenius. Let $(a_0, \dots, a_n) \in \mathcal{J}_{n+1}(A)$, i.e., $a_0^{p^n} = 0$. Choosing $\tilde{a}_i$ with $\tilde{a}_i^{p^{n+1}} = a_i$ we have $(a_0, \dots, a_n) = [\tilde{a}_0]^{p^{n+1}} + \dots + p^n[\tilde{a}_n]^p$ and $F(a_0, \dots, a_n) = (a_0^p, \dots, a_n^p) = [\tilde{a}_0]^{p^{n+2}} + \dots + p^n[\tilde{a}_n]^{p^2} = [\tilde{a}_0]^{p^{n+2}} + p(\dots)$. But as the Teichmüller lift commutes with multiplication, it follows that $[\tilde{a}_0]^{p^{n+2}} = [a_0]^p = p\gamma_p[a_0]$, since $[a_0]$ is an element of the PD ideal $\mathcal{J}_{n+1}(A)$. This shows $F(a_0, \dots, a_n) \in p\mathcal{O}_{n+1}^{\mathrm{cris}}$.

By the exactness criterion and 6.1.14 it is easy to see that $\ker(F : p^n\mathcal{O}_{n+1}^{\mathrm{cris}} \to \mathcal{O}_{n+1}^{\mathrm{cris}}) = \mathcal{J}_1$. Thus $\mathcal{J}_1 \subset \ker F$ and $F : \mathcal{J}_{n+1} \to p\mathcal{O}_{n+1}^{\mathrm{cris}}$ factors via $\mathcal{I}_n$. Composing the map $F : \mathcal{I}_n \to p\mathcal{O}_{n+1}^{\mathrm{cris}}$ with the inverse of the isomorphism $\underline{p} : \mathcal{O}_n^{\mathrm{cris}} \to p\mathcal{O}_{n+1}^{\mathrm{cris}}$ we get the morphism $p^{-1}F$ which makes the above diagram commute by construction. □

For the morphism $p^{-1}F$ there is a short exact sequence which will be fundamental for the proof of the conjecture of Birch and Swinnerton Dyer discussed in the next section:

Proposition 6.1.22 ([Bau92, 3.4 (ii)]). *On* syn(X) *there is an exact sequence*

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathcal{I}_n \xrightarrow{1-p^{-1}F} \mathcal{O}_n^{\mathrm{cris}} \longrightarrow 0.$$

Proof. This proof is quite complicated and therefore omitted here. See [Bau92, 3.4, 3.5]. □

7. The Proof of BSD

This section will sketch as an example of the usefulness of the syntomic cohomology a proof of the conjecture of Birch and Swinnerton-Dyer for abelian varieties with good reduction everywhere over function fields in characteristic $p > 0$ as given in [Bau92]. In this section, p will denote an odd prime. The following objects will play an important role:

(i) Let K be an algebraic function field in one variable with field of constants $\mathbb{F}_q$, $q = p^r$.

(ii) Let $S/\mathbb{F}_q$ be the smooth projective integral curve corresponding to K.

(iii) Let A_K/K be an abelian variety over K, which has good reduction everywhere.

(iv) Let $\pi : A \longrightarrow S$ an abelian scheme with generic fiber $\pi^{-1}(s) = A_K$.

The goal of [Bau92] is to prove the conjecture of Birch and Swinnerton-Dyer as stated in [Tat66, conj. B], which reads in our case as follows:

Theorem 7.0.1. *Let A_K/K be of dimension d. Let ρ be the order of the zero of the L-function $L^*(s)$ of A_K/K at $s = 1$. Let g be the genus of K. Let $h : A_K(K) \times \hat{A}_K(K) \longrightarrow \mathbb{R}$ denote the height pairing. If $Ш(A_K/K)(\ell)$ is finite for one prime ℓ, then the whole $Ш(A_k/K)$ is finite and*

$$\left| \lim_{s \to 1} \frac{L^*(s)}{(s-1)^\rho} \right| = \frac{\# Ш(A_K/K) \cdot |\det h|}{\# \operatorname{Tor} A_K(K) \cdot \# \operatorname{Tor} \hat{A}_K(K)}.$$

Up to p-part, this is already proven by Tate ([Tat66, Thm 5.2]) for elliptic curves and by Schneider ([Sch82, Theorem]) for arbitrary abelian varieties using étale cohomology. Thus, it is sufficient to consider the p-part.

7.1. The L-Function of an Abelian Variety in Terms of Syntomic Cohomology Groups

To A_K/K one attaches the Hasse-Weil L-function:

$$\begin{aligned} L(s) &= L_A(q^{-s}), \text{ where} \\ L_A(t) &= \prod_{x \in |S|} \det(1 - \varphi^{-\deg x} \cdot t^{\deg x} | H^1_{\text{ét}}(\bar{A}_x, \mathbb{Q}_\ell))^{-1}. \end{aligned}$$

The first step is to transform the L-function into a function in terms of crystalline cohomology groups:

Proposition 7.1.1 ([Bau92, 2.4]). *$L(s) = L_{\mathrm{CRIS}}(p^{-s})$, where*

$$L_{\mathrm{CRIS}}(t) = \prod_{j=0}^{2} \det(1 - \Phi t | H^j_{\mathrm{CRIS,ZAR}}(S/\mathbb{Z}_p, \mathcal{E}_{\mathbb{Q}_p}))^{(-1)^{j+1}}.$$

with $H^j_{\mathrm{CRIS,ZAR}}(S/\mathbb{Z}_p \mathcal{E}_{\mathbb{Q}_p})$ is defined as follws (see [Bau92, 2.1]): Let $\mathcal{E} = \mathbf{D}(A) = \mathcal{E}\mathrm{xt}^1_{S/\mathbb{Z}_p}(u^*_{S/\mathbb{Z}_p} A, \mathcal{O}_{S/\mathbb{Z}_p})$ be the Dieudonné module of A/S. Through the closed immersion $i : \Sigma_n := \mathrm{Spec}(\mathbb{Z}/p^n\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}_p)$ define $\mathcal{E}_n := i^*_{\mathrm{CRIS}} \mathcal{E} \simeq \mathcal{E}\mathrm{xt}(u^*_{S/\Sigma_n} A, \mathcal{O}_{S/\Sigma_n})$ [BBM82, 2.3.6]. Then let $H^*_{\mathrm{CRIS}}(S/\mathbb{Z}_p, \mathcal{E}_{\mathbb{Q}_p}) := \varprojlim_n H^*((S/\Sigma_n)_{\mathrm{CRIS,ZAR}}, \mathcal{E}_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ where the transition maps are induced by $i^*_{\mathrm{CRIS}} \mathcal{E}_n \simeq \mathcal{E}_{n-1}$ with respect to the closed immersion $i : \Sigma_{n-1} \hookrightarrow \Sigma_n$. In addition, Φ denotes the Frobenius endomorphism as defined in [Bau92, 2.3].

Now we can pass over to the syntomic site using the morphisms constructed earlier, especially the morphism $u : (S/\Sigma)_{\mathrm{CRIS,SYN}} \longrightarrow S_{\mathrm{SYN}}$. Let $\mathcal{O}^{\mathrm{cris}}_n = u_* \mathcal{O}_{S/\Sigma_n}$, as defined in 6.1.4. Then, using the morphisms of topoi and checking some Frobenius compatibilities one translates the L-function first to an L-function L_{SYN} for the big syntomic site and gets finally

Proposition 7.1.2 ([Bau92, 2.7]). *Let F denote the endomorphism which is induced on $H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{O}^{\mathrm{cris}}_n))$ by the Frobenius of $\mathcal{O}^{\mathrm{cris}}_n$. Further let*

$$H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Q}_p})) := \varprojlim_n H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{O}^{\mathrm{cris}}_n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where the transition maps are induced by the natural epimorphisms $\mathcal{O}^{\mathrm{cris}}_{n+1} \longrightarrow \mathcal{O}^{\mathrm{cris}}_n$. Then the Hasse-Weil L-function of A_K/K is given by $L(s) = L_{\mathrm{syn}}(p^{-s})$ where

$$L_{\mathrm{syn}}(t) := \prod_{j=0}^{2} \det(1 - Ft | H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Q}_p})))^{(-1)^{j+1}}.$$

7.2. Calculating the Syntomic Cohomology Groups

The calculation of the syntomic cohomology groups consists of several steps. First, the factors of the syntomic L-function are expressed in terms of certain $\#\ker/\#\,\mathrm{coker}$-fractions of some morphisms. Plugging in some exact sequences, this can be expressed in terms of the size of some cohomology groups. In the last step, these sizes of cohomology groups can be identified with the arithmetic invariants appearing in the conjecture of Birch and Swinnerton-Dyer.

Application of $\mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, -)$ to the exact sequences 6.1.20 and 6.1.22 gives two short exact sequences of syntomic sheaves ([Bau92, 3.7]):

$$0 \longrightarrow \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{I}_n) \xrightarrow{\ i\ } \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{O}^{\mathrm{cris}}_{n+1}/\mathcal{J}_1) \longrightarrow \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathbb{G}_{\mathbb{A}}) \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mu_{p^n}) \longrightarrow \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{I}_n) \xrightarrow{1-p^{-1}F} \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{O}_n^{\mathrm{cris}}) \longrightarrow 0.$$

Now one defines

$$\begin{aligned} H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p})) &:= \varprojlim_n H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{O}^{\mathrm{cris}}_n)) \\ H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p})) &:= \varprojlim_n H^j(S_{\mathrm{syn}}, \mathcal{E}\mathrm{xt}^1_{S_{\mathrm{syn}}}(A, \mathcal{I}_n)), \end{aligned}$$

which gives rise to two important long exact sequences (we suppress the index in S_{syn} to simplify notation):

Lemma 7.2.1 ([Bau92, 3.8])**.** *There are long exact sequences of finitely generated $\mathbb{Z}_p$-modules from the two short exact sequences above:*

$$\ldots \longrightarrow H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p})) \xrightarrow{i} H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p})) \longrightarrow H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathbb{G}_a)) \longrightarrow \ldots$$

$$\ldots \longrightarrow H^j(S, T_p\hat{A}) \longrightarrow H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p})) \stackrel{1-p^{-1}F}{\longrightarrow} H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p})) \longrightarrow \ldots$$

Handling the determinants: Crucial for handling the determinants of the cohomology groups is the notion of a *quasi-isomorphism* of abelian groups:

Definition 7.2.2. A morphism f of abelian groups is called *quasi-isomorphism* if $\ker f$ and $\operatorname{coker} f$ are finite. In this case one defines

$$z(f) := \frac{\#\ker f}{\#\operatorname{coker} f}.$$

The importance of this definitions becomes apparent in the following

Lemma 7.2.3 ([Tat66, z.1])**.** *Let ℓ be an arbitrary prime, A and B finitely generated $\mathbb{Z}_\ell$-modules of the same rank, and (a_i), resp. (b_i) bases for A_{Tor} and B_{Tor}. Let $f : A \longrightarrow B$ be $\mathbb{Z}_\ell$-linear with $f_{\mathrm{Tor}}(a_i) = \sum_j z_{ij} b_j$, where f_{Tor} denotes the induced morphism $A_{\mathrm{Tor}} \to B_{\mathrm{Tor}}$. Then f is a quasi-isomorphism if and only if $\det(z_{ij}) \neq 0$, in which case*

$$|z(f)|_\ell = \left| \det(z_{ij}) \frac{\#B_{\mathrm{Tor}}}{\#A_{\mathrm{Tor}}} \right|_\ell .$$

Furthermore, this is compatible with composition:

Lemma 7.2.4 ([Tat66, z.2])**.** *With the same notation as above: Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be homomorphisms of $\mathbb{Z}_\ell$-modules. If any two of the three maps f, g, and $g \circ f$ are quasi-isomorphisms, the third is a quasi-isomorphism, too and one has $z(g \circ f) = z(g)z(f)$.*

To simplify notation, we define

$$P_j(t) := \det(1 - Ft|H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Q}_p}))).$$

With this definition we have $L_{\mathrm{syn}}(t) = \frac{P_1(t)}{P_0(t)P_2(t)}$ and $L(s) = L_{\mathrm{syn}}(p^{-s})$. By defining furthermore

$$P(t) = \frac{P_1(t)}{(1 - t^{-1}p^{-1})^{\rho}} \in \mathbb{Q}_p(t),$$

we have $P(p^{-1}) \neq 0$ ([Bau92, 3.14]) and

$$\lim_{s\to 1} \frac{L(s)}{(1 - p^{s-1})^{\rho}} = \frac{P(t)}{P_0(t)P_2(t)}.$$

Now we can try to calculate these polynomials:

Lemma 7.2.5 ([Bau92, 3.12]). *For $j = 0, 2$ the morphisms $i_j := i$ and $\varphi_j := 1 - p^{-1}F$ from* (7.2.1) *and* (7.2.1) *are quasi-isomorphisms and one has*

$$|P_j(p^{-1})|_p = z(\varphi_j)z(i_j)^{-1}.$$

For P some more work is needed:

Definition 7.2.6. We define the Frobenius invariants $H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p}))^{1-p^{-1}F}$ to be the kernel of the map

$$1 - p^{-1}F : H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p})) \longrightarrow H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p})).$$

Furthermore we define the Frobenius coinvariants $H^j(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p}))_{1-p^{-1}F}$ to be the cokernel of the above map.

Lemma 7.2.7 ([Bau92, 3.15]). *Let*

$$f : H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p}))^{1-p^{-1}F} \longrightarrow H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p}))_{1-p^{-1}F}$$

be induced by $i : H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p})) \longrightarrow H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}^{\mathrm{cris}}_{\mathbb{Z}_p}))$ (see 7.2.1). Then f is a quasi-isomorphism if and only if $\rho = \mathrm{rk}_{\mathbb{Z}_p} H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p}))^{1-p^{-1}F}$ and in this case one has

$$|P(p^{-1})|_p = z(f)z(i_1)^{-1}.$$

Noting that $\mathrm{rk}_{\mathbb{Z}_p} H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p}))^{1-p^{-1}F} = \mathrm{rk}_{\mathbb{Z}_p} H^1(S, T_p\hat{A})$ ([Bau92, 3.17]), we can collect what we have achieved so far in the following form:

Proposition 7.2.8 ([Bau92, 3.17]). *One has $\rho = \mathrm{rk}_{\mathbb{Z}_p} H^1(S, T_p\hat{A})$ if and only if f is a quasi-isomorphism and in this case*

$$\left|\lim_{s\to 1} \frac{L(s)}{(1 - p^{s-1})^{\rho}}\right|_p = \frac{z(\varphi_0)\cdot z(i_1)\cdot z(\varphi_2)}{z(i_0)\cdot z(f)\cdot z(i_2)}.$$

The next step is to actually calculate the z's. With the Frobenius invariants and coinvariants (7.2.6), the long exact sequences (7.2.1) give rise to short exact sequences ([Bau92, 3.11])

$$
\begin{tikzcd}
0 \arrow[r] & H^0(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}_{\mathbb{Z}_p}^{\mathrm{cris}}))_{1-p^{-1}F} \arrow[r] & H^1(S, T_p\hat{A}) \arrow[r, "\alpha"] & H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p}))^{1-p^{-1}F} \arrow[r] \arrow[dll, "f"'] & 0 \\
0 \arrow[r] & H^1(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{O}_{\mathbb{Z}_p}^{\mathrm{cris}}))_{1-p^{-1}F} \arrow[r, "\beta"'] & H^2(S, T_p\hat{A}) \arrow[r] & H^3(S, \mathcal{E}\mathrm{xt}^1(A, \mathcal{I}_{\mathbb{Z}_p}))^{1-p^{-1}F} \arrow[r] & 0,
\end{tikzcd}
$$

which allow to proof the following:

Proposition 7.2.9 ([Bau92, 3.17])**.** *One has $\rho = \mathrm{rk}_{\mathbb{Z}_p} H^1(S, T_p\hat{A})$ if and only if f is a quasi-isomorphism and in this case*

$$
\left| \lim_{s \to 1} \frac{L(s)}{(1 - p^{s-1})^\rho} \right|_p = \prod_{j=1}^{3} \# \operatorname{Tor} H^j(S, T_p\hat{A})^{(-1)^j} \prod_{j=0}^{1} \# H^j(S, \mathbf{R}^1\pi_* \mathcal{O}_A)^{(-1)^j} \cdot z((\beta f \alpha)_{Tor})^{-1}.
$$

Let $\langle \, , \rangle_q : H^1(S, T_p\hat{A}) \times H^1(S, T_pA) \to H^2(S, \mathbb{Z}_p(1)) \to \mathbb{Z}_p$ be the cup product pairing induced by the exact Kummer sequence in [Bau92, 3.1]. Some tedious computations allow to show

Proposition 7.2.10 ([Bau92, 3.18])**.** *The morphism f is a quasi isomorphism if and only if $\langle \, , \rangle_q$ is non-degenerate and in this case*

$$
z((\beta f \alpha)_{Tor}) = |r^\rho \cdot \det \langle \, , \rangle_q|_p,
$$

where $r = [\mathbb{F}_q : \mathbb{F}_p]$.

All of this allows to formulate the theorem

Theorem 7.2.11. *One has $\rho \geq \mathrm{rk}_{\mathbb{Z}_p} H^1(S, T_p\hat{A})$. Furthermore $\rho = \mathrm{rk}_{\mathbb{Z}_p} H^1(S, T_p\hat{A})$ if and only if $\langle \, , \rangle_q$ is non-degenerate and in this case one has*

$$
\left| \lim_{s \to 1} \frac{L(s)}{(1 - p^{s-1})^\rho} \right|_p = |r^\rho \cdot \det \langle \, , \rangle_q|_p^{-1} \cdot \prod_{j=1}^{3} \# \operatorname{Tor} H^j(S, T_p\hat{A})^{(-1)^j} \cdot \prod_{j=0}^{1} \# H^j(S, \mathbf{R}^1\pi_* \mathcal{O}_A)^{(-1)^j}.
$$

With this important theorem we can start to calculate the cardinalities and approach the conjecture of Birch and Swinnerton-Dyer. First, using results from Milne, Bauer proofs the following lemma:

Lemma 7.2.12 ([Bau92, 4.2]). *If $Ш(A_K/K)(p)$ is finite, then*

$$\prod_{j=1}^{3} \#\operatorname{Tor} H^j(S, T_p\hat{A})^{(-1)^j} = \frac{\#Ш(A_K/K)(p)}{\#A_K(K)(p) \cdot \#\hat{A}_K(K)(p)}.$$

By the Riemann-Roch theorem, Bauer calculates the next factor:

Lemma 7.2.13 ([Bau92, 4.3]). *Let $e : S \to A$ be the unit section and let $\omega_A := e^*\Omega^1_{A/S}$. Then*

$$\prod_{j=0}^{1} \#H^j(S, \mathbf{R}^1\pi_*\mathcal{O}_A)^{(-1)^j} = q^{-\deg\omega_A + d(1-g)},$$

where g denotes the genus of K and d the dimension of A_K.

Finally, via the Yoneda pairing, Bauer relates $\langle\, ,\rangle_q$ to the Néron-Tate height paring:

Lemma 7.2.14 ([Bau92, 4.5]). *Let $h : A_K(K) \times \hat{A}_K(K) \to \mathbb{R}$ be the Néron-Tate height pairing. If $Ш(A_K/K)(p)$ is finite, then $\langle\, ,\rangle_q$ is non-degenerate and*

$$|\det\langle\, ,\rangle_q|_p = |(\log q)^{-\operatorname{rk}_{\mathbb{Z}} A_K(K)} \cdot \det h|_p.$$

Because of

$$\left|\lim_{s\to 1} \frac{L(s)}{(1-p^{s-1})^\rho}\right| = (\log p)^{-\rho} \left|\lim_{s\to 1} \frac{L(s)}{(s-1)^\rho}\right|,$$

this can be combined with the result for the non-p-part in [Sch82, Theorem] to get

Theorem 7.2.15 ([Bau92, 4.7]). *Let A_K/K be an abelian variety of dimension d possessing good reduction everywhere. Let ρ be the order of the zero of the Hasse-Weil L-function $L(s)$ of A_K/K at $s = 1$. Let $S/\mathbb{F}_q$ be the model of K, g be the genus of K, and A/S be an abelian scheme with generic fiber A_K. Further let $e : S \to A$ be the unit section, $\omega_A := e^*\Omega^1_{A/S}$, and let $h : A_K(K) \times \hat{A}_K(K) \to \mathbb{R}$ denote the height pairing. Then one has $\rho = \operatorname{rk}_{\mathbb{Z}} A_K(K)$ if and only if $Ш(A_K/K)(\ell)$ is finite for one prime ℓ. In this case $Ш(A_K/K)$ is finite too and it holds*

$$\left|\lim_{s\to 1} \frac{L(s)}{(s-1)^\rho}\right| = \frac{\#Ш(A_K/K) \cdot |\det h|}{\#\operatorname{Tor} A_K(K) \cdot \#\operatorname{Tor} \hat{A}_K(K)} \cdot q^{-\deg\omega_A + d(1-g)}.$$

Since for Tate's L-series $L^*(s)$ one has

$$\lim_{s\to 1} \frac{L(s)}{L^*(s)} = q^{-\deg\omega_A + d(1-g)},$$

this results in the conjecture of Birch and Swinnerton-Dyer as formulated in [Tat66, Conjecture B]:

Theorem 7.2.16 ([Bau92, 4.8]). *Under the assumptions of 7.2.15 one has the following statement: If $Ш(A_K/K)(\ell)$ is finite for one prime ℓ then $Ш(A_K/K)$ is finite too and it holds*

$$\left|\lim_{s\to 1} \frac{L^*(s)}{(s-1)^\rho}\right| = \frac{\#Ш(A_K/K) \cdot |\det h|}{\#\operatorname{Tor} A_K(K) \cdot \#\operatorname{Tor} \hat{A}_K(K)}.$$

The following diagram gives an overview of the complete proof of the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic $p > 0$ as given in [Bau92]. The numbers on the arrows refer to the corresponding statements in [Bau92].

$$\left|(\log p)^{-\rho} \lim_{s\to 1} \frac{L(s)}{(s-1)^{\rho}}\right|_p^{-1} \overset{3.1}{=\!=} \left|\lim_{s\to 1} \frac{L(s)}{(1-p^{s-1})^{\rho}}\right|_p^{-1} \overset{2}{=} \left|\lim_{s\to 1} \frac{L_A(q^{-s})}{(1-p^{s-1})^{\rho}}\right|_p^{-1}$$

$$\overset{2.4}{=\!=} \left|\lim_{s\to 1} \frac{L_{\mathrm{CRIS}}(p^{-s})}{(1-p^{s-1})^{\rho}}\right|_p^{-1} \overset{2.6}{=\!=} \left|\lim_{s\to 1} \frac{L_{\mathrm{SYN}}(p^{-s})}{(1-p^{s-1})^{\rho}}\right|_p^{-1} \overset{2.7}{=\!=} \left|\lim_{s\to 1} \frac{L_{\mathrm{syn}}(p^{-s})}{(1-p^{s-1})^{\rho}}\right|_p^{-1}$$

$$\overset{\text{after } 3.11}{=\!=\!=} \left| P_0(p^{-1}) \cdot P_2(p^{-1}) \cdot \frac{1}{P(p^{-1})} \right|_p$$

3.12 3.12 3.15

$$= \frac{z(\varphi_0)}{z(i_0)} \cdot \frac{z(\varphi_2)}{z(i_2)} \cdot \frac{z(i_1)}{z(f)}$$

3.11 ($j = 3$) 3.16 ($j = 1, 2$) 3.8.1 3.16 3.16 3.11

$$= \prod_{j=1}^{3} \#\,\mathrm{Tor}\, H^j(S, T_p\hat{A})^{(-1)^j} \cdot \prod_{j=0}^{1} \# H^j(S, \mathbf{R}^1\pi_*\mathcal{O}_A)^{(-1)^j} \cdot z((\beta f \alpha)_{\mathrm{Tor}})^{-1} \cdot \frac{\frac{\#\ker(1-p^{-1}F)}{\#\,\mathrm{coker}(1-p^{-1}F)}}{\frac{\#\ker(1-p^{-1}F)}{\#\,\mathrm{coker}(1-p^{-1}F)}}$$

3.18

$$= \frac{\#\,\mathrm{Tor}\, H^2(S, T_p\hat{A})}{\#\,\mathrm{Tor}\, H^3(S, T_p\hat{A}) \cdot \#\,\mathrm{Tor}\, H^1(S, T_p\hat{A})} \cdot \prod_{j=0}^{1} \# H^j(S, \mathbf{R}^1\pi_*\mathcal{O}_A)^{(-1)^j} \cdot |r^{\rho} \cdot \det\langle\ ,\ \rangle_q|_p^{-1}$$

4.3 4.4, 4.5

$$= \frac{\#\text{Ш}(A_K/K)(p)}{\# A_K(K)(p) \cdot \#\hat{A}_K(K)(p)} \cdot q^{-\deg \omega_A + d(1-g)} \cdot |(\log p)^{-\rho} \det h|_p^{-1}$$

A. Appendix

Lemma A.1.1. *Let R be a ring and $I \subseteq R$ an ideal. Then one has a canonical morphism of R/I-modules*

$$I \otimes_R R/I \cong I/I^2.$$

Proof. There is a short exact sequence of R-modules

$$0 \longrightarrow I^2 \xrightarrow{i} I \xrightarrow{\pi} I/I^2 \longrightarrow 0,$$

which gives raise to an exact sequence of R/I-modules

$$I^2 \otimes_R R/I \xrightarrow{i'} I \otimes_R R/I \xrightarrow{\pi'} I/I^2 \otimes_R R/I \longrightarrow 0.$$

Let $xy \otimes a \in I^2 \otimes_R R/I$ with $x, y \in I$, $a \in R/I$. Then one has $i'(xy \otimes a) = xy \otimes a = x \otimes ya = 0$ and thus $i' = 0$. This shows that $\pi' : I \otimes_R R/I \to I/I^2 \otimes_R R/I$ is an isomorphism. Since I annihilates I/I^2, we have furthermore $I/I^2 \otimes_R R/I \cong I/I^2 \otimes_{R/I} R/I \cong I/I^2$. This shows the lemma. $\square$

Lemma A.1.2. *Let k be a perfect field of characteristic $p > 0$, $W_n := W_n(k)$ the ring of Witt vectors of length n. Let $w_n : W_n \to k, (x_0, \dots, x_{n-1}) \mapsto x_0^{p^n}$ denote the n-th ghost component. Then for all $(x_0, \dots, x_{n-1}) \in W_n$ one has*

$$(x_0, \dots, x_{n-1}) = \hat{x}_0^{p^n} + \dots + p^{n-1}\hat{x}_{n-1}^{p},$$

where $\hat{x}_i$ denotes a lift of $x_i \in k$ under w_n.

Proof. This map is well-defined: Since k is perfect, one has $\ker(w_n) = (p)$, therefore $\hat{p}^i \hat{x}_i^{p^{n-1}}$ does not depend on the choice of the lift by 5.2.8. Hence, let without restriction $\hat{x}_i = [x_i^{p^{-n}}]$. Then one has $\hat{p}^i \hat{x}_i^{p^{n-i}} = p^i [x_i^{p^{-n}}]^{p^{n-i}} = p^i [x_i^{n-i}] = (0, \dots, 0, x_i, 0, \dots, 0)$ as the Teichmüller lift commutes with products. Now the statement follows from [FOed, p. 12, 0.12]. $\square$

Proposition A.1.3. *Let k be a perfect field of characteristic $p > 0$, $W_n := W_n(k)$ the ring of Witt vectors of length n. Let $\rho_k : W_n \to W_{n-k}$ be given by $(x_0, \dots, x_{n-1}) \mapsto (x_0^{p^k}, \dots, x_{n-k-1}^{p^k})$. The ring W_{n-k} is an W_{n+1}-module via ρ_k. One has an exact sequence of W_{n+1}-modules*

$$0 \longrightarrow W_1(k) \xrightarrow{\nu^n} W_{n+1}(k) \xrightarrow{\rho_1} W_n(k) \longrightarrow 0.$$

Proof. The exactness of the sequence is clear, as k is perfect. The map ρ_1 is W_{n+1}-linear, as it is a ring homomorphism ($\rho_1 = (- \mod p^n) \circ \sigma$) and W_n is a W_{n+1}-module via ρ_1. The map ν_n is additive by [FOed, p. 12]. Let $y = (y_0, \ldots, y_n) \in W_{n+1}$ and $(x_0) \in W_1$. Then one has $y \cdot (x_0) = (y_0^n x_0) \mapsto (0, \ldots, y_0^n x_0)$. On the other hand one has $y \cdot (0, \ldots, x_0) = p^n \hat{x}_0^p \cdot (\sum_i p^i \hat{y}_i^{p^{n+1-i}}) = p^n \hat{x}_0^p \hat{y}_0^{p^{n+1}} = (0, \ldots, x_0 y_0^{p^n})$. □

Lemma A.1.4. *Consider the sheaf of real-valued $\mathcal{C}^\infty$-functions on $\mathbb{R}$. Let F be the ring of germs of this sheaf in 0. This local ring contains the maximal ideal $\mathfrak{m} = \{f \mid f(0) = 0\}$. (To simplify notation, germs will be written by representatives). Furthermore F contains the function $t : x \mapsto x$. Then $\mathfrak{m}$ is generated by t.*

Proof. Let $f \in \mathfrak{m}$. It suffices to show that the function $\frac{f(x)}{x}$ can be $\mathcal{C}^\infty$-extended in 0. Therefore, it suffices to show that the functions $\left(\frac{f(x)}{x}\right)^{(k)}$ are continuous extendable in 0 for all $k \geq 0$. One has:

$$\begin{aligned}
\left(\frac{f(x)}{x}\right)^{(k)} &= \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(x) \frac{(-1)^{k-i}(k-i)!}{x^{k-i+1}} \\
&= \sum_{i=0}^{k} (-1)^{(k-i)} \frac{k!}{i!} \frac{f^{(i)}(x)}{x^{k-i+1}} \\
&= \frac{\sum_{i=0}^{k} (-1)^{k-i} \frac{k!}{i!} x^i f^{(i)}(x)}{x^{k+1}}.
\end{aligned}$$

The limit in 0 can be calculated by de l'Hospital. With

$$\begin{aligned}
\left(\sum_{i=0}^{k} (-1)^{k-i} \frac{k!}{i!} x^i f^{(i)}(x)\right)' &= \sum_{i=0}^{k} (-1)^{k-i} \frac{k!}{i!} \left(i x^{i-1} f^{(i)}(x) + x^i f^{(i+1)}(x)\right) \\
&= x^k f^{(k+1)}(x),
\end{aligned}$$

one gets

$$\lim_{x \to 0} \left(\frac{f(x)}{x}\right)^{(k)} \to \lim_{x \to 0} \frac{x^k f^{(k+1)}(x)}{(k+1) x^{k+1}} = \frac{f^{(k+1)}(0)}{k+1}.$$

Thus, there is a $\mathcal{C}^\infty$-function g such that $gt = f$. □

Lemma A.1.5. *Let $U \to V$ and $U \to W$ be immersions over Y. Then $U \to V \times_Y W$ is an immersion, too.*

Proof. The induced morphism $U \to V \times_Y W$ factors as $U \to U \times_Y U \to V \times_Y U \to V \times_Y W$ as can be seen in the following diagram:

$$\begin{tikzcd}
 & U \arrow[r, hook] & V \arrow[r, equal] & V \arrow[dr] & \\
U \arrow[ur, equal] \arrow[dr, equal] \arrow[r, "\Delta"] & U \times_Y U \arrow[u] \arrow[d] \arrow[r] & V \times_Y U \arrow[u] \arrow[d] \arrow[r] & V \times_Y W \arrow[u] \arrow[d] & Y. \\
 & U \arrow[r, equal] & U \arrow[r, hook] & W \arrow[ur] &
\end{tikzcd}$$

The diagonal $\Delta_{U/Y} : U \to U \times_Y U$ is an immersion ([GW10, 9.5]) and $U \times_Y U \to V \times_Y U$, $V \times_Y U \to V \times_Y W$ are so, because immersions are stable under composition and base change ([GW10, 4.30]). Thus, the composition is an immersion, too. □

Proposition A.1.6. *Let $\mathcal{P}$ be the category of PD rings and let (A, I, γ) be a PD ring. Let $\mathcal{C}$ be the category of pairs (R, I) where R is a ring and $I \subset R$ an ideal with morphisms $f : (R, I) \to (R', I')$ consisting of homomorphisms $f : R \to R'$ with $f(I) \subseteq I'$. Then taking the PD envelope is a functor $\mathcal{C}/A \to \mathcal{P}/A$ which is left adjoint to the forgetful functor $\mathcal{P}/A \to \mathcal{C}/A$.*

Proof. This follows directly from the universal property of the PD envelope ([BO78, 3.119]):

$$\mathrm{Hom}_{\mathcal{C}/A}((B, J), (C, K)) = \mathrm{Hom}_{\mathcal{P}/A}((\mathcal{D}_{B,\gamma}(J), \bar{J}, [\,]), (C, K, \gamma))$$

□

Lemma A.1.7. *Let $i : X \to Y$ be a closed immersion over $W_n(k)$, $U \subset Y$ open with $i(X) \subset U$. Then one has $D_{X,(p)}(Y) = D_{X,(p)}(U)$.*

Proof. Let X, U, Y be affine. Obviously $U \to Y$ is flat. Therefore, by [BO78, 3.21], we have

$$D_{X,(p)}(U) = D_{X,(p)}(Y) \times_Y U.$$

It remains to show that $D_{X,(p)}(Y) \times_Y U = D_{X,(p)}(Y)$. But, as we have topological identities $X \to D_{X,(p)}(Y)$ and $X \to D_{X,(p)}(U)$, the morphism $D_{X,(p)}(Y) \to Y$ factors through U and we get a morphism $u : D_{X,(p)}(Y) \to D_{X,(p)}(Y) \times_Y U$ via the universal property of the fiber product:

$$\begin{tikzcd}
D_{X,(p)}(Y) \arrow[dr, dotted, "u"] \arrow[drr, "\mathrm{id}"] \arrow[ddr] & & \\
 & D_{X,(p)}(Y) \times_Y U \arrow[r, "p_1"'] \arrow[d, "p_2"] & D_{X,(p)}(Y) \arrow[d] \\
 & U \arrow[r] & Y
\end{tikzcd}$$

Obviously we have $p_1 \circ u = \mathrm{id}$ by construction. Since the morphism $u \circ p_1$ commutes over $D_{X,(p)}(Y) \to Y$ and $U \to Y$, it is unique with this property and therefore the identity, too.

Now let X, U, Y be arbitrary. The sheaf giving rise to $D_{X,(p)}(Y)$ is quasi-coherent on Y ([BO78, 3.30]), therefore we can check the proposition on affine coverings. $\square$

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Erklärung

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