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MASTERARBEIT

Epsilon-Isomorphisms and Twists

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Abstract

This master thesis considers aspects of Fukaya's and Kato's equivariant ε-isomorphism conjecture ([FK06] Conj 3.4.3). We start with some preliminaries on determinant functors, p-adic Hodge theory and Galois cohomology. Then we elaborate Fukaya's and Kato's construction of ε-isomorphisms for de Rham representations and incorporate ideas of Nakamura [Nak17] concerning the multiplicativity. Assuming the existence of an equivariant ε-isomorphism for an instance of a certain class of triples $(\Lambda, \mathbb{T}(T), \xi)$ we establish in the main chapter of this work that the equivariant ε-isomorphism exists for a twisted triple $(\Lambda, \mathbb{T}(T(\chi)), \xi)$ as well. We do this by viewing the twist as a base change as in section 4.5 of [LVZ15] and check that the properties in the equivariant ε-isomorphism conjecture are compatible with this base change. Our arguments show with little gerneralisation effort that the properties in [FK06] Conj 3.4.3 are compatible with any base change of adic rings. As an outlook we present ideas how a stronger connection concerning the specialisation to ε-isomorphisms of de Rham representations could be established between the equivariant ε-isomorphisms for $\mathbb{T}(T)$ and the twisted representation $\mathbb{T}(T(\chi))$. Here, we assume that the character χ is finite and unramified.

Zusammenfassung

Diese Masterarbeit behandelt Aspekte der äquivarianten ε-Isomorphismen Vermutung von Fukaya und Kato ([FK06] Conj. 3.4.3). Wir beginnen mit einigen Grundlagen zu Determinanten-Funktoren, p-adischer Hodge Theorie und Galois Kohomologie. Dann führen wir Fukayas und Katos Konstruktion von ε-Isomorphismen zu de Rham Darstellungen unter Berücksichtigung von Nakamuras Ideen zur Multiplikativität in [Nak17] aus. Im Hauptteil der Arbeit zeigen wir ausgehend von der Existenz eines äquivarianten ε-Isomorphismus zu einem Tripel $(\Lambda, \mathbb{T}(T), \xi)$ einer gewissen Klasse von Triplen auch die Existenz des äquivarianten ε-Isomorphismus zum getwisteten Tripel $(\Lambda, \mathbb{T}(T(\chi)), \xi)$. Die Kernidee ist, den Twist wie in Sektion 4.5 von [LVZ15] als Basiswechsel aufzufassen. Wir prüfen dann, dass die Eigenschaften der äquivarianten ε-Isomorphismen Vermutung kompatibel mit diesem Basiswechsel sind. Mit wenig Aufwand sieht man mit unseren Argumenten auch, dass die Eigenschaften in [FK06] Conj. 3.4.3 kompatibel mit allgemeinem Basiswechsel adischer Ringe ist. Als Ausblick halten wir abschließend einige Gedanken zu einer stärkeren, die Spezialisierung zu ε-Isomorphismen von de Rham Darstellungen betreffenden Verbindung zwischen den äquivarianten ε-Isomorphismen zu $\mathbb{T}(T)$ und dem getwisteten $\mathbb{T}(T(\chi))$ fest. Dazu nehmen wir an, der Charakter χ sei endlich und unverzweigt.

Contents

Introduction

Motivation

The thesis deals with the equivariant ε-isomorphism conjecture of Fukaya and Kato ([FK06] Conj 3.4.3). For its relation to the ε-conjectures by Fontaine/Perrin-Riou or Benois/Berger see chapter 2 of [Izy12]. Fukaya and Kato formulate their ε-conjecture in a very general setting, working, for instance, over non-commutative rings, to relate the (non-commutative) Equivariant Tamagawa Number Conjecture (ETNC) of Burns and Flach ([BF01]) with the non-commutative Iwasawa Main Conjecture with p-adic L-functions ([Ven05a] and [CFK$^+$05]). The connection of these topics has been elaborated in the survey [Ven05b]. We briefly recall some aspects.
For a motive M Bloch/Kato ([BK07] Conj. 5.15) and Fontaine/Perrin-Riou ([FPR94]) conjectured a relation between the leading coefficient of the L-function, $L(M, s)$, associated to M at zero, $L^*(M)$, and Galois cohomology groups of integral representations associated to M up to some period and regulator. This strong conjecture, the Tamagawa Number Conjecture (TNC), implies, for instance, in the case of elliptic curves the Birch and Swinnerton-Dyer conjecture ([Ven05b] 3.1). Fukaya and Kato generalised it to an equivariant version, the ζ-isomorphism conjecture, in their [FK06] Conj. 2.3.2 extending the ETNC of Burns and Flach. One can view the ζ-isomorphism conjecture as an interpolation of the TNC for twists of M by representations of a p-adic Lie group.
Conjecturally, $L(M, s)$ satisfies a functional equation relating it to the L-function of the Kummer dual $M^*(1)$. Taking the leading coefficients of this functional equation implies the equation

$$L^*(M) = (-1)^\eta \varepsilon(M) \cdot \frac{L^*_\infty(M * (1))}{L^*_\infty(M)} \cdot L^*(M^*(1)).$$

One reason for the importance of the equivariant ε-isomorphism conjecture is that assuming the existence of the functional equation, it implies the equivalence of the ζ-isomorphism conjecture for M and the ζ-isomorphism conjecture for $M^*(1)$ (and hence also the ETNC for the respective motives) via the functional equation and establies a functional equation between the ζ-isomorphisms for M and $M^*(1)$ (see [Ven05b] thm 5.11).
In [CFK$^+$05] the existence of certain p-adic L-functions is conjectured. Assuming their ζ- and equivariant ε-conjectures Fukaya and Kato can construct these p-adic L-functions (see §4 of [FK06], in particular 4.1.3 and section 4.3).

Some known results

The equivariant ε-isomorphism conjecture predicts the existence of ε-isomorphisms with certain properties for tripels $(\Lambda, \mathbb{T}, \xi)$, where Λ is an adic ring (see definition 1.3.9), for instance the Iwasawa algebra $\mathbb{Z}_p[\![G]\!]$ of a Galois group G of a p-adic Lie extension of $\mathbb{Q}_p$. $\mathbb{T}$ is a Λ-linear representation of $G_{\mathbb{Q}_p}$, often an induction from a G_K-representation with $K/\mathbb{Q}_p$ finite. There are several results which establish the existence of ε-isomorphisms for some choices of G and

$\mathbb{T}$. One of the first results was Benois' and Berger's proof of the existence of ε-isomorphisms for $\mathbb{T}$ being any crystalline representation and $G = \mathrm{Gal}(F/K)$ with $K/\mathbb{Q}_p$ finite unramified and $F \subset K_\infty = \cup_{n=1}^{\infty} K(\zeta_{p^n})$ a finite subextension ([BB05a]). Venjakob proved in [Ven13] the equivariant ε-isomorphism conjecture for $\mathbb{T}$ being the Iwasawa deformation of $\mathbb{Z}_p(\eta)(r)$, where η is an unramified character, K any unramified extension of $\mathbb{Q}_p$ and $G = \mathrm{Gal}(K_\infty/K)$. Together with Izychev ([IV16]) he later proved the case where $\mathbb{T}$ is a p-adic Tate module of a one-dimensional Lubin-Tate group defined over $\mathbb{Z}_p$ and $G = \mathrm{Gal}(F/K)$, where both F and K are finite extensions of $\mathbb{Q}_p$ and F/K is at most tamely ramified. Loeffler, Venjakob and Zerbes [LVZ15] considered the situation of $\mathbb{T}$ being a crystalline representation and G an abelian p-adic, p-torsion free Lie-extension. They also worked with more general coefficients for Λ. This work was extended by Bellovin and Venjakob [BV16] using Wach modules. Bley and Cobbe [BC16] proved the existence of equivariant ε-isomorphisms for $\mathbb{T} = \mathbb{Z}_p(\eta)(1)$, where η is an unramified character, K an unramified extension of $\mathbb{Q}_p$ and $G = \mathrm{Gal}(F/K)$ with F/K abelian, weakly and wildly ramified so that the residue degrees of $K/\mathbb{Q}_p$ and F/K are coprime. Nakamura ([Nak17]) considered ε-isomorphisms for rank one (φ, Γ)-modules over the Robba ring. Our own work considers the situation of L and K being finite extensions of $\mathbb{Q}_p$, T a finitely generated projective $\mathcal{O}_L$-module with continuous G_K-action, G chosen so that $\Lambda = \mathcal{O}_L[\![G]\!]$ is an adic ring and $\chi : G_K \to \mathcal{O}_L^\times$ a continuous character. We show that the existence of the ε-isomorphism for $(\Lambda, \mathbb{T}(T), \xi)$ implies the existence of the ε-isomorphism for $(\Lambda, \mathbb{T}(T(\chi)), \xi)$, where $\mathbb{T}(-)$ is $\mathrm{Ind}_K^{\mathbb{Q}_p}(\Lambda^\natural \otimes_{\mathcal{O}_L} -)$. The thesis thus extends a few of the above results.

Structure of the thesis

In the first chapter, we recall some preliminaries. We treat determinant functors using Deligne's category of virtual objects, collect some key aspects of p-adic Hodge theory and of continuous Galois cohomology, in particular Shapiro's lemma and Tate duality.

The second chapter begins with a review of Deligne's local constants. Next, we elaborate the construction of ε-isomorphisms for de Rham respresentations including the correction factor proposed by Nakamura [Nak17]. Lastly, we state of the equivariant ε-isomorphism conjecture.

The third chapter contains the main result of the thesis. By viewing a twist with a continuous character χ as a base change of the ring Λ, we show that the existence of an ε-isomorphism for a triple $(\Lambda, \mathbb{T}(T), \xi)$ implies the existence of the ε-isomorphism for a twisted triple $(\Lambda, \mathbb{T}(T(\chi)), \xi)$. With little extra effort, our arguments show that the properties in the equivariant ε-isomorphism conjecture are compatible with base change in general.

The last chapter is an outlook. We present some rough ideas on a stronger connection between $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ and $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ regarding the specialisation to ε-isomorphisms of de Rham representations. Here, we assume χ to be finite and unramified.

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Chapter 1

Preliminaries

1.1 Determinant functors

In this section, we will describe a method of defining a "determinant" for isomorphisms in general exact categories. For our purposes we will need a determinant functor for the exact category $\mathrm{PMod}(\Lambda)$ of finitely generated projective modules over a not necessarily commutative (but always unital) ring Λ. The first work on determinant functors was Knudsen's and Mumford's [KM77], which Knudsen generalised in [Knu02]. We follow mainly the approach taken by Deligne in [Del87]. There is also more modern work on deteminant functors on triangulated categories by Breuning [Bre11] and on Waldhausen categories by Muro, Tonks and Witte [MTW15], whose generality we won't need in our work.
In the prototypical setting of determinants of L-vector space isomorphisms, the determinant is an element of $L^\times$, which is just $K_1(L)$. It will be this intuition that carries over to general exact categories.

1.1.1 K-theory

We now recall properties of the K_0 and K_1 group of a ring following the explicit descriptions in section 1.1 of [FK06] and touch on general K-theory of exact categories by mentioning some results from [Wei13].

Definition 1.1.1. *The group $K_0(\Lambda)$, or* Grothendieck group*, is the (additively written) abelian group with generators $[P]$ for P an object of $\mathrm{PMod}(\Lambda)$ and relations $[P \oplus Q] = [P] + [Q]$ as well as $[P] = [Q]$ for isomorphic Λ-modules P and Q.*

Definition 1.1.2. *The group $K_1(\Lambda)$, or* Whitehead group*, is the (multiplicatively written) abelian group generated by elements $[P, \alpha]$, where P is an object of $\mathrm{PMod}(\Lambda)$ and α an automorphism of P, subject to the following relations:*

(1) If $[P, \alpha]$ and $[Q, \beta]$ are elements of $K_1(\Lambda)$ and $\phi : P \to Q$ a Λ-isomorphism such that the diagram

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\phi"'] & P \arrow[d, "\phi"] \\ Q \arrow[r, "\beta"] & Q \end{tikzcd}$$

commutes, then $[P, \alpha] = [Q, \beta]$.

(2) $[P, \alpha\beta] = [P, \alpha] \cdot [P, \beta]$

(3) $[P \oplus Q, \alpha \oplus \beta] = [P, \alpha] \cdot [Q, \beta]$.

The following theorem gives an alternative definition of $K_1(\Lambda)$. It is more prominent in the literature and provides a good intuition for $K_1(\Lambda)$:

Theorem 1.1.3. *Let* $\mathrm{GL}(\Lambda) := \mathrm{colim}_n \mathrm{GL}_n(\Lambda)$*, where the transition maps are given by*

$$\mathrm{GL}_n(\Lambda) \to \mathrm{GL}_{n+1}(\Lambda), g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the map

$$\omega : \mathrm{GL}(\Lambda)/[\mathrm{GL}(\Lambda), \mathrm{GL}(\Lambda)] \to K_1(\Lambda), g \mapsto [\Lambda^n, g]$$

is an isomorphism of abelian groups.

Proof. ω is well-defined since $K_1(\Lambda)$ is commutative and because by the third relation in definition 1.1.2 we have

$$\left[\Lambda^{n+1}, \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] = [\Lambda^n \oplus \Lambda, g \oplus \mathrm{id}_\Lambda] = [\Lambda, g] \cdot [\Lambda, \mathrm{id}_\Lambda]$$

and $[P, \mathrm{id}_P]$ is the neutral element in $K_1(\Lambda)$ for any P in $\mathrm{PMod}(\Lambda)$. It is a homomorphism by relation (2) of definition 1.1.2. By choosing a complement $P \oplus Q \cong \Lambda^n$ for a finitely generated projective Λ-module P, one sees that $[P, \alpha] = [\Lambda^n, \alpha \oplus \mathrm{id}_Q]$, so ω is surjective. The remaining injectivity is intuitive, since in $\mathrm{GL}(\Lambda)/[\mathrm{GL}(\Lambda), \mathrm{GL}(\Lambda)]$ all the relations of $K_1(\Lambda)$ hold. A formal proof is given in [Ros94] theorem 3.1.7. combined with Whitehead's lemma, see proposition 2.1.4 in [Ros94]. $\square$

We have a determinant map on $K_1(\Lambda)$ if Λ is commutative:

Lemma 1.1.4. *Let* Λ *be commutative. Then the determinant homomorphisms* $\det : \mathrm{GL}_n(\Lambda) \to \Lambda^\times$ *induce a map* $\det : K_1(\Lambda) \to \Lambda^\times$ *with section* $\Lambda^\times \to K_1(\Lambda), \lambda \mapsto [\Lambda, \lambda]$.

Proof. The determinant functors $\det : \mathrm{GL}_n(\Lambda) \to \Lambda^\times$ are compatible with the inclusions $\mathrm{GL}_n(\Lambda) \hookrightarrow \mathrm{GL}_{n+1}(\Lambda)$ and $\Lambda^\times$ is commutative by assumption. $\square$

If Λ is also semi-local, det is an isomorphism:

Definition 1.1.5. *Let* Λ *be a (not necessarily commutative) ring. Its* Jacobson radical $J(\Lambda)$ *is the two sided ideal* $\{\lambda \in \Lambda | 1 + \Lambda\lambda\Lambda \subset \Lambda^\times\}$.
The ring Λ *is called* semi-local *if* $\Lambda/J(\Lambda)$ *is a left-semisimple ring, i.e. if every left-submodule of* $\Lambda/J(\Lambda)$ *is a direct summand.*

Remark 1.1.6. Any left-semisimple ring is also right-semisimple by corollary 3.7 in [Lam01]. Therefore, it makes sense to simply speak of semi-local rings.

We give some examples of semi-local rings that are important in this work.

Example 1.1.7.
(1) Every field is semi-local.
(2) Every local ring Λ is semi-local, since $J(\Lambda)$ is the maximal ideal and thus $\Lambda/J(\Lambda)$ is a field.
(3) (Left)-Artinian rings are semi-local (see theorem 4.14 in [Lam01]).

(4) If L/k and K/k are field extensions with K/k finite. Then $L \otimes_k K$ is semi-local. In fact, $L \otimes_k K$ is Artinian as finite ring extension of the Artinian ring L.
(5) Every adic ring is semi-local (see definition 1.3.9 and lemma 1.3.11).

Lemma 1.1.8. *Let Λ be a semi-local ring. Then the homomorphism $\Lambda^\times \to K_1(\Lambda), \lambda \mapsto [\Lambda, \lambda]$ is surjective. If Λ is commutative, the map is an isomorphism and hence so is* det.

Proof. This is lemma 1.4 in chapter III of [Wei13] □

Remark 1.1.9. By the above lemma 1.1.8 the determinant can be seen as a map to $K_1(\Lambda)$ in the case of semi-local commutative rings. Using $K_1(\Lambda)$ as the target for a determinant is further motivated by exercise 1.2 in chapter III of [Wei13] where a unique group homomorphism $\det : \mathrm{GL}(\Lambda) \to K_1(\Lambda)$, which satisfies some key properties of the commutative determinant, is constructed for semi-local non-commutative rings.

Lemma 1.1.10. *Let Λ and Λ' be two rings and Y a finitely generated projective Λ'-module with a commuting right action of Λ. Then there are base change morphisms $K_0(\Lambda) \to K_0(\Lambda')$ and $K_1(\Lambda) \to K_1(\Lambda')$ induced by the functor $Y \otimes_\Lambda - : \mathrm{PMod}(\Lambda) \to \mathrm{PMod}(\Lambda')$.*
In particular, if $Y = \Lambda'$ and the right action of Λ is given by a ring homomorphism $f : \Lambda \to \Lambda'$, we denote these base change homomorphisms $K_0(\Lambda) \to K_0(\Lambda')$ and $K_1(\Lambda) \to K_1(\Lambda')$ both by f^*. Moreover, we have the following commutative diagram:

$$\begin{tikzcd} \Lambda^\times \arrow[r, "f"] \arrow[d] & \Lambda'^\times \arrow[d] \\ K_1(\Lambda) \arrow[r, "f^*"] & K_1(\Lambda') \end{tikzcd}$$

in which the vertical maps are surjective (isomorphisms) if Λ is semi-local (and commutative).

Proof. For a projective Λ-module P the Λ'-module $Y \otimes_\Lambda P$ is projective, since if $P \oplus Q \cong \Lambda^n$, then $(Y \otimes_\Lambda P) \oplus (Y \otimes_\Lambda Q) \cong Y^n$. The functor $Y \otimes_\Lambda -$ is additive and thus induces group homomorphisms

$$K_0(\Lambda) \to K_0(\Lambda'), [P] \mapsto [Y \otimes_\Lambda P] \text{ and } K_1(\Lambda) \to K_1(\Lambda'), [P, \alpha] \mapsto [Y \otimes_\Lambda P, \mathrm{id}_Y \otimes \alpha].$$

If $f : \Lambda \to \Lambda'$ is a ring homomorphism, we get an induced group homomorphism $\mathrm{GL}(f) : \mathrm{GL}(\Lambda) \to \mathrm{GL}(\Lambda')$, which maps each entry of a matrix via f. Let $g \in \mathrm{GL}_n(\Lambda)$ represent an element of $\mathrm{GL}(\Lambda)/[\mathrm{GL}(\Lambda), \mathrm{GL}(\Lambda)]$. One the one hand, $\omega(\mathrm{GL}(f)(g))$ is $[\Lambda'^n, \mathrm{GL}(f)(g)]$. On the other hand, we have $f^*(\omega(g)) = f^*([\Lambda^n, g]) = [\Lambda' \otimes_f \Lambda^n, \mathrm{id}_{\Lambda'} \otimes g]$. The isomorphism $\Lambda'^n \cong \Lambda' \otimes_f \Lambda^n$ shows that both elements are the same in $K_1(\Lambda')$. Thus, the following diagram commutes

$$\begin{tikzcd} \Lambda^\times \arrow[r, "f"] \arrow[d, hook] & \Lambda'^\times \arrow[d, hook] \\ \mathrm{GL}(\Lambda) \arrow[r, "\mathrm{GL}(f)"] \arrow[d, "\omega"] & \mathrm{GL}(\Lambda') \arrow[d, "\omega"'] \\ K_1(\Lambda) \arrow[r, "f^*"] & K_1(\Lambda'). \end{tikzcd}$$

The semi-local case follows directly from lemma 1.1.8. □

Lemma 1.1.11. *Let $A \hookrightarrow B$ and $A \hookrightarrow C$ be extensions of (not necessarily commutative) rings and τ a ring automorphism of B. Let further P be a finitely generated projective C-module and f a $B \otimes_A C$-automorphism of $B \otimes_A P$. Then $(\tau \otimes \mathrm{id}_P) f (\tau^{-1} \otimes \mathrm{id}_P)$ is a $B \otimes_A C$-automorphism of $B \otimes_A P$ and in $K_1(B \otimes_A C)$ we have*

$$\big[B \otimes_A P, (\tau \otimes \mathrm{id}_P) f (\tau^{-1} \otimes \mathrm{id}_P)\big] = (\tau \otimes \mathrm{id}_P)^* \big([B \otimes_A P, f]\big).$$

Proof. The map $(\tau \otimes \mathrm{id}_P) f (\tau^{-1} \otimes \mathrm{id}_P)$ is clearly a C-linear bijection and it is $B \otimes_A C$-linear because f is, so that the action of τ cancels on scalars from B. Choosing a complement of P, we can assume it to be free. Given a C-base $p_1, \dots p_n$ of P we obtain $1 \otimes p_1, \dots 1 \otimes p_n$ as a $B \otimes_A C$-base of $B \otimes_A P$. This base is invariant under $\tau^{-1} \otimes \mathrm{id}_P$. So the matrix representing $(\tau \otimes \mathrm{id}_P) f (\tau^{-1} \otimes \mathrm{id}_P)$ with respect to the above base is just the matrix representing f mapped element-wise via $\tau \otimes \mathrm{id}_P$, i.e. $(\tau \otimes \mathrm{id}_P)^* \big([B \otimes_A P, f]\big)$. □

K-theory is neither only defined for rings nor limited to the groups K_0 and K_1. To illustrate, we cite some properties of Quillen's K-theory for exact categories.

Definition 1.1.12. *An* exact category *is an additive category $\mathcal{E}$ admitting a small skeleton which is a full subcategory of an abelian category $\mathcal{A}$ and closed under extensions. That is, for each short exact sequence $0 \to E' \to E \to E'' \to 0$ in $\mathcal{A}$, where E' and E'' are isomorphic to objects in $\mathcal{E}$, there is an object in $\mathcal{E}$ which is isomorphic to E in $\mathcal{A}$.*
So an exact category comes with a collection of exact sequences, namely those sequences that consists of objects in $\mathcal{E}$ and are short exact sequences in $\mathcal{A}$.
An exact functor *$F : \mathcal{E} \to \mathcal{E}'$ is an additive functor between exact categories $\mathcal{E}$ and $\mathcal{E}'$ which maps short exact sequences to short exact sequences.*

We give some examples of exact categories:

Example 1.1.13.
1. Every small abelian category is an exact category.
2. If Λ is a ring, the category $\mathrm{PMod}(\Lambda)$ of all finitely generated projective Λ-modules is exact as the full subcategory of the abelian category $\mathrm{Mod}(\Lambda)$ of all Λ-modules. Note that in $\mathrm{PMod}(\Lambda)$ every short exact sequence splits. We call such an exact category split exact. $\mathrm{PMod}(\Lambda)$ is not an abelian category in general, since not all cokernels of homomorphisms between finitely generated projective modules are projective.
3. The category of torsion abelian groups is an exact category via the embedding into the category of abelian groups. However, for instance the following sequence does not split

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{pr} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Quillen described in [Qui73] an approach to associate K_i groups to an exact category for $i \in \mathbb{N}_0$.

Theorem 1.1.14. *For every exact category $\mathcal{E}$, there is a topological space $BQ\mathcal{E}$, which is the geometric realisation of an auxiliary category $Q\mathcal{E}$, such that $K_n(\mathcal{E}) \coloneqq \pi_{n+1}(BQ\mathcal{E})$ yields a functor from the category of exact categories and exact functors to the category of abelian groups. Moreover, for $\mathcal{E} = \mathrm{PMod}(\Lambda)$ we have $K_0(\mathrm{PMod}(\Lambda)) = K_0(\Lambda)$ and $K_1(\mathrm{PMod}(\Lambda)) = K_1(\Lambda)$.*

Proof. The construction of the category $Q\mathcal{E}$ is described in chapter IV §6 of [Wei13]. Definition 6.3. and the following remarks establish the functoriality. Finally, corollary 7.2 together with definition 1.1 [ibid.] establish that the conceptual and the explicit construction of K_0 and K_1 of a ring coincide. □

Remark 1.1.15. For every exact category $\mathcal{E}$, one can explicitly describe K_0 as the free group on isomorphism classes of objects of $\mathcal{E}$ modulo the relations given by the short exact sequences. For a ring this reflects the explicit construction of K_0 in definition 1.1.1. One could be inclined to describe $K_1(\mathcal{E})$ explicitly in a similar way as in definition 1.1.2. This however only yields the correct K_1 group only if $\mathcal{E}$ is split exact (see theorem 3.3 in [She82] and §5 in [Ger73] for counter examples of non split exact categories).

Remark 1.1.16. The group homomorphisms in lemma 1.1.10 are the homomorphisms induced by the exact functor $Y \otimes_\Lambda - : \mathrm{PMod}(\Lambda) \to \mathrm{PMod}(\Lambda')$ via the functoriality in theorem 1.1.14.

1.1.2 Picard Categories

The target category of a determinant functor will be a certain Picard category. In this subsection we discuss some concepts of Picard categories in general. We follow the material in [Bre11] 2.1. and 2.3. For more details, see also the material in appendix A of [Knu02]. Picard categories are monoidal categories with additional structure.

Definition 1.1.17. *An* AC tensor category *is a category $\mathcal{C}$ with a covariant bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ subject to coherent associativity and commutativity constraints.*

Remark 1.1.18. In the light of theorem 4.2. of [Mac63] the coherence of the associativity and commutativity constraints is equivalent to the associativity constraint satisfying the hexagon axiom ([Mac63] (3.5)), the commutativity constraint being self-inverse ([Mac63] (4.2)) and both of them satisfying the hexagon axiom ([Mac63] (4.5))

Definition 1.1.19. *A* (commutative) Picard category *is a non-empty AC tensor category $\mathcal{P}$ in which every morphism is an isomorphism and for which the functors $A \otimes -$ are auto-equivalences for every object A of $\mathcal{P}$.*

Remark 1.1.20. Let $\mathcal{P}$ be a Picard category.

(1) A unit in $\mathcal{C}$ is an object $\mathbb{1}$ together with natural isomorphisms $\lambda_A : A \xrightarrow{\sim} \mathbb{1} \otimes A$, satisfying the conditions in [Mac63] (5.2) and (5.3). A morphism of units $f : (\mathbb{1}, \lambda) \to (\mathbb{1}', \lambda')$ is a morphism $f : \mathbb{1} \to \mathbb{1}'$ which satisfies $(f \otimes \mathrm{id}_A) \circ \lambda_A = \lambda'_A$ for all objects A in $\mathcal{P}$. Every Picard category has a unit object and is thus automatically a monoidal category. The unit object is unique up to isomorphism of unit objects ([Riv06] I. 1.3.2.1). This isomorphism is itself unique. For if α and $\beta : \mathbb{1} \to \mathbb{1}'$ are two isomorphisms of units, then we have

$$(\alpha \otimes \mathrm{id}_{\mathbb{1}'}) \circ \lambda_{\mathbb{1}'} = \lambda'_{\mathbb{1}'} = (\beta \otimes \mathrm{id}_{\mathbb{1}'}) \circ \lambda_{\mathbb{1}'}.$$

Cancelling $\lambda'_{\mathbb{1}'}$ and using the faithfulness of $- \otimes \mathrm{id}_{\mathbb{1}'}$, we get $\alpha = \beta$.

(2) Let A be an object of $\mathcal{P}$. An inverse of A is an object B together with an isomorphism $\mu_A : A \otimes B \to \mathbb{1}$. We denote an inverse of A by A^{-1} and call an application of μ a trivialisation of A. Since $A \otimes -$ is an auto-equivalence for every A, every object of a Picard category has an inverse, which is by commutativity a two-sided inverse. As for

units, the full- and faithfulness of $A \otimes -$ shows that inverses are unique up to unique isomorphism of inverses (i.e. isomorphisms consistent with the μ's). It is clear that A is an inverse of A^{-1}. Moreover, inverses commute with the product structure in the following way: For all objects A and B an inverse of $A \otimes B$ is $A^{-1} \otimes B^{-1}$ as well as $B^{-1} \otimes A^{-1}$. Note however, that inverses might not be coherent with the associativity constraints, see [Del87] (4.1.1).

(3) Taking inverses with respect to $- \otimes -$ even is a functor. Given a choice of inverses for every object of $\mathcal{P}$ we need to define $f^{-1} : A^{-1} \to B^{-1}$ for a morphism $f : A \to B$ in such a way that the diagram

$$\begin{tikzcd} A \otimes A^{-1} \arrow[rr, "f \otimes f^{-1}"] \arrow[dr, "\mu_A"'] & & B \otimes B^{-1} \arrow[dl, "\mu_B"] \\ & \mathbb{1} & \end{tikzcd}$$

commutes. Therefore, we set f^{-1} to be the unique morphism $A^{-1} \to B^{-1}$ which is mapped to $\overline{f \otimes \mathrm{id}_{B^{-1}}} \circ \overline{\mu_B} \circ \mu_A$ under the fully faithful functor $\mathrm{id}_A \otimes -$. Here and later on $\bar{f}$ means the inverse of the morphism f with respect to composition. Writing $f \otimes f^{-1} = f \otimes \mathrm{id}_{B^{-1}} \circ \mathrm{id}_A \otimes f^{-1}$ shows that the above diagram commutes. Clearly, $\mathrm{id}_A^{-1} = \mathrm{id}_{A^{-1}}$ by the faithfulness of $\mathrm{id}_A \otimes -$. Taking inverses also agrees with composition. For this we write $f \otimes f^{-1} = \mathrm{id}_B \otimes f^{-1} \circ f \otimes \mathrm{id}_{A^{-1}}$ to see that $\mathrm{id}_B \otimes f^{-1} = \overline{\mu_B} \circ \mu_A \circ \overline{f} \otimes \mathrm{id}_{A^{-1}}$. So, we get that on the one hand $\mathrm{id}_B \otimes (g^{-1} \circ f^{-1}) = \overline{g} \otimes \mathrm{id}_{C^{-1}} \circ \overline{\mu_C} \circ \mu_A \circ \overline{f} \otimes \mathrm{id}_{A^{-1}}$. On the other hand, we have $\mathrm{id}_C \otimes (g \circ f)^{-1} = \overline{\mu_C} \circ \mu_A \circ \overline{(g \circ f)} \otimes \mathrm{id}_{A^{-1}}$. Since $g \circ \mathrm{id}_B \circ \overline{g} = \mathrm{id}_C$, we get that $\mathrm{id}_C \otimes (g^{-1} \circ f^{-1}) = \mathrm{id}_C \otimes (g \circ f)^{-1}$. Using the faithfulness of $\mathrm{id}_C \otimes -$, we conclude that taking inverses indeed is a functor.

Definition 1.1.21. *For a Picard category $\mathcal{P}$, we define* homotopy groups of $\mathcal{P}$ *as follows: $\pi_0(\mathcal{P})$ is the group with the isomorphism classes of objects of $\mathcal{P}$ as elements and multiplication induced by $- \otimes -$. Furthermore, we set $\pi_1(\mathcal{P})$ to be the group* $\mathrm{Aut}(\mathbb{1})$ *of automorphisms of a fixed unit (general automorphisms, not automorphisms of units).*

Remark 1.1.22. Via the map $\mathrm{Aut}(\mathbb{1}) \to \mathrm{Aut}(A), f \mapsto \overline{\lambda_A} \circ f \otimes \mathrm{id}_A \circ \lambda_A$ and its inverse $\mathrm{Aut}(A) \to \mathrm{Aut}(\mathbb{1}), g \mapsto \mu_A \circ g \otimes \mathrm{id}_{A^{-1}} \circ \overline{\mu_A}$ we can canonically identify the group of automorphisms of any object in $\mathcal{P}$ with that of a unit. So we can canonically view $\pi_1(\mathcal{P})$ as $\mathrm{Aut}(A)$ for any object A.

The "value" of a determinant functor is a morphism in a Picard category. We will now see how we can interpret this value as an element of $\pi_1(\mathcal{P})$ (and later for universal Picard categories as an element of a K_1 group). For this we need the notion of a torseur.

Definition 1.1.23. *Let X be a set and G a group. X is called a G-*torseur *if X is non-empty, and G operates freely and transitively on X.*

Since the torseurs in our work will always be over commutative groups, we do not need to distinguish left and right actions. The following lemma shows in which situations we will consider torseurs.

Lemma 1.1.24.

(1) For all objects A and B of a Picard category $\mathcal{P}$ there are either no morphisms from A to B or $\mathcal{P}(A,B)$ is a $\pi_1(\mathcal{P})$-torseur.

(2) Let $\phi : A \to B$ and $\psi : B \to C$ be two composable morphisms in $\mathcal{P}$ and α an element of $\pi_1(\mathcal{P})$. Then $\alpha(\psi \circ \phi) = (\alpha\psi) \circ \phi = \psi \circ (\alpha\phi)$.

(3) $\pi_0(\mathcal{P})$ and $\pi_1(\mathcal{P})$ are commutative.

Proof.

(1) All morphisms in $\mathcal{P}$ are isomorphisms, so that if $\mathcal{P}(A,B)$ is not empty, it is automatically an $\mathrm{Aut}(A)$-left and an $\mathrm{Aut}(B)$-right torseur. More explicitly, let $\phi : A \to B$ and $\alpha : \mathbb{1} \to \mathbb{1}$. Then the image of ϕ under the action of α is

$$\begin{tikzcd}
A \arrow[r, "\lambda_A"] \arrow[drr, bend right, "\phi"'] & \mathbb{1}\otimes A \arrow[r, "\alpha\otimes\phi"] \arrow[d, "\mathrm{id}\otimes\phi"] & \mathbb{1}\otimes B \arrow[r, "\overline{\lambda_B}"] & B \\
& \mathbb{1}\otimes B \arrow[r, equal] \arrow[dr, "\overline{\lambda_B}"] & \mathbb{1}\otimes B \arrow[u, "\alpha\otimes\mathrm{id}"'] & \\
& & B \arrow[u, "\lambda_B"'] \arrow[uur, bend right, "\alpha\in\mathrm{Aut}(B)"'] &
\end{tikzcd}.$$

The left square commutes because the λ's are natural . The long right arrow is α viewed as an element of $\mathrm{Aut}(B)$ according to remark 1.1.22. So the whole diagram commutes. A similar diagram could be drawn with A at the bottom and α acting as element of $\mathrm{Aut}(A)$.

(2) The discussion above shows that both $(\alpha\psi) \circ \phi$ and $\phi \circ (\alpha\phi)$ are given by $A \xrightarrow{\phi} B \xrightarrow{\alpha\in\mathrm{Aut}(B)} B \xrightarrow{\psi} C$, which, again by the above diagram, is the same as viewing α as automorphism of A (or C) and pre- (post-) composing it with $\psi \circ \phi$, which is $\alpha(\psi \circ \phi)$.

(3) The commutativity of $\pi_0(\mathcal{P})$ is due to the commutativity constraints on $- \otimes -$. For the commutativity of $\pi_1(\mathcal{P}) = \mathrm{Aut}(\mathbb{1})$ we recall the fact ((5.2) in [Mac63]) that $\lambda_{\mathbb{1}} = c(\mathbb{1},\mathbb{1}) \circ \lambda_{\mathbb{1}}$, where $c(\mathbb{1},\mathbb{1})$ is the commutativity constraint $\mathbb{1}\otimes\mathbb{1} \to \mathbb{1}\otimes\mathbb{1}$ (since we are working in a Picard category, $\lambda_{\mathbb{1}}$ is an isomorphism and hence $c(\mathbb{1},\mathbb{1}) = \mathrm{id}_{\mathbb{1}\otimes\mathbb{1}}$). For α and β in $\mathrm{Aut}(\mathbb{1})$ we thus get together with the above diagram for $A = \mathbb{1} = B$:

$$\alpha \circ \beta = \overline{\lambda_{\mathbb{1}}} \circ \alpha \otimes \beta \circ \lambda_{\mathbb{1}} = \overline{\lambda_{\mathbb{1}}} \circ \beta \otimes \alpha \circ \lambda_{\mathbb{1}} = \beta \circ \alpha.$$

□

We will need the following lemma later:

Lemma 1.1.25. *Let X be a G-torseur and $G \to G'$ a group homomorphism. Then the quotient*

$$X' := X \overset{G}{\times} G' := X \times G'/(x * g, g') \sim (x, gg')$$

is a G'-torseur.

Proof. Clearly, X' is not empty. We define a G'-action on X' via $(x, g_1)g_2 := (x, g_1g_2)$. Let (x, g_1) represent an element of X' on which $g_2 \in G'$ acts as the identity. Since G acts freely on X, the equation $(x, g_1) = (x, g_1g_2)$ also holds in $X \times G'$. Therefore, $g_2 = e$ and G' acts freely on X'. Let (x_1, g_1) and (x_2, g_2) represent two elements of X'. Since G acts transitively on X, there is an element $g \in G$ such that $x_1 = x_2g$. But then $(x_2, g_2)g_2^{-1}gg_1$ represents (x_1, g_1) and G' acts transitively on X'. □

Now we discuss some subtleties concerning the commutativity constraints.

Definition 1.1.26. *We write $c(A,B)$ for the commutativity constraint $A \otimes B \to B \otimes A$. Usually, we will omit it from the notation, but there are subtleties involved if $A = B$ since $c(A,A)$ does not need to be the identity. We will sometimes refer to it as the* commutativity constraint *of A. We write $\varepsilon(A)$ for the self-inverse element of $\pi_1(\mathcal{P})$ corresponding to $c(A,A) \in \mathrm{Aut}_{\mathcal{P}}(A \otimes A)$.*

Remark 1.1.27. Keeping track of the $\varepsilon(B)$'s, we have the following identities in a Picard category (see [Ven05b] Remark 1.2).

(1) The inverse $f^{-1} : A^{-1} \to B^{-1}$ of a morphism $f : A \to B$ can be written as

$$\varepsilon(B) f^{-1} = \overline{\lambda_{B^{-1}}} \circ c(B^{-1}, \mathbb{1}) \circ \mathrm{id}_{B^{-1}} \cdot \mu_A \circ \overline{\mathrm{id}_{B^{-1}} \cdot f \cdot \mathrm{id}_{A^{-1}}} \circ \overline{\mu_{B^{-1}} \cdot \mathrm{id}_{A^{-1}}} \circ \lambda_{A^{-1}}.$$

Here and in the following we will often write "$\cdot$" instead of "$\otimes$" for the product structure in a Picard category or omit it entirely. This should not cause confusion with the $\pi_1(\mathcal{P})$-torseur structure of the morphisms.

(2) Moreover, we can write a composition $A \xrightarrow{f} B \xrightarrow{g} C$ as

$$\varepsilon(B) \cdot \overline{\lambda_C} \circ \mu_{B^{-1}} \cdot \mathrm{id}_C \circ \mathrm{id}_{B^{-1}} \cdot c(B,C) \circ \mathrm{id}_{B^{-1}} \cdot g \cdot f \circ \overline{\mu_{B^{-1}}} \cdot \mathrm{id}_A \circ \lambda_A.$$

Since all morphisms in a Picard category are isomorphisms, we have $\varepsilon(B) = \varepsilon(C)$ for any morphism $g : B \to C$. So if we write the composition of three morphisms as product of those morphisms, the factors $\varepsilon(B)$ and $\varepsilon(C)$ cancel, since they are self-inverse.

In the next section, we introduce the category of virtual objects, which will be the only Picard category of interest in this thesis. In this setting $\varepsilon(B)$ gives rise to signs, see remark 1.1.44.

We saw in remark 1.1.20 that Picard categories are monoidal categories. Subsequently, we will also need the notion of monoidal functors, which we indroduce following [Bre11] 2.2.

Definition 1.1.28. *Let $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \boxtimes)$ be two monoidal categories with commutativity constraints. A* monoidal functor *$(F, m) : (\mathcal{C}, \otimes) \to (\mathcal{D}, \boxtimes)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with natural isomorphisms $m_{A,B} : F(A) \boxtimes F(B) \to F(A \otimes B)$, which respects the associativity and commutativity constraints ([Riv06] I 4.2.1 and 4.2.2) and which maps each unit of $(\mathcal{C}, \otimes)$ to a unit of $(\mathcal{D}, \boxtimes)$.*
Let $(F, m), (G, n) : (\mathcal{C}, \otimes) \to (\mathcal{D}, \boxtimes)$ be two monoidal functors. A morphism of monoidal functors $\mu : (F, m) \to (G, n)$ is a natural transformation $t : F \to G$ such that for all objects A, B of $\mathcal{C}$ we have $t_{A \otimes B} \circ m_{A,B} = n_{A,B} \circ (t_A \boxtimes t_B)$.
The category of monoidal functors from $(\mathcal{C}, \otimes)$ to $(\mathcal{D}, \boxtimes)$ is denoted $\mathrm{Hom}^{\otimes}(\mathcal{C}, \mathcal{D})$.

Remark 1.1.29. If $(F, m) : (\mathcal{A}, \otimes) \to (\mathcal{B}, \boxtimes)$ and $(G, n) : (\mathcal{B}, \boxtimes) \to (\mathcal{C}, *)$ are monoidal functors, then the composition $(G \circ F, Gm \circ n_{F-,F-}) : (\mathcal{A}, \otimes) \to (\mathcal{C}, *)$ is again a monoidal functor.

Lemma 1.1.30. *Let $(F, m) : \mathcal{P} \to \mathcal{Q}$ be a monoidal functor between Picard categories. Then F commutes with units and inverses up to unique isomorphisms of units and inverses.*

Proof. A monoidal functor is defined to map units to units and these are unique up to unique isomorphism of units by remark 1.1.20. Let A be an object of $\mathcal{P}$. Then the isomorphism

$$\mu'_{F(A)} : F(A) \otimes_{\mathcal{Q}} F(A^{-1}) \xrightarrow{m_{A,A^{-1}}} F(A \otimes_{\mathcal{P}} A^{-1}) \xrightarrow{F(\mu_A)} F(\mathbb{1}) \to \mathbb{1},$$

where the last morphism is the commutativity of F with units, exhibits $F(A^{-1})$ as an inverse of $F(A)$. Thus the uniqueness of inverses in remark 1.1.20 proves the claim. □

Example 1.1.31. By remark 1.1.20 (2) and (3) taking inverses with respect to the product structure of a Picard category is a monoidal functor.

1.1.3 Determinant functors

Now we have introduced the necessary concepts to be able to define a determinant functor. We follow the presentation in [BF01] 2.3 and [Del87]. Let $\mathcal{E}$ be an exact category, let $(\mathcal{E}, is)$ be its subcategory of all objects and all isomorphisms and $\mathcal{P}$ a Picard category.

Definition 1.1.32. *A* determinant functor *from $\mathcal{E}$ to $\mathcal{P}$ consists of the following data:*

- *a functor $d : (\mathcal{E}, is) \to \mathcal{P}$,*
- *an isomorphism $d(\Sigma) : d(B) \to d(A) \otimes d(C)$ for every short exact sequence $\Sigma : A \to B \to C$ in $\mathcal{E}$ which is natural in isomorphisms of exact sequences,*
- *for each zero object 0 in $\mathcal{E}$ an isomorphism $\zeta(0) : d(0) \to \mathbb{1}$ such that for every isomorphism $f : A \to B$ in $\mathcal{E}$, we have*

$$d(f) = d(A) \xrightarrow{d(0\to A\to B)} d(0) \otimes d(B) \xrightarrow{\zeta(0)\otimes \mathrm{id}} \mathbb{1} \otimes d(B) \xrightarrow{\lambda_B^{-1}} d(B) \text{ and}$$
$$d(f^{-1}) = d(B) \xrightarrow{d(A\to B\to 0)} d(A) \otimes d(0) \xrightarrow{\mathrm{id}\otimes\zeta(0)} d(A).$$

which satisfies associativity and commutativity:

Associativity:

Let $\Sigma_4 : B/A \to C/A \to C/B$ be the exact sequence induced by three short exact sequences

$$\Sigma_1 : A \to B \to B/A$$
$$\Sigma_2 : B \to C \to C/B$$
$$\Sigma_3 : A \to C \to C/A$$

then the following diagram commutes in $\mathcal{P}$:

$$\begin{tikzcd} d(C) \arrow[r, "d(\Sigma_2)"] \arrow[d, "d(\Sigma_3)"'] & d(B) \otimes d(C/B) \arrow[d, "d(\Sigma_1)\otimes \mathrm{id}"] \\ d(A) \otimes d(C/A) \arrow[r, "\mathrm{id}\otimes d(\Sigma_4)"] & d(A) \otimes d(B/A) \otimes d(C/B) \end{tikzcd}$$

Commutativity:

If $\Sigma_1 : A \to A \oplus B \to B$ and $\Sigma_2 : B \to A \oplus B \to A$ are exact sequences given by inclusion and projection, the following diagram commutes in $\mathcal{P}$:

$$\begin{tikzcd} & d(A \oplus B) \arrow[dl, "d(\Sigma_1)"'] \arrow[dr, "d(\Sigma_2)"] & \\ d(A) \otimes d(B) \arrow[rr, "c\big(d(A), d(B)\big)"] & & d(B) \otimes d(A). \end{tikzcd}$$

Given an exact category $\mathcal{E}$ and a Picard category $\mathcal{P}$, a morphism between two determinant functors $d, d' : \mathcal{E} \to \mathcal{P}$ is a natural transformation $t : d \to d'$ such that for each exact sequence $\Sigma : A \to B \to C$ we have $d'(\Sigma) \circ t_B = (t_A \cdot t_B) \circ d(\Sigma)$. The category of determinant functors of $\mathcal{E}$ with values in $\mathcal{P}$ is denoted $\det(\mathcal{E}, \mathcal{P})$.

[Del87] showed in §4.2-4.5 that there is a universal determinant functor:

Definition 1.1.33. *For an exact category $\mathcal{E}$ we define a category $V(\mathcal{E})$, the* category of virtual objects *as the category with closed loops based at 0 in the topological space $BQ\mathcal{E}$ as objects and homotopy classes of homotopies between such loops as morphisms. The product structure of $V(\mathcal{E})$ is given by composition of loops.*

Theorem 1.1.34. *Let $\mathcal{E}$ be an exact category. Then there exists a universal determinant functor $d : \mathcal{E} \to V(\mathcal{E})$, such that for all Picard categories $\mathcal{P}$ there is an equivalence of categories*

$$\begin{aligned} \mathrm{Hom}^{\otimes}(V(\mathcal{E}), \mathcal{P}) &\to \det(\mathcal{E}, \mathcal{P}) \\ F &\mapsto F \circ d. \end{aligned}$$

Lemma 1.1.35.

(1) Let $\mathcal{E}^{op}$ be the opposite category of an exact category. It is also exact and $V(\mathcal{E}) = V(\mathcal{E}^{op})$.

(2) An exact functor $F : \mathcal{E} \to \mathcal{E}'$ induces a monoidal functor $V(F) : V(\mathcal{E}) \to V(\mathcal{E}')$ such that the following diagram commutes up to isomorphism of determinant functors

$$\begin{tikzcd} (\mathcal{E}, is) \arrow[r, "d"] \arrow[d, "F"'] & V(\mathcal{E}) \arrow[d, "V(F)"] \\ (\mathcal{E}', is) \arrow[r, "d'"] & V(\mathcal{E}'). \end{tikzcd}$$

Similarly, if F is a contravariant exact functor, it induces a contravariant, monoidal functor $V(F) : V(\mathcal{E}) \to V(\mathcal{E}')$ so that for an isomorphism $f : A \to B$ in $\mathcal{E}$, we have a natural isomorphism between $V(F) \circ d$ and $d' \circ F$. For a short exact sequence $\Sigma : 0 \to A \to B \to C \to 0$ in $\mathcal{E}$, the map $\overline{d'(F(\Sigma))}$ is $V(F)(d(\Sigma)) \circ c\big(d'(F(A)), d'(F(C))\big)$.

(3) Let $F : \mathcal{E} \to \mathcal{E}'$ and $G : \mathcal{E}' \to \mathcal{E}''$ be exact functors. Then $V(G \circ F)$ and $V(G) \circ V(F)$ are isomorphic as monoidal functors.

(4) We have $K_i(\mathcal{E}) \xrightarrow{\sim} \pi_i(V(\mathcal{E}))$ for $i \in \{0, 1\}$. The map for $i = 0$ is induced by d. If $\mathcal{E}$ is split exact, the isomorphism for $i = 1$ is given by d and remark 1.1.22. That is to say, that the automorphisms of an object A of $\mathcal{E}$ are mapped like this:

$$\mathrm{Aut}_{\mathcal{E}}(A) \xrightarrow{d} \mathrm{Aut}_{V(\mathcal{E})}(d(A)) \cong \mathrm{Aut}_{V(\mathcal{E})}(\mathbb{1}) = \pi_1(V(\mathcal{E})).$$

In particular, each object of $V(\mathcal{E})$ is isomorphic to an object $d(A) \otimes d(B)^{-1}$ for some objects A and B of $\mathcal{E}$.

(5) Let $K_i(F) : K_i(\mathcal{E}) \to K_i(\mathcal{E}')$ for $i = 0, 1$ be the homomorphisms induced by an exact functor F. The monoidal functor $V(F)$ induces homomorphisms $\pi_i(V(F)) : \pi_i(V(\mathcal{E})) \to \pi_i(V(\mathcal{E}'))$ for $i = 0, 1$. These are identified using the isomorphism above.

Proof. This proof elaborates parts of section 4.11 in [Del87].

(1) The auxiliary categories $Q\mathcal{E}$ and $Q\mathcal{E}^{op}$ are isomorphic by exercise 6.3 in chapter IV of [Wei13]. So $BQ\mathcal{E}$ and $BQ\mathcal{E}^{op}$ are the same topological space.

(2) Since F is exact, $d' \circ F$ is a determinant functor. The universality of d (theorem 1.1.34) yields a monoidal functor $V(F)$ which makes the diagram commute up to isomorphism in $\det(\mathcal{E}, V(\mathcal{E}))$. If $F : \mathcal{E} \to \mathcal{E}'$ is contravariant, we can use the universality of d to obtain a covariant monoidal functor $\widetilde{V(F)}$ so that the outer rectangle of the following diagram commutes up to isomorphism of determinant functors

$$\begin{tikzcd}
(\mathcal{E}, is) \arrow[r, "d"] \arrow[d, "F"] & V(\mathcal{E}) \arrow[d, dotted, "V(F)"] \arrow[dd, bend left, "\widetilde{V(F)}"] \\
(\mathcal{E}', is) \arrow[r, "d'"] \arrow[d, "inv"] & V(\mathcal{E}') \arrow[d, "inv"] \\
({\mathcal{E}'}^{op}, is) \arrow[r, "(d')^{op}"] & V(\mathcal{E}')^{op}.
\end{tikzcd}$$

Here, inv the contravariant functor that inverts the morphisms of a groupoid. A short exact sequence $\Sigma : A \to B \to C$ in $\mathcal{E}$ is mapped to the morphism $c\big(d'(F(A)), d'(F(C))\big) \circ d'(F(\Sigma))$ in $V(\mathcal{E}')^{op}$. This turns the composition of the left and bottom morphisms into a determinant functor and thus induces the existence of the monoidal covariant functor $\widetilde{V(F)}$. The desired covariant, monoidal functor $V(F)$ is defined by post-composition with inv.

(3) By the previous statement, we have that both $V(GF) \circ d$ and $V(G)V(F) \circ d$ are isomorphic as determinant functors to $d \circ GF$. The category equivalence of theorem 1.1.34 shows that $V(GF)$ and $V(G)V(F)$ are isomorphic as monoidal functors.

(4)&(5) The isomorphism stems from the topological construction of $V(\mathcal{E})$, see [Del87] 4.2., from which the description of the isomorphism for $i = 0$ also follows. For the case of $i = 1$ see [BF01] 2.3. which also states the compatibility of the induced morphisms. In particular, by $K_0(\mathcal{E}) \underset{\sim}{\xrightarrow{d}} \pi_0(V(\mathcal{E}))$ every object of $V(\mathcal{E})$ is isomorphic to

$$d\left(\sum_{i=1}^{n}[A_i] - \sum_{j=1}^{m}[B_j]\right) = d\left(\left[\bigoplus_{i=1}^{n} A_i\right]\right) \otimes d\left(\left[\bigoplus_{j=1}^{m} B_j\right]\right)^{-1}$$

for objects A_i and B_j in $\mathcal{E}$.

□

Remark 1.1.36. The construction of the covariant functor associated to a contravariant, exact functor $F : \mathcal{E} \to \mathcal{E}'$ in part (2) of lemma 1.1.35 is slightly different than in [Del87] section 4.11. There, Deligne uses part 1 of lemma 1.1.35 and defines a contravariant functor $(\mathcal{E}, is) \to V(\mathcal{E})$ presumably as

$$(\mathcal{E}, is) \xrightarrow{F} (\mathcal{E}', is) \xrightarrow{inv} ({\mathcal{E}'}^{op}, is) \xrightarrow{d_{{\mathcal{E}'}^{op}}} V({\mathcal{E}'}^{op}) = V(\mathcal{E}').$$

For the two constructions to be the same, it seems that a commutative diagram of the form

$$\begin{tikzcd}
({\mathcal{E}'}^{op}, is) \arrow[r, "d_{{\mathcal{E}'}^{op}}"] \arrow[d, "(d')^{op}"] & V({\mathcal{E}'}^{op}) \arrow[d, equal] \\
V(\mathcal{E}')^{op} \arrow[r, dotted] & V(\mathcal{E}')
\end{tikzcd}$$

is needed The obvious choice for the dotted arrow seems to be inv, but this messes up the co-/contravariance from the upper left to the lower right corner.

Corollary 1.1.37.

(1) For A and B in $\mathrm{Ob}(\mathcal{E})$ we have that $d(A) \cong d(B)$ if and only if $[A] = [B]$ in $K_0(\mathcal{E})$.

(2) Let M and N be objects of $V(\mathcal{E})$. Then there is a morphism from M to N if and only if $M \cong d(P)d(Q)^{-1}$ and $N \cong d(P')d(Q')^{-1}$ with $[P] + [Q'] = [P'] + [Q]$ in $K_0(\mathcal{E})$.

(3) For all objects M and N of $V(\mathcal{E})$ the morphisms from M to N, $V(\mathcal{E})(M, N)$, are either empty or a $K_1(\mathcal{E})$-torseur.

Remark 1.1.38. By the third part of corollary 1.1.37, we have reached our goal of the section. The determinant of an isomorphism is at least non-canonically an element of $K_1(\mathcal{E})$. Each automorphism group of an object in $V(\mathcal{E})$ can be even canonically identified with $K_1(\mathcal{E})$ by sending the identity to $1 \in K_1(\mathcal{E})$. Roughly speaking, one can say that the information of $V(\mathcal{E})$ is in the morphisms rather than in the objects.

Proof.

(1)&(2) By lemma 1.1.35 (2) we have $K_0(\mathcal{E}) \cong \pi_0(V(\mathcal{E}))$ with the isomorphism induced by d. (2) follows from (1) and part (4) of the above lemma 1.1.35.

(3) This is immediate from lemma 1.1.35 (4) and lemma 1.1.24.

□

From now on let Λ be a (not necessarily commutative) ring. We will need determinant functors in this work only for the case where $\mathcal{E} = \mathrm{PMod}(\Lambda)$ is the category of finitely generated projective modules over Λ. For this case Fukaya and Kato ([FK06] §1.2) introduce an ad hoc construction of a universal determinant functor category, which they claim to be equivalent to the category of virtual objects (ibid. 1.2.10). Unfortunately, there are some gaps in their presentation. Therefore, we work with the setting of Deligne's virtual category.

Example 1.1.39. Burns and Flach show in [BF01] 2.5 (see also [Del87] 4.13) that if Λ is a commutative ring and local or semisimple, the category of graded line bundles on $\mathrm{Spec}(\Lambda)$ is equivalent to $V(\Lambda)$ and the determinant functor is $d(P) = \left(\bigwedge_{\Lambda}^{\mathrm{rk}_\Lambda P} P, \mathrm{rk}(P)\right)$. For Λ being a field this shows that we recover the usual definition of a determinant.

In the following, we will collect some properties of the universal determinant functor for $\mathcal{E} = \mathrm{PMod}(\Lambda)$ and in particular see how we can extend determinant functors to the derived category. Most of these properties are mentioned in [FK06] 1.2 and [Ven05b] §1. We set $V(\Lambda) := V(\mathrm{PMod}(\Lambda))$ and denote $d : \mathrm{PMod}(\Lambda) \to V(\Lambda)$ by d_Λ.

Lemma 1.1.40.

(1) We have $d_\Lambda(P \oplus Q) \cong d_\Lambda(P) \otimes d_\Lambda(Q)$ naturally in objects P, Q of $\mathrm{PMod}(\Lambda)$.

(2) Let Λ' be another ring, Y a finitely generated projective Λ'-module carrying a Λ-right-module structure which commutes with the action of Λ'. Then there is a monoidal functor $Y \otimes_\Lambda - : V(\Lambda) \to V(\Lambda)$ such that the following diagram commutes up to isomorphism of determinant functors:

$$\begin{tikzcd}
(\mathrm{PMod}(\Lambda), is) \arrow[r, "d_\Lambda"] \arrow[d, "Y\otimes_\Lambda -"'] & V(\Lambda) \arrow[d, "Y\otimes_\Lambda -"] \\
(\mathrm{PMod}(\Lambda'), is) \arrow[r, "d_{\Lambda'}"] & V(\Lambda').
\end{tikzcd}$$

If $Y = \Lambda'$ and Λ acts via a ring homomorphism on Λ', then we denote $\Lambda' \otimes_\Lambda d_\Lambda(-)$ by $d_\Lambda(-)_{\Lambda'}$.

(3) *The functor $\mathrm{Hom}_\Lambda(-,\Lambda) : \mathrm{PMod}(\Lambda) \to \mathrm{PMod}(\Lambda^\circ)$ induces a contravariant monoidal functor $(-)^* : V(\Lambda) \to V(\Lambda^\circ)$ such that*

$$\begin{tikzcd}
(\mathrm{PMod}(\Lambda), is) \arrow[r, "d_\Lambda"] \arrow[d, "\mathrm{Hom}_\Lambda(-,\Lambda)"'] & V(\Lambda) \arrow[d, "(-)^*"] \\
(\mathrm{PMod}(\Lambda^\circ), is) \arrow[r, "d_{\Lambda^\circ}"] & V(\Lambda^\circ)
\end{tikzcd}$$

commutes up to natural isomorphism and we have for a short exact sequence Σ in $\mathrm{PMod}(\Lambda)$ that $d_{\Lambda^\circ}(\Sigma^*) = \overline{d_\Lambda(\Sigma)^*}$. We sometimes also denote the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$ itself by $(-)^*$.

Proof.

(1) This is just an instance of the naturality of d_Λ on short exact sequences.

(2) The functor $Y \otimes_\Lambda - : \mathrm{PMod}(\Lambda) \to \mathrm{PMod}(\Lambda')$ is exact. So by lemma 1.1.35 (2) it induces a functor monoidal $V(\Lambda) \to V(\Lambda')$, which we also call $Y \otimes_\Lambda -$.

(3) We first show that the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$ is well defined. For $P \in \mathrm{PMod}(\Lambda)$ we equip $\mathrm{Hom}_\Lambda(P,\Lambda)$ with a Λ°-action by letting $\lambda \in \Lambda^\circ$ act from the left on $f \in \mathrm{Hom}_\Lambda(P,\Lambda)$ by right multiplication in Λ, i.e. $(\lambda \cdot f)(p) = f(p) \cdot \lambda$. Since $\mathrm{Hom}_\Lambda(-,\Lambda)$ is an additive functor, it maps finitely generated projective Λ-modules to finitely generated, projective Λ°-modules. So it is well-defined. Finally, it is exact since $\mathrm{Hom}_\Lambda(-,\Lambda)$ is left-exact on the category of all Λ-modules and it is also right-exact because every exact sequence in $\mathrm{PMod}(\Lambda)$ splits and the splitting is preserved under the additive functor $\mathrm{Hom}_\Lambda(-,\Lambda)$. All that remains to do is to invoke the contravariant part of lemma 1.1.35 part (2)

□

The universal determinant functor d_Λ extends to perfect complexes. The key ingredient is the work of Knudsen and Mumford [KM77] Proposition 4 and Theorem 2, where they extend d_Λ in the case of Λ being commutative. The following (which is structered after [BF01] section 2.4) is a slight generalisation for not necessarily commutative rings.

Definition 1.1.41. *Let $C^p(\Lambda)$ be the full subcategory of the category of complexes of Λ-modules, $C(\Lambda)$, consisting of bounded complexes of finitely generated projective Λ-modules. We also call it the* category of strictly perfect complexes. *Its subcategory, where only quasi-isomorphisms are considered will be denoted $(C^p(\Lambda), qis)$. For an object A of $\mathrm{PMod}(\Lambda)$ we denote by $A[r]$ the complex which is A in degree r and zero elsewhere.*
Let $D(\Lambda)$ be the derived category of the category of all Λ-modules. Then we define $D^p(\Lambda)$ as the full subcategory of the derived category $D(\Lambda)$ consisting of those complexes that are quasi-isomorphic to a strictly perfect complex. We call it category of perfect complexes.
We also define the homotopy category $K(\Lambda)$ as the category with complexes of Λ-modules as objects and with homotopy classes of complex morphisms as morphisms. $K^p(\Lambda)$ shall denote the full subcategory generated by strictly perfect complexes.

Proposition 1.1.42. *The functor $d_\Lambda : (\mathrm{PMod}(\Lambda), is) \to V(\Lambda)$ factors as*

$$(\mathrm{PMod}(\Lambda), is) \xrightarrow{A \mapsto A[0]} (C^p(\Lambda), qis) \to (D^p(\Lambda), is) \to V(\Lambda).$$

For every short exact sequence of perfect complexes $\Sigma = \Sigma(f,g) : 0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ there is an isomorphism $d_\Lambda(\Sigma) : d_\Lambda(C) \to d_\Lambda(C') \otimes d_\Lambda(C'')$, which is natural in morphisms of short exact sequences of complexes that consist of quasi-isomorphisms. The following properties hold in addition:

(1) For each acyclic complex C there is an isomorphism $\zeta_C : d_\Lambda(C) \to \mathbb{1}$.

(2) If f (or g) in Σ is a quasi-isomorphism, then

$$\overline{d_\Lambda(f)} = \lambda_{d_\Lambda(C')} \circ c(d_\Lambda(C'), \mathbb{1}) \circ \mathrm{id} \otimes \zeta_{C''} \circ d_\Lambda(\Sigma)$$
$$(\text{or } d_\Lambda(g) = \lambda_{d_\Lambda(C'')} \circ \zeta_{C'} \otimes \mathrm{id} \circ d_\Lambda(\Sigma)).$$

(3) The determinant functor commutes with base change. That is to say that for Λ' another ring, Y a finitely generated projective Λ'-module carrying a Λ right structure that commutes with the Λ'-structure and Σ a short exact sequence of complexes, we have $d_\Lambda(Y \otimes_\Lambda \Sigma) = Y \otimes_\Lambda d_\Lambda(\Sigma)$ and the following diagram commutes (up to isomorphism of determinant functors in the right square):

$$\begin{tikzcd}
(\mathrm{PMod}(\Lambda), is) \arrow[r] \arrow[d, "Y\otimes_\Lambda -"'] & (D^p(\Lambda), is) \arrow[r, "d_\Lambda"] \arrow[d, "Y\otimes^{\mathbb{L}}_\Lambda -"] & V(\Lambda) \arrow[d, "Y\otimes_\Lambda -"] \\
(\mathrm{PMod}(\Lambda'), is) \arrow[r] & (D^p(\Lambda'), is) \arrow[r, "d_{\Lambda'}"] & V(\Lambda').
\end{tikzcd}$$

The corresponding statement holds for the duality functors $\mathrm{Hom}_\Lambda(-,\Lambda)$.

(4) For every commutative nine term diagram of complexes

$$\begin{tikzcd}
C_1' \arrow[r, "f_1"] \arrow[d, "f'"] & C_1 \arrow[r, "g_1"] \arrow[d, "f"] & C_1'' \arrow[d, "f''"] \\
C_2' \arrow[r, "f_2"] \arrow[d, "g'"] & C_2 \arrow[r, "g_2"] \arrow[d, "g"] & C_2'' \arrow[d, "g''"] \\
C_3' \arrow[r, "f_3"] & C_3 \arrow[r, "g_3"] & C_3''
\end{tikzcd}$$

in which all rows and columns are short exact sequences of complexes the following diagram commutes up to a commutativity constraint:

$$\begin{tikzcd}
d_\Lambda(C_2) \arrow[r, "d_\Lambda(\Sigma(f_2,g_2))"] \arrow[d, "d_\Lambda(\Sigma(f,g))"] & d_\Lambda(C_2') \otimes d_\Lambda(C_2'') \arrow[d, "d_\Lambda(\Sigma(f',g'))\otimes d_\Lambda(\Sigma(f'',g''))"] \\
d_\Lambda(C_1) \otimes d_\Lambda(C_3) \arrow[r, "d_\Lambda(\Sigma(f_1,g_1))\otimes d_\Lambda(\Sigma(f_3,g_3))"] & d_\Lambda(C_1')d_\Lambda(C_1'')d_\Lambda(C_3')d_\Lambda(C_3'').
\end{tikzcd}$$

(5) Let C be a perfect complex such that $H^q(C)[0]$ is a perfect complex for all q. Then we have a canonical isomorphism $d_\Lambda(C) \xrightarrow{\sim} \bigotimes_q d_\Lambda(H^q(C))^{(-1)^q}$ which is natural in quasi-isomorphisms.

Proof. This is just a slightly extended version of proposition 2.1 in [BF01]. The key ingredients of the proof are theorems 1 and 2 in [KM77] where the extension of a determinant functor for a commutative Λ is considered. The claim about the duality functors follows from lemma 1.1.40 (3) together with the explicit description of the extension of d_Λ in the following remark 1.1.43. □

We collect some cornerstones of the proof of proposition 1.1.42 which provide some intuition about the extension of d_Λ to the category of perfect complexes in the following remark, whose content can mostly be found in [FK06] 1.2.[3, 8, 9] and the proof of theorem 1 in [KM77]:

Remark 1.1.43.

(1) For C an object of $C^p(\Lambda)$ we have $d_\Lambda(C) = \bigotimes_q d_\Lambda(C^q)^{(-1)^q}$.

(2) Let $\Sigma : 0 \to C' \to C \to C'' \to 0$ be a short exact sequence of complexes in $C^p(\Lambda)$. Denote the degree-wise short exact sequences in $\mathrm{PMod}(\Lambda)$ by $\Sigma_q : 0 \to C'^q \to C^q \to C''^q \to 0$. Then $d_\Lambda(\Sigma) = \bigotimes_q d_\Lambda(\Sigma^q)^{(-1)^q}$.

(3) Let C be an acyclic complex in $C^p(\Lambda)$ and I^q the image of the differential $C^q \to C^{q+1}$. Then we have short exact sequences $\Sigma_q : 0 \to I^q \to C^{q+1} \to I^{q+1} \to 0$ in $\mathrm{PMod}(\Lambda)$ for all q. Starting at the non-zero I^q with highest superscript one sees that all I^q are projective as the kernel of a surjective map between projective modules. They are finitely generated, since they are quotients of finitely generated modules. So we can apply d_Λ to the I^q's. Up to commutativity, the isomorphism ζ_C is given by

$$\zeta_C : d_\Lambda(C) = \bigotimes_q d_\Lambda(C^{q+1})^{(-1)^q} \xrightarrow{\bigotimes_q d_\Lambda(\Sigma^q)^{(-1)^q}} \big(d_\Lambda(I^q) d_\Lambda(I^{q+1})\big)^{(-1)^q} \xrightarrow{\bigotimes_q \mu_{I^q}^{(-1)^q}} \mathbb{1}.$$

(4) Let $\alpha : C \to C'$ be a quasi-isomorphism in $C^p(\Lambda)$. Then the mapping cone of C'' of α is acyclic and we have an exact sequence $\Sigma : 0 \to C' \to C'' \to C[1] \to 0$ given by the canonic inclusion and projection. This yields a map $\psi : d_\Lambda(C')d_\Lambda(C[1]) \xrightarrow[d_\Lambda(\Sigma)]{} d_\Lambda(C'') \xrightarrow[\zeta_{C''}]{} \mathbb{1}$. Then $d_\Lambda(\alpha)$ is given as $\overline{\psi \otimes \mathrm{id}_{d_\Lambda(C)}}$.

(5) In order to extend d_Λ to perfect complexes, note that the category $C(\Lambda)$ satisfies the fourth Grothendieck axiom, AB4, of [Gro57] section 1.5. For such abelian categories Böksted and Neeman prove in the dual of their proposition 2.12 in [BN93] that the functor $K(P) \hookrightarrow K(\Lambda) \to D(\Lambda)$, where $K(P)$ is some subcategory of $K(\Lambda)$, which contains $K^p(\Lambda)$ as full subcategory, is an equivalence of categories. Since $D^p(\Lambda)$ is the essential image of $K^p(\Lambda) \hookrightarrow K(P) \to D(\Lambda)$, we get an equivalence of categories $K^p(\Lambda) \to D^p(\Lambda)$. Finally, we remark that by [KM77] Proposition 2 the determinant functor on $(C^p(\Lambda), qis)$ is insensitive to homotopy, so that it factors over $K^p(\Lambda)$. Hence, we can define d_Λ on $D^p(\Lambda)$ by choosing a quasi-inverse $D^p(\Lambda) \to K^p(\Lambda)$. Any such choice will yield canonically isomorphic determinant functors $d_\Lambda : D^p(\Lambda) \to V(\Lambda)$.

We conclude the section with a warning concerning the signs introduced by the commutativity constraints.

Remark 1.1.44. In section 4.9 of [Del87] Deligne shows that for an object A of an exact category $\mathcal{E}$ the commutativity constraint $c(d(A), d(A))$ corresponds to the symmetry isomorphism $d(A \oplus A \to A \oplus A)$ and that $\varepsilon(d(A))$ corresponds to $d(A \xrightarrow{\cdot(-1)} A) \in \mathrm{Aut}_{V(\mathcal{E})}(A) = K_1(\mathcal{E})$. If $\mathcal{E}$ has a well-defined rank function, we get that $\varepsilon(A) = (-1)^{\mathrm{rk}\, A} \in K_1(\mathcal{E})$. This can be extended to perfect complexes via the Euler-Poincaré characteristic. See also remark B.0.2 in [LVZ15] for more details. Note how this matches up with the commutativity constraint $V \otimes W = (-1)^{\mathrm{rk}\, V \cdot \mathrm{rk}\, W} W \otimes V$ of graded line bundles in example 1.1.39.

1.2 p-adic Hodge theory

We collect some results from p-adic Hodge theory in this section. The main references are [FO18] and [BC09].

1.2.1 Admissible representations

The theory of admissible representations can be found in [BC09] section I.5. Let F be a field and G a group. Let B be an F-algebra which has an F-linear G-action. Assume that the ring $E := B^G$ is a field. Finally, let V be an object in the category $\mathrm{Rep}_F(G)$ of finite dimensional F-vector spaces with an F-linear action of G.

Definition 1.2.1. *In the above setting B is called* (F, G)-regular*, if*

(1) B is a domain,

(2) $B^G = \mathrm{Frac}(B)^G$ and

(3) each $b \in B$ for which there is some $\sigma \in G$ and $f \in F$ with $\sigma(b) = fb$ is a unit in B.

We define a functor $D_B : \mathrm{Rep}_F(G) \to \mathrm{Vec}_E$ with target in the category Vec_E of finite dimensional E-vector spaces by $V \mapsto D_B(V) := (B \otimes_F V)^G$. This functor indeed produces finite dimensional E-vector spaces as the next theorem states the injectivity of the map

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_F V, b \otimes \sum_i b_i \otimes v_i \mapsto \sum_i bb_i \otimes v_i.$$

Definition 1.2.2. *An object of* $\mathrm{Rep}_F(V)$ *is called* B-admissible *if* $\dim_E(D_B(V)) = \dim_F V$*.*

Theorem 1.2.3. *We keep the above notation. In all but (1) we assume B to be (F, G)-regular.*

(1) If B is a domain and $E = \mathrm{Frac}(B)^G$, then the map α_V is injective and in particular, $\dim_E D_B(V) \leq \dim_F V$.

(2) V is B-admissible if and only if α_V is an isomorphism.

(3) We denote by $\mathrm{Rep}_F^B(G)$ the full subcategory of $\mathrm{Rep}_F(G)$ consisting of B-admissible representations. It is closed under (sub)quotients, tensor products and duals. On $\mathrm{Rep}_F^B(G)$ the functor $D_B(-)$ is exact, faithful and compatible with tensor products and duals.

(4) Let L be a finite extension of F. Further, let V, V' be two objects of $\mathrm{Rep}_L(G)$, which are B-admissible as F-representations.

(a) There is a natural isomorphism of $E \otimes_F L$-modules

$$\nu : D_B(V^{*_L}) \to D_B(V)^{*_{E\otimes_F L}}, \sum_i b_i \otimes \phi_i \mapsto \left(\sum_j b'_j \otimes v_j \mapsto \sum_{ij} b_i b'_j \otimes \phi_i(v_j) \right).$$

(b) Assume either that E/F is a Galois extension, L is separable over F and $B_\sigma := B \otimes_{E\cap L,\sigma} L$ is a domain for each F-linear embedding $\sigma : E \cap L \hookrightarrow E$ or assume that $E = F$. Then $D_B(V)$ is a free $E \otimes_F L$-module of rank $\dim_L V$. Moreover, we have a natural $E \otimes_F L$-isomorphism

$$D_B(V) \otimes_{E\otimes_F L} D_B(V') \to D_B(V \otimes_L V')$$
$$\left(\sum_i b_i \otimes v_i \right) \otimes \left(\sum_j b'_j \otimes v'_j \right) \mapsto \sum_{ij} b_i b'_j \otimes v_i \otimes v'_j.$$

In particular, $V \otimes_L V'$ and V^{*_L} are B-admissible as F-representations.

Proof. Parts (1) to (3) can be found in theorem 5.2.1 of [BC09]. We prove the last part.

4.(a) We have a $B \otimes_F L$-homomorphism

$$B \otimes_F \mathrm{Hom}_L(V, L) \to \mathrm{Hom}_{B\otimes_F L}(B \otimes_F V, B \otimes_F L)$$
$$b \otimes \phi \mapsto (b' \otimes v \mapsto b'b \otimes \phi(v)).$$

It is surjective: Choose an element $\psi \in \mathrm{Hom}_{B\otimes_F L}(B\otimes_F V, B\otimes_F L)$ and an L-base $v_1, \dots, v_n$ of V. The $B \otimes_F L$-linear map ψ is defined by $\psi(1 \otimes v_i) = \sum_j b_{ij} \otimes l_j$. Then the element $\sum_{i,j} b_{ij} \otimes l_j v_i^*$ is a preimage of ψ. Since both sides have the same $B \otimes_F L$-rank $\dim_L V$, the map is an isomorphism. It becomes G-equivariant, if we equip $\mathrm{Hom}_{B\otimes_F L}(B \otimes_F V, B \otimes_F L)$ with the G-action $\psi \mapsto \sigma \circ \psi \circ \sigma^{-1}$, where σ acts trivially on the factor L. Restricting this isomorphism to the G-invariants, we get an $E \otimes_F L$-isomorphism

$$D_B(V^{*_L}) \xrightarrow{\sim} \mathrm{Hom}_{B\otimes_F L}(B \otimes_F V, B \otimes_F L)^G.$$

By restricting the source of elements of $\mathrm{Hom}_{B\otimes_F L}(B \otimes_F V, B \otimes_F L)^G$ to G-invariants, we get an $E \otimes_F L$-linear map

$$f : \mathrm{Hom}_{B\otimes_F L}(B \otimes_F V, B \otimes_F L)^G \to \mathrm{Hom}_{E\otimes_F L}(D_B(V), E \otimes_F L).$$

We compose the constructed maps to get the map $D_B(V^{*_L}) \to D_B(V)^{*_{E\otimes_F L}}$ from the statement. To finish the proof that it is an isomorphism, we construct an inverse to f. Since V is B-admissible as F-representation, any E-base $d_1, \dots, d_m$ of $D_B(V)$ is a B-base of $B \otimes_F V$. We define an inverse of f by sending ϕ to the $B \otimes_F L$-linear map given on d_i by $\phi(d_i)$. This map only depends on ϕ and not on the choice of a base. One can easily check that it also is G-equivariant and that we have thus constructed the desired inverse to f.

4.(b) The statement for the tensor products can be proven as in theorem 5.2.1 of [BC09] once it has been established that $D_B(V)$ is a free $E \otimes_F L$-module. If $E = F$, this is clear. Let us consider the case where E/F is Galois, L/F separable and B_σ is a domain for each F-linear embedding $\sigma : E \cap L \hookrightarrow E$. The reasoning is modelled after the proof of VII §2 lemma 1 in [Ven17]. By the Galois theory lemma 1.2.4 below, we know that $A_\sigma := E \otimes_\sigma L$ is a field for each σ and that $E \otimes_F L \cong \prod_\sigma A_\sigma$. So $D_B(V)$ is a free $E \otimes_F L$-module of rank $\dim_L V$ if and only if each $D_B(V) \otimes_{E\otimes_F L} A_\sigma$ is an A_σ-vector space of dimension $\dim_L V$. Consider the A_σ-vector space $D_{B_\sigma}(V) := (B_\sigma \otimes_L V)^G$. It is clearly isomorphic as A_σ-vector space to $D_B(V) \otimes_{E\otimes_F L} A_\sigma$. We have an E-isomorphism $D_B(V) \cong \oplus_\sigma D_B(V) \otimes_{E\otimes_F L} A_\sigma$. Together with the B-admissibility of V as F-representation we get

$$\begin{aligned}\dim_F(V) = \dim_E(D_B(V) = \sum_\sigma \dim_E(D_B(V)) \otimes_{E\otimes_F L} A_\sigma) = \sum_\sigma \dim_E(D_{B_\sigma}(V)) \\ = \sum_\sigma [L : E \cap L] \cdot \dim_{A_\sigma}(D_{B_\sigma}(V)).\end{aligned}$$

By assumption B_σ is a domain and by definition we have $B_\sigma^G = E \otimes_{E\cap L,\sigma} L = A_\sigma$, which is a field. Hence $\mathrm{Frac}(B_\sigma)^G = B_\sigma^G$ and we are allowed to apply part (1) of theorem 1.2.3 with $B = B_\sigma$, $E = A_\sigma$ and $F = L$ and get

$$\dim_{A_\sigma}(D_{B_\sigma}(V)) \leq \dim_L V.$$

Adding these inequalities for the σ's, of which there are $[E \cap L : F]$ many, and multiplying with $[L : E \cap L]$, we get

$$\dim_F(V) = [L : F] \cdot \dim_L V \leq \sum_\sigma [L : E \cap L] \cdot \dim_{A_\sigma}(D_{B_\sigma}(V)).$$

But from above, we know that equality holds. So equality must hold in the inequality for each σ, which finishes the proof.

□

The following lemma is an easy result from Galois theory:

Lemma 1.2.4. *Let F be a field with an algebraic closure $\overline{F}$ and let K/F be a Galois extension and L/F a finite separable subextension inside $\overline{F}$.*

(1) For an F-homomorphism $\sigma : K \cap L \hookrightarrow K$ let $\sigma' : L \hookrightarrow \overline{F}$ be an extension to L. Then, we have an isomorphism

$$K \otimes_{K \cap L, \sigma} L \to K\sigma'(L), k \otimes l \mapsto k\sigma'(l).$$

(2) The F-homomorphism

$$K \otimes_F L \to \prod_\sigma K \otimes_{K \cap L, \sigma} L, k \otimes l \mapsto (k \otimes l)_\sigma,$$

where the product runs over all F-homomorphisms $K \cap L \hookrightarrow K$, is an isomorphism.

(3) The isomorphism above is $\mathrm{Gal}(K/F)$-equivariant. Here, $\tau \in \mathrm{Gal}(K/F)$ sends $(k \otimes l)_\sigma$ of $\prod_\sigma K \otimes_{K \cap L, \sigma} L$ to $(\tau(k) \otimes l)_{\tau\sigma}$. In particular, if K contains L and we let $\tau \in \mathrm{Gal}(K/F)$ act on $\prod_\sigma K$ by $\tau((a_\sigma)_\sigma) = (\tau(a_{\tau^{-1}\sigma}))_\sigma$, the following diagram commutes:

$$\begin{tikzcd}
K \otimes_F L \arrow[r] \arrow[d, "\tau\otimes\mathrm{id}"] & \prod_\sigma K \otimes_{K\cap L,\sigma} L \arrow[r] \arrow[d, "\tau"] & \prod_\sigma K \arrow[d, "\tau"] \\
K \otimes_F L \arrow[r] & \prod_\sigma K \otimes_{K\cap L,\sigma} L \arrow[r] & \prod_\sigma K.
\end{tikzcd}$$

Proof.

(1) This can be proven by elementary Galois theory. A prove using Galois descent can be found in [BCH03] chapter V, §10, theorem 5.

(2) Since K is normal over F, any F-homomorphism $K \cap L \to \overline{F}$ factors over K. In particular, source and target of the morphism in question have the same K-dimension. Now, assume that L is Galois and generated by some element $\alpha \in \overline{F}$ whose minimal polynomial over F is denoted f. Over $K \cap L$ the polynomial f factors as $\prod_\sigma f_\sigma$ with the product running over $\mathrm{Gal}(K \cap L, F)$. The f_σ are irreducible monic polynomials over $K \cap L$ of the same degree and such that applying $\tau \in \mathrm{Gal}(K \cap L, F)$ to the coefficients of f_σ yields $f_{\tau\sigma}$. Let f_e be the factor containing α. Because of $(K \cap L)[X]/(f_\tau) \xrightarrow[\sigma]{\sim} (K \cap L)[X]/(f_{\sigma\tau})$, we have

$$\begin{aligned}
(K \cap L) \otimes_F L &\cong (K \cap L) \otimes_F F[X]/(f) \\
&\cong (K \cap L)[X]/(f) \\
&\cong \prod_{\sigma \in \mathrm{Gal}(K \cap L, F)} (K \cap L)[X]/(f_\sigma) \xrightarrow[\sim]{\prod_\sigma \sigma^{-1}} \prod_\sigma (K \cap L)[X]/(f_e) \cong \prod_\sigma L.
\end{aligned}$$

Tensoring this over $K \cap L$ with K yields exactly the isomorphism $K \otimes_F L \to \prod_\sigma K \otimes_{K \cap L, \sigma} L$. If L is not Galois over F, we consider the normal closure L' of L in $\overline{F}$. Then the diagram

$$\begin{tikzcd}
K \otimes_F L \arrow[r] \arrow[d, hook] & \prod_{\sigma: K\cap L \to K} K \otimes_{K\cap L,\sigma} L \arrow[d] \\
K \otimes_F L' \arrow[r, "\sim"] & \prod_{\sigma' \in \mathrm{Gal}(K\cap L', F)} K \otimes_{K\cap L',\sigma'} L'
\end{tikzcd}$$

commutes. The morphism on the right is given by

$$K \otimes_{K\cap L,\sigma} L \hookrightarrow K \otimes_{K\cap L,\sigma} L' \xrightarrow{\sim} \prod_{\substack{\sigma' s.t. \\ \sigma'|_{K\cap L}=\sigma}} K \otimes_{K\cap L',\sigma'} L'.$$

Its second map is an isomorphism by the above arguments and because L'/F is Galois. We conclude that the upper morphism in the above square is injective and an isomorphism for dimension reasons.

(3) The action of τ on $\prod_\sigma K \otimes_{K\cap L,\sigma} L$ is well defined. Let $k \in K$ and $l \in K \cap L$. Then $(k \otimes l)_\sigma$ is mapped to $(\tau(k) \otimes l)_{\tau\sigma}$ and $(k\sigma(l) \otimes 1)_\sigma$ is mapped to $(\tau(k\sigma(l)) \otimes 1)_{\tau\sigma} = (\tau(k) \otimes l)_{\tau\sigma}$, this proves that the action is well-defined and the map is equivariant.

□

1.2.2 Period rings

Period rings are originally due to Fontaine ([Fon94] Exp II and III). We use the more modern textbooks [FO18] and [BC09] as main reference.
From now on, we fix a prime number p and some separable closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We want to apply the mechanics of admissible representations to Fontaine's period rings.
First, we fix some notation concerning the absolute Galois group of $\mathbb{Q}_p$. For a field K we denote by G_K the absolute Galois group $\mathrm{Gal}(K^{sep}, K)$.

Recall the local Kronecker-Weber theorem, which is a standard result of local class field theory (see for instance [CF10] p.146).

Theorem 1.2.5 (Local Kronecker-Weber theorem)**.** *Every finite abelian extension K of $\mathbb{Q}_p$ is contained in some $\mathbb{Q}_p(\zeta)$ for ζ a root of unity in $\overline{\mathbb{Q}_p}$.*
In particular, the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$ is $\cup_{n\in\mathbb{N}} \mathbb{Q}_p(\zeta_n)$.

We get the following corollary concerning the structure of $\mathrm{Gal}(\mathbb{Q}_p^{ab}, \mathbb{Q}_p) = G_{\mathbb{Q}_p}/\overline{[G_{\mathbb{Q}_p}, G_{\mathbb{Q}_p}]}$.

Corollary 1.2.6. *The restriction homomorphism yields an isomorphism*

$$Gal(\mathbb{Q}_p^{ab}, \mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p), \sigma \mapsto (\sigma|_{\mathbb{Q}_p^{nr}}, \sigma|_{\mathbb{Q}_{p,\infty}}),$$

where $\mathbb{Q}_{p,\infty}$ is the union of all $\mathbb{Q}_p(\zeta_{p^n})$ and $\mathbb{Q}_p^{nr}$ the maximal unramified extension of $\mathbb{Q}_p$.

Proof. By the local Kronecker-Weber theorem, we have $\mathbb{Q}_p^{ab} = \mathbb{Q}_p^{nr}\mathbb{Q}_{p,\infty}$ since $\mathbb{Q}_p^{nr} = \cup_{p\nmid n} \mathbb{Q}_p(\zeta_n)$. The field $\mathbb{Q}_p^{nr}$ is unramified over $\mathbb{Q}_p$ and $\mathbb{Q}_{p,\infty}$ is totally ramified, so they are linearly disjoint and the Galois group splits as in the statement. □

Definition 1.2.7. *The* cyclotomic character *is the continuous homomorphism $\chi_{cycl} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ that maps σ to the element $(\chi_{cycl}(\sigma)_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p$ for which $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{cycl}(\sigma)_n}$. Clearly, it induces*

an isomorphism $\chi_{cycl} : \mathrm{Gal}(\mathbb{Q}_{p,\infty}, \mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Z}_p^\times$.
We write $\mathbb{Z}_p(1)$ *for the rank one* $\mathbb{Z}_p$*-module on which* $G_{\mathbb{Q}_p}$ *acts via* χ_{cycl}*. If* $K/\mathbb{Q}_p$ *is finite,* Λ *is a ring containing* $\mathbb{Z}_p$ *and* T *is a finitely generated projective* Λ*-linear* G_K*-representation, we write* $T(1)$ *for the representation* $T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ *with diagonal action of* G_K*.*

Remark 1.2.8. Let $K/\mathbb{Q}_p$ be finite, Λ a ring containing $\mathbb{Z}_p$ and T a finitely generated projective Λ-linear G_K-representation. There is a G_K-equivariant Λ-isomorphism

$$T(-1)^* \to T^*(1), \phi \mapsto \phi(- \otimes_{\mathbb{Z}_p} \xi^{-1}) \otimes \xi$$

where ξ is a base of $\mathbb{Z}_p(1)$. We will often use this isomorphism implicitly.

Definition 1.2.9. *We call an element* $\sigma \in G_{\mathbb{Q}_p}$ (arithmetic) Frobeniuslift*, or just* arithmetic Frobenius *by abuse of notation, if* $\sigma|_{\mathbb{Q}_p^{nr}}$ *corresponds to the Frobenius homomorphism of* $\mathbb{F}_p$*, i.e. the map* $x \mapsto x^p$*, under* $\mathrm{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p) \cong G_{\mathbb{F}_p}$*. We denote an arithmetic Frobenius often by* ϕ*. The inverse* ϕ^{-1} *of a arithmetic Frobenius is called* geometric Frobenius *and often denoted* Fr.

Remark 1.2.10. The sequence $0 \to I \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 0$ is exact. So an arithmetic (and hence a geometric) Frobenius is only unique up to I. By corollary 1.2.6 and definition 1.2.7 an arithmetic (geometric) Frobenius is uniquely determined by its value under the cyclotomic character as element in $G_{\mathbb{Q}_p}^{ab}$.

Now we summarise the main properties of Fontaine's period rings.

Proposition 1.2.11. *There is a topological field* B_{dR} *with a continuous action of* $G_{\mathbb{Q}_p}$ *which has the following properties:*

(1) B_{dR} *is the field of fractions of a discrete valuation ring* B_{dR}^+*, from which the* $G_{\mathbb{Q}_p}$*-action is induced.*
(2) There is a canonic $G_{\mathbb{Q}_p}$*-equivariant embedding* $\mathbb{Z}_p(1) \hookrightarrow B_{dR}^+$*. The* $\mathbb{Z}_p$*-bases of* $\mathbb{Z}_p(1)$ *map to uniformisers of* B_{dR}^+*. We will denote such a uniformiser, which is unique up to* $\mathbb{Z}_p^\times$*-multiples, by* t*.*
(3) There is a $G_{\mathbb{Q}_p}$*-stable filtration on* B_{dR}^+ *given by* $t^i B_{dR}^+$*.*
(4) The residue field of B_{dR}^+ *is* $\mathbb{C}_p$*.*
(5) $B_{\mathrm{dR}}^{G_K} = K$ *and* $B_{\mathrm{dR}}^{G_{\widehat{K^{nr}}}} = \widehat{K^{nr}}$ *for* K *a finite extension of* $\mathbb{Q}_p$*.*
(6) B_{dR} *is* $(\mathbb{Q}_p, G_K)$*- and* $(\mathbb{Q}_p, G_{\widehat{K^{nr}}})$*-regular for any finite extension* K *of* $\mathbb{Q}_p$*.*

Proof. The content of this proposition can be found in [BC09] propositions 4.4.6, 4.4.8., page 61, theorem 4.4.13 and example 5.1.3.. $\square$

Remark 1.2.12. Note that the topology of B_{dR}, that we consider, differes from the topology as discretely valuated field, see [FO18] remark 5.14 for more details.

Definition 1.2.13. *Let* K *be a finite extension of* $\mathbb{Q}_p$*. Then we denote by* K_0 *the maximal unramified subextension of* $K/\mathbb{Q}_p$ *and by* $(\widehat{K^{nr}})_0$ *the field* $\widehat{\mathbb{Q}_p^{nr}}$ *(see page 50 of [BC09]).*

Proposition 1.2.14. *There is a* $G_{\mathbb{Q}_p}$*-stable subring* B_{cris} *of* B_{dR} *with the following properties:*

(1) B_{cris} *contains the image of* $\mathbb{Z}_p(1)$*, in particular the element* t*.*
(2) $B_{\mathrm{cris}}^{G_K} = K_0$ *and* $B_{\mathrm{cris}}^{G_{\widehat{K^{nr}}}} = \widehat{\mathbb{Q}_p^{nr}}$ *for* K *a finite extension of* $\mathbb{Q}_p$*.*

(3) There is an injective ring endomorphism φ of B_{cris}, the Frobenius, which acts on $\mathbb{Q}_p^{nr}$ as the arithmetic Frobenius ϕ and on t by multiplication with p. It is $G_{\mathbb{Q}_p}$-equivariant.
(4) B_{cris} is $(\mathbb{Q}_p, G_K)$- and $(\mathbb{Q}_p, G_{\widehat{K^{nr}}})$-regular for any finite extension K of $\mathbb{Q}_p$.

Proof. This theorem is a summary of some of the statements in [BC09] definition 9.1.4, theorem 9.1.5., proposition 9.1.6, theorem 9.1.8 and the remarks prior to it. □

Proposition 1.2.15. *There is a $G_{\mathbb{Q}_p}$-stable subring B_{st} of B_{dR} with the following properties:*
(1) There is an element u of B_{st} such that $B_{\mathrm{st}} = B_{\mathrm{cris}}[u]$.
(2) φ extends to B_{st} via $\varphi(u) = pu$ and remains injective.
(3) There is a B_{cris}-linear, $G_{\mathbb{Q}_p}$-equivariant derivation $N = -\frac{d}{du}$, called the monodromy operator, for which the relation $N\varphi = p\varphi N$ holds.
(4) $B_{\mathrm{st}}^{G_K} = K_0$ and $B_{\mathrm{st}}^{G_{\widehat{K^{nr}}}} = \widehat{\mathbb{Q}_p^{nr}}$ for K a finite extension of $\mathbb{Q}_p$.
(5) B_{st} is $(\mathbb{Q}_p, G_K)$- and $(\mathbb{Q}_p, G_{\widehat{K^{nr}}})$-regular for any finite extension K of $\mathbb{Q}_p$.

Proof. Again we refer to [BC09] for the proof. It can be found in definition 9.2.3, remark 9.2.4, theorem 9.2.10 and proposition 9.2.11. □

Lemma 1.2.16. *Let K be a finite extension of $\mathbb{Q}_p$. Then the maps $K \otimes_{K_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ and $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ are injective.*

Proof. This is proven in [BC09] theorem 9.1.5 and theorem 9.2.10. □

Proposition 1.2.17. *The sequence $0 \to \mathbb{Q}_p \to B_{\mathrm{cris}} \xrightarrow{(1-\varphi,\bar{1})} B_{\mathrm{cris}} \oplus B_{\mathrm{dR}}/B_{dR}^{+} \to 0$ is exact as a sequence of $G_{\mathbb{Q}_p}$-modules.*

Proof. Follows from [BK07] proposition 1.17. equation (1.17.2). □

Definition 1.2.18. *Let K be a finite extension of $\mathbb{Q}_p$ or the completion of the maximal unramified extension of a finite extension of $\mathbb{Q}_p$. We denote the category of finite dimensional continuous $\mathbb{Q}_p$-linear representations of G_K over $\mathbb{Q}_p$ by $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. For L a finite extension of $\mathbb{Q}_p$, we define a subcategory $\mathrm{Rep}_L(G_K)$ of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ by all those representations which are L-vector spaces and carry an L-linear G_K-action.*
We call an object V of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ de Rham *(*semi-stable, crystalline*) if it is B_{dR}- (B_{st}-, B_{cris}-) admissible. It is called* potentially semi-stable*, if it is semi-stable as G_F-representation for some finite extension F of K. We denote the category of such representations $\mathrm{Rep}_{\mathrm{dR}}(G_K)$ ($\mathrm{Rep}_{\mathrm{st}}(G_K)$, $\mathrm{Rep}_{\mathrm{cris}}(G_K)$, $\mathrm{Rep}_{\mathrm{pst}}(G_K)$) and the corresponding functors to the categories of finite dimensional K- (K_0-, K_0-, $\mathbb{Q}_p^{nr}$-) vector spaces by $D_{\mathrm{dR},K}$ ($D_{\mathrm{st},K}$, $D_{\mathrm{cris},K}$, $D_{\mathrm{pst},K} = \bigcup_{\substack{F/K \\ finite}} D_{\mathrm{st},F}$).*

Remark 1.2.19. We overloaded the notation $\mathrm{Rep}_F(G)$. In the following it will always stand for the category of continuous representations. We omit the subscript K in $D_{?,K}$ if $K = \mathbb{Q}_p$.

Remark 1.2.20. Let ? be any of dR, st, cris or pst. Let L be a finite extension of $\mathbb{Q}_p$. We always consider objects in $\mathrm{Rep}_L(G_K)$ as $\mathbb{Q}_p$-vector spaces when we apply $D_?$. Nevertheless, if V is in $\mathrm{Rep}_L(G_K)$ then $D_?(V)$ will have the structure of a $K \otimes_{\mathbb{Q}_p} L$ ($K_0 \otimes_{\mathbb{Q}_p} L$, $K_0 \otimes_{\mathbb{Q}_p} L$ or $\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L$)-module. This structure is well behaved if the conditions in theorem 1.2.3 (4)(b) are satisfied, for instance if B is any of B_{cris}, B_{st} or B_{dR}, the field F is $\mathbb{Q}_p$ and $G = G_{\mathbb{Q}_p}$. By lemma 1.2.16 it is also satisfied if instead G is the absolute Galois group G_K of a finite extension K of L.

Additional structure on $D_?(V)$

The vector spaces $D_?(V)$ for ? any of dR, st, cris or pst come with additional structure inherited form the rings $B_?$. We will discuss some of it in this subsection.

Definition 1.2.21. *Using the $G_{\mathbb{Q}_p}$-stable filtration of B_{dR} we define a filtration on $D_{\mathrm{dR}}(V)$ by $D^i_{\mathrm{dR}}(V) = (t^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$. We call $D_{\mathrm{dR}}(V)/D^0_{\mathrm{dR}}(V)$ the tangent space of V and denote it by $t(V)$. Similar definitions apply for $D_{\mathrm{dR},K}(V)$.*

Definition 1.2.22. *Let V and V' be filtered vector spaces. Then the filtration on their tensor product is given by $\mathrm{Fil}^k(V \otimes V') := \sum_{i+j=k} \mathrm{Fil}^i(V) \otimes \mathrm{Fil}^j(V')$. The filtration on the dual vector space is set to be $\mathrm{Fil}^i(V^*) := (\mathrm{Fil}^{1-i}(V))^{\perp} = \{\psi \in V^* | \mathrm{Fil}^{1-i}(V) \subset \ker(\psi)\}$.*

Definition 1.2.23. *Let K be a field. The category Fil_K is the category with finite dimensional filtered K-vector spaces as objects and K-linear homomorphisms that respect the filtration as morphisms.*

Proposition 1.2.24. *The functor $D_{\mathrm{dR},K} : \mathrm{Rep}_{\mathrm{dR}}(G_K) \to \mathrm{Fil}_K$ sends short exact sequences to short exact sequences and is compatible with tensor products and duals. Moreover, D_{dR} maps $\mathrm{Rep}_L(G_{\mathbb{Q}_p}) \cap \mathrm{Rep}_{\mathrm{dR}}(G_{\mathbb{Q}_p})$ to Fil_L and is compatible with short exact sequences, tensor products and duals of filtered L-vector spaces.*

Proof. The first part is proposition 6.3.3 in [BC09]. The part on L-linear de Rham representations follows immediately since the $D^i_{\mathrm{dR}}(V)$ are L-vector spaces. □

Definition 1.2.25. *A de Rham representation V has* Hodge-Tate weight *r if $D^{r+1}_{\mathrm{dR}}(V) \subsetneq D^r_{\mathrm{dR}}(V)$. The* multiplicity of a Hodge-Tate weight *r is $\dim_{\mathbb{Q}_p} D^r_{\mathrm{dR}}(V)/D^{r+1}_{\mathrm{dR}}(V)$. If V is also in $\mathrm{Rep}_L(G_{\mathbb{Q}_p})$, then we set $h_L(r)_V := \dim_L D^r_{\mathrm{dR}}(V)/D^{r+1}_{\mathrm{dR}}(V)$ and $t_{H,L}(V)$ to be the unique Hodge-Tate weight of $\bigwedge_L^{\dim_L D_{\mathrm{dR}}(V)} D_{\mathrm{dR}}(V)$. We often omit the subscript L.*

By proposition 6.45 of [FO18], we have the following

Lemma 1.2.26. *For an L-linear de Rham representation V we have $t_H(V) = \sum_{i\in\mathbb{Z}} i \cdot h_L(i)_V$.*

Proposition 1.2.27. *Let V be an object of $\mathrm{Rep}_L(G_K)$. The morphisms $\varphi, N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ induce morphisms $\varphi, N : D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K}(V)$ and $\varphi, N : D_{\mathrm{pst},K}(V) \to D_{\mathrm{pst},K}(V)$ and $\varphi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ induces $\varphi : D_{\mathrm{cris},K}(V) \to D_{\mathrm{cris},K}(V)$. The induced map N is $K_0 \otimes_{\mathbb{Q}_p} L$ ($\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L$)-linear and the induced maps φ are $\phi \otimes \mathrm{id}$-semi-linear $K_0 \otimes_{Qp} L$ ($\mathbb{Q}_P^{nr} \otimes_{\mathbb{Q}_p} L$)-isomorphisms. The endomorphism $N : D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K}(V)$ is nilpotent, i.e. $N^k = 0$ for some $k \in \mathbb{N}$. We have $D_{\mathrm{cris},K}(V) = D_{\mathrm{st},K}(V)^{N=0}$.*

Proof. Since φ and N are G_K-equivariant, they induce the corresponding morphisms on $D_{\mathrm{st},K}(V)$, $D_{\mathrm{pst},K}(V)$ and $D_{\mathrm{cris},K}(V)$. The nilpotency of N can be deduced from lemma 8.2.8 in [BC09]. The fact that $D_{\mathrm{cris},K}(V) = D_{\mathrm{st},K}(V)^{N=0}$ follows from $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$. Finally, φ is not only injective but bijective because of [BC09] exercise 7.4.10. □

Lemma 1.2.28. *Let ξ be a basis of $\mathbb{Z}_p(1)$ and V an object in $\mathrm{Rep}_L(G_K)$.*

(1) Let ? be any of dR, st, cris or pst. The assignment $\sum_i b_i \otimes v_i \otimes \xi \mapsto \sum_i b_i t \otimes v_i$ induces an $K \otimes_{\mathbb{Q}_p} L$ ($K_0 \otimes_{\mathbb{Q}_p} L$, $K_0 \otimes_{\mathbb{Q}_p} L$, $\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L$)-linear isomorphism $\vartheta : D_{?,K}(V(1)) \to D_{?,K}(V)$ of (non-linear) $G_{\mathbb{Q}_p}$-representations.

(2) ϑ *sends* $D^i_{\mathrm{dR},K}(V(1))$ *to* $D^{i+1}_{\mathrm{dR},K}(V)$.
(3) *The morphisms* $N \circ \vartheta$ *and* $\vartheta \circ N : D_{\mathrm{st},K}(V(1)) \to D_{\mathrm{st},K}(V)$ *agree. In addition,* $\varphi \circ \vartheta$ *and* $p\vartheta \circ \varphi$ *are the same as morphisms* $D_{\mathrm{st},K}(V(1)) \to D_{\mathrm{st},K}(V)$ *or* $D_{\mathrm{cris},K}(V(1)) \to D_{\mathrm{cris},K}(V)$. *In particular, the diagram*

$$\begin{array}{ccc} D_{\mathrm{st},K}(V(1)) & \xrightarrow{\vartheta\circ N} & D_{\mathrm{st},K}(V) \\ \varphi\downarrow & & \downarrow\varphi \\ D_{\mathrm{st},K}(V(1)) & \xrightarrow{\vartheta\circ N} & D_{\mathrm{st},K}(V) \end{array}$$

commutes. Similar statements hold for $D_{\mathrm{pst},K}$.
(4) *There is a short exact sequence of* $(K_0 \otimes_{\mathbb{Q}_p} L, \varphi, G_K)$*-modules:*

$$0 \to D_{\mathrm{cris},K}(V(1)) \to D_{\mathrm{st},K}(V(1)) \xrightarrow{N\circ\vartheta} D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K}(V)/ND_{\mathrm{st},K}(V) \to 0.$$

Proof. The content of the third part can be found in [FPR94] I. 2.1.8 or [Fon94] Exp VIII 2.2.5. The action of G_K on both t and ξ is by the cyclotomic character χ_{cycl}, which shows the existence of ϑ and its $G_{\mathbb{Q}_p}$-equivariance. The shift in the filtration is clear. By proposition 1.2.14 (2), we have $\varphi(t) = pt$ and $N(tb_i) = tN(b_i)$ since t lies in B_{cris} (1.2.15 (3)). This implies the relations of ϑ with φ and N. The third part is due to the relation $N\varphi = p\varphi N$ (1.2.15 (3)). The short exact sequence follows immediately. □

Corollary 1.2.29. *Let* V *be in* $\mathrm{Rep}_L(G_{\mathbb{Q}_p}) \cap \mathrm{Rep}_{\mathrm{dR}}(G_{\mathbb{Q}_p})$. *Then the pairing*

$$D_{\mathrm{dR}}(V) \otimes_L D_{\mathrm{dR}}(V^*(1)) \to D_{\mathrm{dR}}(L(1)) = L$$

is perfect and a homomorphism of filtered L*-vector spaces, where* L *has the unique filtration jump at* -1. *This induces a natural isomorphism* $\psi_{\mathrm{dR},V} : D^0_{\mathrm{dR}}(V) \to t(V^*(1))^*$ *and an isomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^0_{\mathrm{dR}}(V) & \longrightarrow & D_{\mathrm{dR}}(V) & \longrightarrow & t(V) & \longrightarrow & 0 \\ & & \Big\downarrow \psi_{\mathrm{dR},V} & & \downarrow\vartheta & & \Big\downarrow \psi^*_{\mathrm{dR},V^*(1)} & & \\ & & & & D_{\mathrm{dR}}(V(-1)) & & & & \\ & & & & \downarrow\nu & & & & \\ 0 & \longrightarrow & t(V^*(1))^* & \longrightarrow & D_{\mathrm{dR}}(V^*(1))^* & \longrightarrow & D^0_{\mathrm{dR}}(V^*(1))^* & \longrightarrow & 0, \end{array}$$

where $\nu : D_{\mathrm{dR}}(V^*(1)^*) \to D_{\mathrm{dR}}(V^*(1))^*$ *is the compatibility of* D_{dR} *with* L*-duals (see theorem 1.2.3 and remark 1.2.20).*

Proof. The isomorphism $\nu : D_{\mathrm{dR}}(V^*) \to D_{\mathrm{dR}}(V)^*$ of filtered L-vector spaces shows that the pairing

$$D_{\mathrm{dR}}(V) \otimes_L D_{\mathrm{dR}}(V^*) \to D_{\mathrm{dR}}(L) = L$$

is perfect (here the filtration jump on L is at zero). Since D_{dR} is compatible with tensor products in Fil_L by proposition 1.2.24 and $V \otimes_L V^* \to L$ is a morphism of L-vector spaces, the above pairing is a homomorphism of filtered L-vector spaces. Applying ϑ^{-1} to $D_{\mathrm{dR}}(V^*)$ and $D_{\mathrm{dR}}(L)$ shows that the pairing in the corollary is perfect and a homomorphism of filtered L-vector spaces. Since the filtration on $D_{\mathrm{dR}}(L(1))$ jumps at -1 from L to 0, the space $D_{\mathrm{dR},0}(V^*(1))$

annihilates $D_{\mathrm{dR},0}(V)$. In other words $D_{\mathrm{dR},0}(V^*(1))$ maps into $\mathrm{Fil}^1(D_{\mathrm{dR}}(V)^*)$ under the isomorphism $D_{\mathrm{dR}}(V^*(1)) \cong D_{\mathrm{dR}}(V)^*$ induced by the perfect pairing. But $\mathrm{Fil}^1(D_{\mathrm{dR}}(V)^*)$ has the same dimension as $D_{\mathrm{dR},0}(V^*(1))$ by the isomorphisms ϑ and ν. The pairing is perfect, so that $D_{\mathrm{dR},0}(V^*(1))$ is precisely the annihilator of $D_{\mathrm{dR},0}(V)$. We get the desired isomorphism

$$\psi_{\mathrm{dR},V} : D_{\mathrm{dR},0}(V) \cong D_{\mathrm{dR}}(V^*(1))^*/D_{\mathrm{dR},0}(V^*(1))^* = t(V^*(1))^*.$$

The naturality is clear. The isomorphism of short exact sequences follows by construction of $\psi_{\mathrm{dR},V}$ and the commutativity of

$$\begin{array}{ccc} D_{\mathrm{dR}}(V) & \xrightarrow{\nu^*} & D_{\mathrm{dR}}(V^*)^* \\ \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\vartheta^*} \\ D_{\mathrm{dR}}(V(-1)) = D_{\mathrm{dR}}(V^*(1)^*) & \xrightarrow{\nu} & D_{\mathrm{dR}}(V^*(1))^*. \end{array}$$

□

Relation between crystalline, semi-stable and de Rham representations

The relation of the period rings implies relations between crystalline, semi-stable and de Rham representations. Before we state it, we use Galois descent and complete unramified descent to see that being de Rham, semi-stable or crystalline is insensitive to (complete) unramified extensions and being de Rham is also insensitive to finite extensions:

Proposition 1.2.30. *Let K'/K be an extension inside $\mathbb{C}_p/\mathbb{Q}_p$, where K and K' are finite over $\mathbb{Q}_p$ or the completion of a maximal unramified extension of a finite extension of $\mathbb{Q}_p$ (individually, so that K could be finite but K' infinite over $\mathbb{Q}_p$). Moreover, let V be in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.*

(1) The map $K' \otimes_K D_{\mathrm{dR},K}(V) \to D_{\mathrm{dR},K'}(V)$ is a $G_{\mathbb{Q}_p}$-equivariant isomorphism in $\mathrm{Fil}_{K'}$. In particular, V is de Rham as G_K-representation if and only if it is de Rham as $G_{K'}$-representation.

(2) If $K' = \widehat{K^{nr}}$ or $K'/\mathbb{Q}_p$ is finite and unramified, the map $K'_0 \otimes_{K_0} D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K'}(V)$ is a K'_0-linear isomorphism of $(\varphi, N, G_{\mathbb{Q}_p})$-modules. In particular, V is G_K semi-stable if and only if it is $G_{K'}$-semi-stable. The same holds for the crystalline case.

(3) If K'/K is finite and V is G_K-semi-stable, then the map $K'_0 \otimes_{K_0} D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K'}(V)$ is an isomorphism of K'_0-linear $(\varphi, N, G_{\mathbb{Q}_p})$-modules. The same holds for the crystalline case.

Note that the $G_{\mathbb{Q}_p}$- and φ-actions are diagonally on the left hand sides of the isomorphisms.

Proof. The first two assertions are mostly propositions [BC09] 6.3.8. and [BC09] 9.3.1. with additional attention paid to the Galois-actions. The finite part of the second assertion follows by Galois descent. We prove the third assertion, parts of which can be found in [Füt18] remark 2.23. The map $K'_0 \otimes_{K_0} D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K'}(V)$ is injective as restriction of the map α in theorem 1.2.3. But by assumption, the left hand side has K'_0-dimension $\dim_{\mathbb{Q}_p}(V)$, which is an upper bound on the dimension of the right hand side by theorem 1.2.3, so it is an isomorphism and visibly compatible with the additional structure. The statements for the crystalline case follow by taking the kernel of N, which is K'-linear, since $K' \subset B_{\mathrm{cris}}$. □

Proposition 1.2.31. *Every crystalline representation is semi-stable. Every (potentially) semi-stable representation is de Rham. Every de Rham representation is potentially semi-stable.*

Proof. All but the last part is clear by $B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}$, proposition 1.2.30, theorem 1.2.3 and lemma 1.2.16. The last part is a theorem by Berger ([Ber02] theorem 0.7). $\square$

Corollary 1.2.32. *Let V be in $\mathrm{Rep}_{\mathrm{dR}}(G_{\mathbb{Q}_p})$. There is a $\mathbb{Q}_p^{nr}$-linear isomorphism of $(\varphi, N, G_{\mathbb{Q}_p})$-modules between $\mathbb{Q}_p^{nr} \otimes_{F_0} D_{\mathrm{st},F}(V)$ and $D_{\mathrm{pst}}(V)$ for some F finite over $\mathbb{Q}_p$. In particular, N on $D_{\mathrm{pst}}(V)$ is nilpotent.*

Proof. Since V is de Rham, it is potentially semi-stable. Choose F so that V is G_F-semi-stable and apply proposition 1.2.30 part (3) and the fact that N is $\mathbb{Q}_p^{nr}$-linear and nilpotent on $D_{\mathrm{st},F}(V)$ by proposition 1.2.27. $\square$

$\mathbf{D_{pst}(V)}$ as Weil-Deligne representation

In this subsection, we will see that for a de Rham representation V the pair $(D_{\mathrm{pst}}(V), N)$ is a Weil-Deligne representation. We make this explicit here, since for Weil-Deligne representations one includes a correction term in the ε-factor, which Fukaya and Kato missed in [FK06] as Nakamura pointed out in [Nak17] remark 3.6.
A good reference for Weil-Deligne representations and their relation to ε-factors is Tate's [Tat79].

Definition 1.2.33. *The* Weil group $W_{\mathbb{Q}_p}$ *is the preimage of $\mathbb{Z}$ under the surjection $G_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p^{nr}, \mathbb{Q}_p) = G_{\mathbb{F}_p} \cong \hat{\mathbb{Z}}$. We endow it with the topology that makes the following sequence an exact sequence of topological groups*

$$1 \to I \to W_{\mathbb{Q}_p} \xrightarrow{v} \mathbb{Z} \to 0,$$

where v is defined by $\sigma|_{\mathbb{Q}_p^{nr}} = \phi^{v(\sigma)}$ with ϕ an arithmetic Frobenius and I carries its subspace topology from $I \subset G_{\mathbb{Q}_p}$. Similarly, we can define a Weil group W_K for a finite extension $K/\mathbb{Q}_p$ as a subgroup of $W_{\mathbb{Q}_p}$ with $(W_{\mathbb{Q}_p} : W_K)$ being the residue degree of $K/\mathbb{Q}_p$.

Definition 1.2.34. *Let E be a field of characteristic 0. A* Weil-Deligne representation of W_K *over E is a pair (V, N), where V is a finite dimensional E-vector space V together with an E-linear action of W_K which is continuous with respect to the discrete topology on V. Moreover, N is a nilpotent E-linear endomorphism of V such that $gNg^{-1} = p^{v(g)}N$ for all g in $W_{\mathbb{Q}_p}$.*

Proposition 1.2.35. *The functor $D_{\mathrm{pst},K}$ sends de Rham representations of G_K, for $K/\mathbb{Q}_p$ finite, to Weil-Deligne representations of W_K over $\mathbb{Q}_p^{nr}$ if one linearises the natural action of W_K by setting $g(d) = g\varphi^{-v(g)}(d)$.*

Proof. This is done in [Füt18] construction 3.22. We give the proof here for the convenience of the reader. Since φ acts on $\mathbb{Q}_p^{nr} \subset B_{\text{cris}}$ as the arithmetic Frobenius, the linearised action is indeed $\mathbb{Q}_p^{nr}$-linear. Let V be a de Rham representation. Then by corollary 1.2.32, we have $D_{\mathrm{pst},K}(V) = \mathbb{Q}_p^{nr} \otimes_{F_0} D_{\mathrm{st},F}(V)$ for some F finite over K, which ensures the finite-dimensionality, and that the monodromy operator is nilpotent. The open subgroup $I_K \cap G_K = I_F = G_{F^{nr}} \subset W_K$ acts trivially on $D_{\mathrm{pst},K}(V)$. As a result, the action of W_K is continuous on the discrete module $D_{\mathrm{pst},K}(V)$. The relation of N with elements of W_K follows from the relation of N with φ and the fact that N commutes with the non-linear W_K-action. $\square$

Lemma 1.2.36. *Let L and K be finite extension of $\mathbb{Q}_p$. Let V be an L-linear de Rham representation of G_K. Let A be $\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L$. Then $D_{\mathrm{pst},K}(V)$ is an A-module and W_K acts A-linearly on $D_{\mathrm{pst}}(V)$. Moreover, $D_{\mathrm{pst},K}(V)$ is free of rank $\dim_L V$ as an A-module.*

Proof. $D_{\mathrm{pst},K}(V)$ is clearly an A-module. The W_K-action is automatically L-linear and was linearised to be also $\mathbb{Q}_p^{nr}$-linear (1.2.35). V is potentially semi-stable, so we can choose some finite extension F/K which contains L such that V is G_F-semi-stable. By remark 1.2.20, the $F_0 \otimes_{\mathbb{Q}_p} L$-module $D_{\mathrm{st},F}(V)$ is free of rank $\dim_L V$. By corollary 1.2.32, we have $D_{\mathrm{pst}}(V) \cong \mathbb{Q}_p^{nr} \otimes_{F_0} D_{\mathrm{st},F}(V)$, whence $D_{\mathrm{pst}}(V)$ is a free A-module of rank $\dim_L V$. □

Corollary 1.2.37. *For each $\mathbb{Q}_p$-linear embedding $\sigma : L \hookrightarrow \overline{\mathbb{Q}_p}$ we have a Weil-Deligne representation $\left(\overline{\mathbb{Q}_p} \otimes_{A,\sigma} D_{\mathrm{pst},K}(V), N\right)$ of W_K over $\overline{\mathbb{Q}_p}$, where $A = \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L$ maps to $\overline{\mathbb{Q}_p}$ via $x \otimes y \mapsto x\sigma(y)$. We will denote it by $D_{\mathrm{pst},K}(V)_\sigma$.*

Proof. The finite dimensionality of $\overline{\mathbb{Q}_p} \otimes_{A,\sigma} D_{\mathrm{pst},K}(V)$ is just lemma 1.2.36. The rest follows from proposition 1.2.35. □

D_{st} on de Rham representations

Let ? be any of dR, st or cris. Nothing prevents us from applying $D_{?,K}$ to an arbitrary representations in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. However, the various properties in theorem 1.2.3 hold a prior only if the representation is $B_?$-admissible. In the following we show that de Rham representations still behave nicely with respect to D_{st}.
We start with a lemma which extends ideas from remark 2.23 and the proof of lemma 3.24 in [Füt18]:

Lemma 1.2.38. *Suppose V is a continuous $\mathbb{Q}_p$-linear representation of G_K and $K'/K/\mathbb{Q}_p$ a tower of finite extensions. Then we have*

$$D_{\mathrm{st},K'}(V)^{I_K} \cong K_0' \otimes_{K_0} D_{\mathrm{st},K}(V) \text{ and } D_{\mathrm{cris},K'}(V)^{I_K} \cong K_0' \otimes_{K_0} D_{\mathrm{st},K}(V).$$

In particular,

$$D_{\mathrm{pst},K}(V)^{I_K} \cong \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{st},K}(V) \text{ and } \left(D_{\mathrm{pst},K}(V)^{I_K}\right)^{N=0} \cong \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{cris},K}(V).$$

If we consider the linearised W_K-action on the left hand sides of the $D_{\mathrm{pst},K}$ statements, then the f-power of the geometric Frobenius Fr^f acts on the right hand sides $\mathbb{Q}_p^{nr}$-linearly as φ^f with f the residue degree of $K/\mathbb{Q}_p$.

Proof. We prove that for any tower of finite extensions $K'/K/\mathbb{Q}_p$, the map

$$K_0' \otimes_{K_0} D_{\mathrm{st},K}(V) \to D_{\mathrm{st},K'}(V), k \otimes \sum_i b_i \otimes v_i \mapsto \sum_i kb_i \otimes v_i$$

yields an isomorphism to $D_{\mathrm{st},K'}(V)^{I_K}$. Since I_K acts trivially on K_0' and $D_{\mathrm{st},K}(V)$, the map lands in $D_{\mathrm{st},K}(V)^{I_K}$. It is also injective as an restriction of the injective map α in theorem 1.2.3. To prove that it is an equality, we consider the following sequence of injections

$$\begin{aligned}
\widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0} D_{\mathrm{st},K}(V) &= \widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0'} K_0' \otimes_{K_0} D_{\mathrm{st},K}(V) \hookrightarrow \widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0'} D_{\mathrm{st},K'}(V)^{I_K} \\
&= \left(\widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0'} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_{K'}}\right)^{I_K} \hookrightarrow (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{I_K} = D_{\mathrm{st},\widehat{K^{nr}}}(V) \\
&\cong (\widehat{K^{nr}})_0 \otimes_{K_0} D_{\mathrm{st},K}(V) = \widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0} D_{\mathrm{st},K}(V)
\end{aligned}$$

The first inclusion is the one from above tensored with the flat K_0'-module $\widehat{\mathbb{Q}_p^{nr}}$. The second inclusion is again a restriction of the injective map α in theorem 1.2.3. The final isomorphism is complete unramified descent as in proposition 1.2.30. For dimension reasons, the injections must be isomorphisms and in particular $K_0' \otimes_{K_0} D_{\mathrm{st},K}(V) \cong D_{\mathrm{st},K'}(V)^{I_K}$. Unpacking the definition of $D_{\mathrm{pst},K}(V)$ yields

$$D_{\mathrm{pst},K}(V)^{I_K} = \bigcup_{\substack{K\subset K'\subset\overline{\mathbb{Q}_p}\\ K'/\mathbb{Q}_p \text{ finite}}} D_{\mathrm{st},K'}(V)^{I_K} \cong \bigcup_{\substack{K\subset K'\subset\overline{\mathbb{Q}_p}\\ K'/\mathbb{Q}_p \text{ finite}}} K_0' \otimes_{K_0} D_{\mathrm{st},K}(V) = \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{st},K}(V).$$

We can proceed entirely analogously with $D_{\mathrm{cris},K}(V)$ by writing $D_{\mathrm{cris},K}(V) = D_{\mathrm{cris},K}(V)^{N=0}$ and noting that all the maps above commute with N, so that the above statements transfer to the kernel of N

Finally, the linearised W_K-action on $D_{\mathrm{pst},K}(V)$ is given by the semi-linear action of W_K corrected by $\varphi^{-v(\sigma)}$ for $\sigma \in W_K$. So, on $D_{\mathrm{st},K}(V)$ the element Fr^f of W_K acts (linearly) under the established isomorphism as φ^f since $v(\mathrm{Fr}) = -1$ and the non-linear action of Fr^f is trivial on $D_{\mathrm{st},K}(V)$. □

Lemma 1.2.39.

(1) Let $\Sigma : 0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence in $\mathrm{Rep}_{\mathrm{dR}}(G_{\mathbb{Q}_p}) \cap \mathrm{Rep}_L(G_{\mathbb{Q}_p})$ and K a finite extension of $\mathbb{Q}_p$. Then the sequence $0 \to D_{\mathrm{pst}}(V_1) \to D_{\mathrm{pst}}(V_2) \to D_{\mathrm{pst}}(V_3) \to 0$ ($0 \to D_{\mathrm{st},K}(V_1) \to D_{\mathrm{st},K}(V_2) \to D_{\mathrm{st},K}(V_3) \to 0$) of A-($K_0 \otimes_{\mathbb{Q}_p} L$-) modules is exact.

(2) Let V be an L-linear de Rham representation of $G_{\mathbb{Q}_p}$ and $K/\mathbb{Q}_p$ finite. Then $D_{\mathrm{st},K}$ is compatible with L-duals, more precisely

$$\nu_K : D_{\mathrm{st},K}(V^{*_L}) \to D_{\mathrm{st},K}(V)^{*_{K_0\otimes_{\mathbb{Q}_p} L}}$$
$$\sum_i b_i \otimes \phi_i \mapsto \left(\sum_j b_j' \otimes v_j \mapsto \sum_{ij} b_i b_j' \phi_i(v_j) \in D_{\mathrm{st}}(L) = L \right)$$

is an isomorphism of $K_0 \otimes_{\mathbb{Q}_p} L$-modules.
By the same formula, $D_{\mathrm{pst}}(-)$ is compatible with duals, i.e. $v : D_{\mathrm{pst}}(V^{*_L}) \xrightarrow{\sim} D_{\mathrm{pst}}(V)^{*_A}$ is an isomorphism of A-modules.
Finally, we have $\nu_{\mathbb{Q}_p} \circ \varphi = (\varphi^{-1})^* \circ \nu_{\mathbb{Q}_p}$ and $\nu_K \circ N = (-N)^* \circ \nu_K$ (compare to the sign convention after definition 8.2.5 of [BC09]).

(3) Let V and V' be two L-linear de Rham representations of G_K, with $K/\mathbb{Q}_p$ finite. Then the natural map $D_{\mathrm{pst},K}V) \otimes_A D_{\mathrm{pst},K}(V') \to D_{\mathrm{pst},K}(V \otimes_L V')$ is an A-linear isomorphism of (φ, N, G_K)-modules, i.e an isomorphism of Weil-Deligne representations of W_K over $\mathbb{Q}_p^{nr}$. On the left hand side φ and G_K act diagonally and N via $\mathrm{id} \otimes N + N \otimes \mathrm{id}$.

Proof. The key to all these results is that de Rham representations are potentially semi-stable.

(1) This part can be found in VII §2 lemma 10 (i) of [Ven17] Choose a finite Galois extension $F/\mathbb{Q}_p$ large enough that V_1, V_2 and V_3 are F-semi-stable. Hence, $D_{\mathrm{st},F}$ is exact on Σ and tensoring the resulting exact sequence of $G_{\mathbb{Q}_p}/G_F = \mathrm{Gal}(F/\mathbb{Q}_p)$-modules with $\mathbb{Q}_p^{nr}$ over F_0 (see 1.2.32) yields the exactness of

$$0 \to D_{\mathrm{pst}}(V_1) \to D_{\mathrm{pst}}(V_2) \to D_{\mathrm{pst}}(V_3) \to 0$$

with linearised $W_{\mathbb{Q}_p}$-action. Since the open subgroup $I_K \cap G_F$ of I_K acts trivially, we can consider the sequence as a sequence of $\mathbb{Q}_p^{nr}[I_K/(I_K \cap G_F)]$-modules. As such it splits by

Maschke's theorem. As a result, taking I_K-invariants is exact and yields together with lemma 1.2.38 the exactness of the sequence

$$0 \to \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{st},K}(V_1) \to \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{st},K}(V_2) \to \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{st},K}(V_3) \to 0.$$

This implies the claim as $\mathbb{Q}_p^{nr}/K_0$ is faithfully flat.

(2) Let F be a finite extension of K such that V is F-semi-stable and F contains L. Then the map $\nu_F : D_{\mathrm{st},F}(V^*) \to D_{\mathrm{st},F}(V)^*$ given by the same formula as ν is an isomorphism by part (4) of theorem 1.2.3 and remark 1.2.20.
For $\sigma \in I_K$ we have $\nu_F \circ \sigma = \sigma^{-1} \circ \nu_F$. To see this, we consider the pairing $D_{\mathrm{st},F}(V^*) \times D_{\mathrm{st},F}(V) \to F_0 \otimes_{\mathbb{Q}_p} L$ induced by ν_F. It sends $(\sigma \times \sigma)\big(\sum_i b_i \otimes \phi_i, \sum_j b'_j \otimes v_j\big)$ to $\sum_{ij} \sigma(b_i b'_j) \otimes \phi_i(v_j) \in F_0 \otimes_{\mathbb{Q}_p} L$. Let $l_1, \dots, l_N$ be a $\mathbb{Q}_p$-base of L. Then there are $q_{ijk} \in \mathbb{Q}_p$ with $\phi_i(v_j) = \sum_k q_{ijk} l_k$. As a result, we get that each $\sum_{ij} \sigma(b_i b'_j) q_{ijk}$ is an element of F_0, on which I_K acts trivially. Since the q_{ijk} are also invariant under I_K, we see that the pairing remains unchanged under precomposing with $\sigma \times \sigma$.
As a result, we get the following commutative diagram

$$\begin{tikzcd}
F_0 \otimes_{K_0} D_{\mathrm{st},K}(V^*) \arrow[r, "\mathrm{id}_{F_0} \otimes \nu_K"] \arrow[dd] & F_0 \otimes_{K_0} D_{\mathrm{st},K}(V)^* \arrow[d, "(\star)"] \\
 & \mathrm{Hom}_{F_0 \otimes_{\mathbb{Q}_p} L}(F_0 \otimes_{K_0} D_{\mathrm{st},K}(V), F_0 \otimes_{\mathbb{Q}_p} L) \arrow[d] \\
D_{\mathrm{st},F}(V^*)^{I_K} \arrow[r, "\nu_{F|}"] & (D_{\mathrm{st},F}(V)^*)^{I_K}.
\end{tikzcd}$$

The unlabelled arrows are the isomorphisms from lemma 1.2.38. For the right one, we use that the I_K action commutes with duals as seen above. The starred arrow is given by $f \otimes \phi \mapsto (f' \otimes d \mapsto f f' \otimes \phi(d))$ and is surjective since $D_{\mathrm{st},K}(V)$ has finite K_0-dimension so that we can construct preimages just as in the proof of part (4) of theorem 1.2.3. The starred arrow thus becomes an isomorphism since both its source and target are F_0-vector spaces of dimension $\dim_{K_0}(D_{\mathrm{st},K}(V)^{*_{K_0 \otimes_{\mathbb{Q}_p} L}})$. The bottom arrow is an isomorphism as we have seen that ν_F commutes with the action of I_K. Since all other maps are isomorphisms, the top arrow is one as well. The field extension F_0/K_0 is faithfully flat. The category of $K_0 \otimes_{\mathbb{Q}_p} L$-modules is balanced and thus the functor $\mathrm{id}_{F_0} \otimes_{K_0} -$ reflects isomorphisms. We conclude that ν_K is an isomorphism. By a limiting process we obtain the result for D_{pst}.
For the commutativity of the monodromy operator with duals, we consider an element $x = \sum b_i \otimes \phi_i$ in $D_{\mathrm{st},K}(V^*)$ and an element $y = \sum b'_j \otimes v_j$ from $D_{\mathrm{st},K}(V)$. We prove that the elements $\nu_K N(x)(y) = \sum_{i,j} N(b_i) b'_j \phi_i(v_j)$ and $(-N)^* \nu_K(x)(y) = \sum_{i,j} -b_i N(b'_j) \phi_i(v_j)$ are the same. With l_k and q_{ijk} as above, we have that the terms $\sum_{i,j} q_{ijk} b_i b'_j$ are in $K_0 \subset B_{\mathrm{cris}} = \ker(N)$. Since N is a B_{cris}-derivation, we get

$$0 = N\left(\sum_{i,j} q_{ijk} b_i b'_j\right) = \sum_{i,j} q_{ijk} N(b_i b'_j) = \sum_{i,j} q_{ijk} \big(b_i N(b'_j) + N(b_i) b'_j\big)$$

which proves the claim after rearranging. So we established the relation $\nu_K N = (-N)^* \nu_K$. By choosing K large enough that V is G_K-semi-stable we get the result for D_{pst}.
The claimed commutativity of φ with the duals can be proven entirely analogously to the

relation of ν_F with $\sigma \in I_K$ above. The point is that φ acts as the identity on $\mathbb{Q}_p \subset B_{\mathrm{st}}$ and hence trivially on the dualising space $D_{\mathrm{st}}(L) = L$.

(3) Choose $F/\mathbb{Q}_p$ finite such that F contains L and K and such that both V and V' are G_F-semi-stable. By remark 1.2.20 and part (4) of theorem 1.2.3, we have $D_{\mathrm{st},F}(V \otimes_L V') \cong D_{\mathrm{st},F}(V) \otimes_{F_0 \otimes_{\mathbb{Q}_p} L} D_{\mathrm{st},F}(V')$ as $F_0 \otimes_{\mathbb{Q}_p} L$-modules. By corollary 1.2.32 we get the desired isomorphism. The compatibility with φ and G_K follows since both act as ring homomorphisms on B_{st} and the compatibility with N, since N is a derivation on B_{st}.

□

1.3 Continuous Galois cohomology

Definition 1.3.1. *Let G be a profinite group and M a topological abelian group on which G acts continuously. Then M is called a* (topological) G-module. *By a continuous action we mean that each $g \in G$ acts as a group automorphism of M and that the map $G \times M \to M, (g, m) \mapsto g(m)$ is continuous.*

Definition 1.3.2. *Let M be a G-module. The complex of* continuous (homogeneous) cochains of G with coefficients in M *is the complex $C(G, M)$ given by*

$$C^i(G, M) = \{f : G^{i+1} \to M | f \text{ is continuous}, \\ \forall \sigma, \sigma_0, \dots, \sigma_i \in G : f(\sigma\sigma_0, \dots, \sigma\sigma_i) = \sigma f(\sigma_0, \dots, \sigma_i)\}$$

and differentials $d^i(f)(\sigma_0, \dots \sigma_i) = \sum_{k=0}^{i} (-1)^i f(\sigma_0, \dots, \widehat{\sigma_k}, \dots, \sigma_i)$.
The cohomology groups of this complex are called the continuous cohomology groups of G with coefficients in M. *We will denote them by $H^i(G, M)$. When M is a Λ-module for some ring Λ and the G-action is Λ-linear, we often denote the complex $C(G, M)$ by $R\Gamma(G, M)$ when it is viewed as an object in the derived category of Λ-modules $D(\Lambda)$.*

Remark 1.3.3. Since we only consider Galois cohomology of continuous group actions in this work, we will usually omit the word “continuous”. For $G = G_K$ an absolute Galois group of a field K, we often denote $C(G, M)$ and $H^i(G, M)$ by $C(K, M)$ and $H^i(K, M)$.

Remark 1.3.4. By [NSW13] 2.7.2 the functor $C(G, -)$ is exact on short exact sequences $0 \to M' \to M \to M'' \to 0$ of G-modules which allow a continuous section $M'' \to M$, that does not need to be a homomorphism. In partiuclar, such sequences induce long exact cohomology sequences. A continuous section $M'' \to M$ exists in most of the cases considered in this work. We usually consider the cohomology of projective Λ-modules which get their topology form the topological ring Λ. By the projectivity, each such sequence allows a Λ-linear section, which is continuous. Alternatively, by [Ser07] Chapter I, §1 Proposition 1, the necessary section exists if the modules in the sequence are profinite, which is also the case in most of this work. See [Wit04] Lemma 5.3.1 for a detailed proof in this case.

A standard result for group cohomology is Shapiro's lemma. To state it, we need the notion of an induced representation.

Definition 1.3.5. *Let H be an open subgroup of a profinite group G and M a topological abelian group with linear topology, which is given by a fundamental system of open subgroups $\mathcal{U}$, on which*

H acts continuously. We define the induced G-module $\mathrm{Ind}_H^G(M)$ *as* $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$ *together with the topology induced by the isomorphism* $\mathrm{Ind}_H^G(M) \cong \prod_{\bar{g} \in G/H} g \otimes M$*, where the latter carries the product topology. The G-action on* $\mathrm{Ind}_H^G(M)$ *is given via left multiplication on the first factor. Moreover, we define the* coinduced G-module $\mathrm{Coind}_H^G(M)$ *as*

$$\mathrm{Coind}_H^G(M) = \{f : G \to M | f \text{ is continuous and } \forall h \in H, g \in G : f(hg) = h(f(g))\}.$$

An element g' of G acts on an element f of $\mathrm{Coind}_H^G(M)$ *via* $(g'f)(g) = f(gg')$*. We give* $\mathrm{Coind}_H^G(M)$ *the compact-open topology, which is given by the subbase consisting of the sets $S(K, U)$. Here, K is a compact subset of G, U is an open subset of M and $S(K, U)$ is defined as the set of all elements of* $\mathrm{Coind}_H^G(M)$ *which map K into U.*

Remark 1.3.6. $\mathrm{Ind}_H^G(M)$ is indeed a topological G-module. Without loss of generality it suffices to consider open subsets of $\mathrm{Ind}_H^G(M)$ of the form $U = \sum_{i=1}^n g_i \otimes U_i$ where U_i are open subsets of M and $g_1, \ldots, g_n$ from a H-left-traversal in G. Let g' be in G and $x = \sum_i g_i \otimes m_i$ in M such that $g'(x)$ is in U. Let p be the permutation of $\{1, \ldots, n\}$ for which there are h_i in H with $g'g_i = g_{p(i)}h_i$. Then $h_i(m_i)$ lies in $U_{p(i)}$. By continuity of the H-action on M, there is an open subgroup N of H, which is after shrinking also normal in G, and open subsets U_i' containing m_i such that $Nh_i(U_i') \subset U_{p(i)}$. Now, we have

$$Ng' \left(\sum_i g_i \otimes U_i' \right) = \sum_i Ng'g_i \otimes U_i' = \sum_i g_{p(i)} \otimes Hh_i(U_i') \subset \sum_i g_{p(i)} \otimes U_{p(i)} = U.$$

Hence, the action of G on $\mathrm{Ind}_H^G(M)$ is continuous.
The case of $\mathrm{Coind}_H^G(M)$ can be deduced from the lemma below which shows that induction and coinduction are isomorphic in our case.

Lemma 1.3.7. *Let H, G and M be as above. We have an isomorphism of topological G-modules*

$$\rho : \mathrm{Ind}_H^G(M) \to \mathrm{Coind}_H^G(M), \sum_{\bar{g} \in H \backslash G} g^{-1} \otimes m_g \mapsto (g \mapsto m_g, H - \text{linearly})$$

where the g's form a right-transversal of H in G.

Proof. First, we observe that any H-linear map from G to M is continuous: Let f be such a map, g be in G and U an element of $\mathcal{U}$. Since H acts continuously on M, there is an open subgroup N of H such that $N(f(g)) \subset f(g) + U$. By the H-linearity of f we obtain $f(Ng) \subset f(g) + U$ and Ng is an open neighbourhood of g in G since N is open in the open subgroup H.
Since H is open in the profinite group G, it is of finite index. Therefore, the obvious inverse $\rho^{-1} : \mathrm{Coind}_H^G(M) \to \mathrm{Ind}_H^G(M), f \mapsto \sum_{\bar{g} \in H \backslash G} g^{-1} \otimes f(g)$ is well defined. In this situation, the above morphism is known to be an isomorphism of G-modules (see for instance [Sha] Proposition 1.5.4). We extend his proof and show that the isomorphism is also a homeomorphism. Firstly, ρ is open. Let U be open in $\mathrm{Ind}_H^G(M)$. Without loss of generality, we can assume that it has the form $\sum_{\bar{g} \in H \backslash G} g^{-1} \otimes U_g$ with U_g open in M. Then its image under ρ is $\cap_{\bar{g} \in H \backslash G} S(\{g\}, U_g)$ which is open as a finite intersection of open sets. Secondly, ρ is also continuous. Since G is profinite, it is Hausdorff. Thus, by [Jac52] lemma 2.1, the sets $S(K, m + U)$ with $U \in \mathcal{U}$ and m an element of M form a subbase of the compact-open topology on $\mathrm{Coind}_H^G(M)$. We choose some

m in M, an open subgroup U in $\mathcal{U}$ and K a compact subset of G. Since H acts continuously on M, we find another element U' of $\mathcal{U}$ and an open subgroup N of H such that $N(U') \subset U$. The open sets Nk for k in K cover K. By compactness, there is a finite subcover $Nk_1, \dots Nk_n$. Let h_i be the element of H such that h_ik_i a member of the chosen right-traversal of H in G. Let f be an element of $S(K, m+U)$. Consider the set $S_f := \cap_{i=1}^n S\big(\{h_ik_i\}, f(h_ik_i) + h_i(U')\big)$. The map f clearly lies in the open set S_f. Moreover, $\rho^{-1}(S_f)$ is $\sum_{i=1}^n (h_ik_i)^{-1} \otimes \big(f(h_ik_i) + h_i(U')\big)$ and hence open. Furthermore, S_f is a subset of $S(K, m+U)$: Let f' be in S_f and k in K with $k = nk_i$ for n in $N \subset H$. Then by the H-linearity of f' and f, by the value of f' on h_ik_i and of f on k and the choice of N and U', we get

$$f'(k) = nh_i^{-1} f'(h_ik_i) \in nh_i^{-1}(f(h_ik_i) + h_i(U')) = f(k) + n(U') \subset m + U + U = m + U$$

and f' belongs to $S(K, m+U)$. To sum up, we saw that S_f is an open neighbourhood of f in $S(K, U)$ with open preimage under ρ. Hence, ρ is continuous. $\square$

Proposition 1.3.8 (Shapiro's lemma)**.** *Let H, G and M be as above. Then there is a quasi-isomorphism $Sh : C(G, \mathrm{Coind}_H^G(M)) \to C(H, M)$ which is natural in M. If Σ is a short exact sequence with a section as in remark 1.3.4, Sh induces an isomorphism between the long exact cohomology sequences to Σ and $\mathrm{Coind}_H^G(\Sigma)$.*

Proof. A slightly more general version of Shapiros lemma is proven in [BW00] IX Lemma 2.2 (2). Since we assume that H is open in G the space G/H is discrete and thus there is a topological section to the projection $G \to G/H$. So the condition in [BW00] IX Lemma 2.2 (2) is satisfied.
The two main steps of the proof are the following: The first step is the Frobenius reciprocity. The map $\mathrm{Coind}_H^G(M) \to M, f \mapsto f(1)$ induces a natural (topological) isomorphism

$$C^{i+1}(G, \mathrm{Coind}_H^G(M)) \to \{f : G^{i+1} \to M | f \text{ is continuous and } H\text{-linear}\} =: \mathscr{H}om_H(G^{i+1}, M)$$

([CW74] Lemma 2). The second step is that the restriction from G^{i+1} to H^{i+1} induces a quasi-isomorphism $\mathscr{H}om_H(G^{\bullet+1}, M) \to C^\bullet(H, M)$. Both of these steps are natural in M, which induces the naturality of the quasi-isomorphism in question. Let Σ be a sequence with a continuous section as in remark 1.3.4. Since $\mathrm{Coind}_H^G(M)$ is topologically the same as $\prod_{H\backslash G} M$ by lemma 1.3.7, the section of Σ thus induces a section of $\mathrm{Coind}_H^G(\Sigma)$. So both Σ and $\mathrm{Coind}_H^G(\Sigma)$ induce long cohomology sequences. By the natruality of Sh, it only remains to prove that Sh is compatible with the connecting homomorphisms. This can be done similarly as in [NSW13] 1.5.2 since Sh commutes with the differentials in $C(G, \mathrm{Coind}_H^G(M))$ and $C(H, M)$. $\square$

1.3.1 Λ-action under Shapiro

In this subsection, we investigate the situation in which we apply Shapiro's lemma to a module with additional structure. More precisely, let G be a profinite group with open normal subgroup H. Furthermore, let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$. Let M be an $\mathcal{O}_L$-module with $\mathcal{O}_L$-linear, continuous H-action and let Λ be the group ring $\mathcal{O}_L[G/H]$. We denote the Λ-module $\Lambda \otimes_{\mathcal{O}_L} M$ with G-action given by $g'(\overline{g} \otimes m) = \overline{g}\overline{g'}^{-1} \otimes g'(m)$ by $\Lambda^\natural \otimes_{\mathcal{O}_L} M$. Then we have two isomorphisms of topological G-modules $\psi : \Lambda \otimes_{\mathcal{O}_L} M \to \mathrm{Ind}_H^G(M), \overline{g} \otimes m \mapsto g^{-1} \otimes g(m)$ and $\rho : \mathrm{Ind}_H^G(M) \to \mathrm{Coind}_H^G(M)$ from lemma 1.3.7. By Shapiro's lemma (1.3.8) the following map is a quasi-isomorphism

$$Sh : C^\bullet(G, \Lambda \otimes_{\mathcal{O}_L} M) \xrightarrow[\psi_*]{\sim} C^\bullet(G, \mathrm{Ind}_H^G(M)) \xrightarrow[\rho_*]{\sim} C^\bullet(G, \mathrm{Coind}_H^G(M))$$
$$\xrightarrow[\left(f \mapsto f(-)(1)\right)]{\sim} \mathscr{H}\!om_H(G^{\bullet+1}, M) \xrightarrow[q-iso]{(H \hookrightarrow G)^*} C^\bullet(H, M).$$

Since the Λ- and G-operations commute on $\Lambda^\natural \otimes_{\mathcal{O}_L} M$ and Λ acts continuously on it, we have a Λ-action on $C^i(G, \Lambda \otimes_{\mathcal{O}_L} M)$ by $(\bar{g}f)(g_0, \ldots, g_i) = \bar{g}f(g_0, \ldots, g_i)$. By transport of structure along the above quasi-isomorphism we get the following Λ-actions on:

$$\begin{aligned}
C^i(G, \mathrm{Ind}_H^G(M)) : \overline{g'}(g \otimes m) &= gg'^{-1} \otimes g'(m) \\
C^i(G, \mathrm{Coind}_H^G(M)) : (\overline{g'}f)(-)(g) &= g'f(-)(g'^{-1}g) \\
\mathscr{H}\!om_H(G^{i+1}, M) : (\overline{g'}f)(g_0, \ldots, g_i) &= g'f(g'^{-1}g_0, \ldots, g'^{-1}g_i) \\
C^i(H, M) : (\overline{g'}f)(h_0, \ldots, h_i) &= g'f(g'^{-1}h_0, \ldots, g'^{-1}h_i) \text{ if } f \text{ is in } \mathrm{im}((H \hookrightarrow G)^*).
\end{aligned}$$

The Λ-action commutes with the differentials and thus induces a Λ-structure on the cohomology groups $H^i(G, \Lambda \otimes_{\mathcal{O}_L} M) \cong H^i(H, M)$.
Let $\chi : G/H \to \mathcal{O}_L^\times$ be a character. Then the morphism

$$\vartheta : \Lambda \otimes_{\mathcal{O}_L} M \to \Lambda \otimes_{\mathcal{O}_L} M(\chi), \bar{g} \otimes m \to \chi(\bar{g})^{-1}\bar{g} \otimes m$$

is an isomorphism of topological G-modules and $\mathrm{id} : M \to M(\chi)$ is an isomorphism of topological H-modules, but the vertical arrows in the following commutative diagram of abelian groups do not preserve the Λ-action, since the twist in the lower row is not accounted for in the upper row:

$$\begin{tikzcd}
C^i(G, \Lambda \otimes_{\mathcal{O}_L} M) \arrow[r, "Sh"] \arrow[d, "\vartheta_*"'] & C^i(H, M) \arrow[d, "\mathrm{id}"] \\
C^i(G, \Lambda \otimes_{\mathcal{O}_L} M(\chi)) \arrow[r, "Sh"] & C^i(H, M(\chi)).
\end{tikzcd}$$

Instead, one has to pull out the twist, to obtain a commutative diagram of Λ-modules:

$$\begin{tikzcd}
C^i(G, \Lambda \otimes_{\mathcal{O}_L} M)(\chi) \arrow[r, "Sh(\chi)"] \arrow[d, "\tilde{\vartheta} := "', "f \otimes e_\chi \mapsto \vartheta \circ f"] & C^i(H, M)(\chi) \arrow[d, "f \otimes e_\chi \mapsto f"] \\
C^i(G, \Lambda \otimes_{\mathcal{O}_L} M(\chi)) \arrow[r, "Sh"] & C^i(H, M(\chi)).
\end{tikzcd}$$

Here Λ operates diagonally on the modules of the upper row and e_χ is a basis of $\mathcal{O}_L(\chi)$.

1.3.2 Adic rings and Galois cohomology

An important result by Fukaya and Kato is that continuous cohomology behaves well with Λ-modules, if Λ is an adic ring. More precisely, they prove in [FK06] Proposition 1.6.5 a more general version of proposition 1.3.12 below. To state it, we introduce the notion of an adic ring, which resembles a ring that is complete with respect to the p-adic topology, after definition 1.4.1 in [FK06]:

Definition 1.3.9. *A (not necessarily commutative) ring Λ is called* adic*, if there is a two-sided ideal $\mathfrak{a}$ of Λ such that Λ is complete in the $\mathfrak{a}$-adic topology and the orders of the quotients $\Lambda/\mathfrak{a}^n$ are finite p-powers for all $n \geq 1$. In this thesis modules over adic rings will always carry the topology that is induced by the topology of the ring.*

Example 1.3.10. Let L be a finite extension of $\mathbb{Q}_p$ with ring of intergers $\mathcal{O}_L$, uniformiser π and let G be a finite group. Then the group ring $\Lambda := \mathcal{O}_L[G]$ is an adic ring. In fact, let $\mathfrak{a}$ be the two-sided ideal $\pi\Lambda$. Then $\Lambda/\mathfrak{a}^n = \mathcal{O}_L/\pi^n[G]$, which is of order $p^{f(L/\mathbb{Q}_p)n|G|}$. Since $\mathcal{O}_L$ is π-adically complete, Λ is complete with respect to the $\mathfrak{a}$-adic topology. This is the main example for an adic ring in this work. For a more general class of adic rings see [FK06] 1.4.2.

Lemma 1.3.11.

(1) Let J be the Jacobson radical of a ring Λ. Then Λ is an adic ring if and only if it is complete in the J-adic topology and the orders of the quotients $\Lambda/J^n\Lambda$ are finite p-powers.

(2) If $\mathfrak{a}$ is an ideal as in definition 1.3.9, then the $\mathfrak{a}$-adic and the $\mathfrak{J}$-adic topologies agree.

(3) An adic ring is semi-local.

Proof. The lemma is proven in [FK06] Lemma 1.4.4 and 1.4.5. We elaborate a bit on the proof. Let Λ be adic with respect to the ideal $\mathfrak{a}$. By [FK06] 1.4.3 (2) we have that if $\mathfrak{a}$ is a two-sided ideal such that Λ is $\mathfrak{a}$-adically complete, then $\mathfrak{a} \subset J$, this implies the correct orders for Λ/J^n and that the $\mathfrak{a}$-adic topology is finer than the J-adic one. Since the ring $\Lambda/\mathfrak{a}$ is finite, it is left artinian. By [Lam01] Theorem 4.12 we get that $J(\Lambda/\mathfrak{a})$ is nilpotent, i.e. $J^n \subset \mathfrak{a}$ for some $n \geq 1$. So, both topologies are the same and in particular, Λ is J-adically complete. $\square$

Proposition 1.3.12. *Let G be a profinite group. Consider the following two conditions on G:*

(i) For every finite, discrete abelian group X of p-power order with a continuous G-action, the cohomology groups $H^i(G, X)$ are finite for all i.

(ii) For every finite, discrete abelian group X of p-power order with a continuous G-action, the cohomology groups $H^i(G, X)$ are zero for i large enough.

Let Λ be an adic ring with Jacobson radical J and $\mathbb{T}$ a finitely generated projective Λ-module with a Λ-linear continuous action of G.

(1) If condition (i) holds, then we have $H^i(G, \mathbb{T}) = \lim_n H^i(G, \mathbb{T}/J^n\mathbb{T})$ for all $i \geq 0$.

(2) If conditions (i) and (ii) hold, then $R\Gamma(G, \mathbb{T})$ is a perfect complex over Λ.

(3) Assume that conditions (i) and (ii) hold. Let Λ' be another adic ring and Y a finitely generated projective Λ'-module on which Λ acts continuously from the right so that the Λ' and Λ structures commute. The action of G on $\mathbb{T}$ turns $Y \otimes_\Lambda \mathbb{T}$ into a topological Λ'-linear G-module and the natural map $y \otimes f \mapsto \big((g_0, \dots, g_i) \mapsto y \otimes f(g_0 \dots, g_i)\big)$ induces a quasi-isomorphism

$$\omega : Y \otimes^L_\Lambda C(G, \mathbb{T}) \xrightarrow{q-iso} C(G, Y \otimes_\Lambda \mathbb{T}).$$

Proof. The proof can be found in [FK06] 1.6.6 - 1.6.9. Here, we will only elaborate on the topological G-module structure of $Y \otimes_\Lambda \mathbb{T}$. As usual for modules over an adic ring, the topology of $Y \otimes_\Lambda \mathbb{T}$ is induced by that of Λ'. Let J' be the Jacobson radical of Λ'. Then the submodules $(J'^n Y) \otimes_\Lambda \mathbb{T}$ form a system of open neighbourhoods of zero.
We show that the action of G on $Y \otimes_\Lambda \mathbb{T}$ is continuous. Let g be an element of G. Further, let $t_1, \dots, t_n$ be generators of $\mathbb{T}$ as Λ-module, let m' be some natural number and $x = \sum_i^n y_i \otimes t_i$ some

element of $Y \otimes_\Lambda \mathbb{T}$. Since Λ acts continuously on Y, there is an $m \geq 1$ such that $y_i\lambda$ lies in $J'^{m'}Y$ for each y_i and λ in J^m. As G acts continuously on $\mathbb{T}$, there is an open subgroup H of G such that each $Hg(t_i)$ lies in $g(t_i) + J^m\,\mathbb{T}$. Now, let g' and x' be such that $g'(x') \in g(x) + J'^{m'}Y \otimes_\Lambda \mathbb{T}$. Then $(Hg', x' + J'^{m'}Y \otimes_\Lambda \mathbb{T})$ is an open neighbourhood of (g', x') and we have

$$\begin{aligned} Hg'(x' + J'^{m'}Y \otimes_\Lambda \mathbb{T}) = Hg'(x') + J'^{m'}Y \otimes_\Lambda Hg'(\mathbb{T}) \subset H(g(x)) + J'^{m'}Y \otimes_\Lambda \mathbb{T} \\ = \sum_{i=1}^n y_i \otimes Hg(t_i) + J'^{m'}Y \otimes_\Lambda \mathbb{T} \subset g(x) + \sum_{i=1}^n y_i \otimes J^m\,\mathbb{T} + J'^{m'}Y \otimes_\Lambda \mathbb{T} \\ \subset g(x) + J'^{m'}Y \otimes_\Lambda \mathbb{T}\,. \end{aligned}$$

Similar arguments show that the map ω is well-defined. It is clear that $\omega(y \otimes f)$ is G -linear. It is also continuous. Let $m' \geq 1$. As before, we find $m \geq 1$ so that $y\lambda$ lies in $y + J'^{m'}Y$ for all λ in J^m. Let $(g_0, \ldots, g_i)$ be in G^{i+1}. Since G acts continuously on $\mathbb{T}$, we find an open subgroup H such that $H(f(g_0, \ldots, g_i)) \subset f(g_0, \ldots, g_i) + J^m\,\mathbb{T}$. Let $(g'_0, \ldots, g'_i)$ be in the preimage of $y \otimes f(g_0, \ldots, g_i) + J'^{m'} \otimes_\Lambda \mathbb{T}$ under $\omega(y \otimes f)$. Then $(Hg'_0, \ldots, Hg'_i)$ is an open neighbourhood of $(g'_0, \ldots, g'_i)$ in G^{i+1} such that

$$\begin{aligned} &\omega(y \otimes f)(Hg'_0, \ldots, Hg'_i) = H\omega(y \otimes f)(g'_0, \ldots, g'_i) \subset H\left(y \otimes f(g_0, \ldots, g_i) + J'^{m'}Y \otimes_\Lambda \mathbb{T}\right) \\ =&y \otimes H(f(g_0, \ldots, g_i)) + J'^{m'}Y \otimes_\Lambda H(\mathbb{T}) \subset y \otimes f(g_0, \ldots, g_i) + y \otimes J^m\,\mathbb{T} + J'^{m'}Y \otimes_\Lambda \mathbb{T} \\ \subset& y \otimes f(g_0, \ldots, g_i) + J'^{m'}Y \otimes_\Lambda \mathbb{T}\,. \end{aligned}$$

□

Remark 1.3.13. For the purposes of this work, we will only apply 1.3.12 when G is the absolute Galois group of a finite extension of $\mathbb{Q}_p$. In this case the conditions (i) and (ii) are fulfilled ([Ser07] II §5.1 proposition 14 and II §5.3 proposition 15).

Lemma 1.3.14. *Let Λ be a topological ring. Let $0 \to T' \xrightarrow{f} T \xrightarrow{g} T'' \to 0$ be a short exact sequence of projective Λ-modules with Λ-linear continuous action of a profinite group G. Then we have an exact sequence of complexes of Λ-modules $0 \to C(G, T') \xrightarrow{f_*} C(G, T) \xrightarrow{g_*} C(G, T'') \to 0$. In particular, there is a long exact cohomology sequence.*

Proof. In [FK06] 1.6.10 Fukaya and Kato show the existence of a section $T'' \to T$ as in 1.3.4. □

1.3.3 Local Tate Duality

Let Λ be an adic ring and $\mathbb{T}$ a finitely generated, projective Λ-module with a continuous Λ-linear action of G_K for some finite extension $K/\mathbb{Q}_p$. We endow the dual module Λ°-module $\mathbb{T}^* = \mathrm{Hom}_\Lambda(\mathbb{T}, \Lambda)$ with the usual G_K-action: An element σ of G_K acts on $f \in T^*$ by $\sigma : f \mapsto f \circ \sigma^{-1}$.
Fukaya and Kato state in [FK06] 1.6.12 (2) the following version of local Tate duality, which [Sha09] proves:

Theorem 1.3.15. *The cup product*

$$C(K, \mathbb{T}) \otimes C(K, \mathbb{T}^*(1)) \to C(K, \Lambda(1))$$

induces an isomorphism $\psi(K, \mathbb{T}) : R\Gamma(K, \mathbb{T}) \to R\,\mathrm{Hom}_{\Lambda^\circ}(R\Gamma(K, \mathbb{T}^*(1)), \Lambda^\circ)[-2]$ in $D^p(\Lambda)$.

Remark 1.3.16. It is easy to see that the isomorphism ψ is natural in $\mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{T}'$ be a homomorphism of finitely generated Λ-modules, which is equivariant with respect to the continuous Λ-linear G_K-actions on $\mathbb{T}$ and $\mathbb{T}'$. Then the diagram

$$\begin{tikzcd} R\Gamma(K,\mathbb{T}) \arrow[r, "\psi(K,\mathbb{T})"] \arrow[d, "R\Gamma(K,f)"'] & R\operatorname{Hom}_{\Lambda^\circ}\big(R\Gamma(K,\mathbb{T}^*(1)),\Lambda^\circ\big)[-2] \arrow[d, "R\operatorname{Hom}_{\Lambda^\circ}(R\Gamma(K,f^*(1)))[-2]"] \\ R\Gamma(K,\mathbb{T}') \arrow[r, "\psi(K,\mathbb{T}')"'] & R\operatorname{Hom}_{\Lambda^\circ}\big(R\Gamma(K,\mathbb{T}'^*(1)),\Lambda^\circ\big)[-2] \end{tikzcd}$$

commutes, since both ways from the upper left to the lower right corner are given by the map $t \mapsto (\phi \mapsto \phi(f \circ t))$ where t is some element of $R\Gamma^i(K,\mathbb{T})$ and ϕ some element of $R\Gamma^{2-i}(K,\mathbb{T}'^*(1))$.

The following lemmata show that the local Tate duality is also compatible with base change. We first establish that duals commute with tensor products by making lemma [LVZ15] 4.6.8. more explicit:

Lemma 1.3.17. *Let Λ be a ring, $\mathbb{T}$ a finitely generated projective Λ°-module. Then $\mathbb{T}^{**} \cong \mathbb{T}$. Let further Λ' be another ring and Y a finitely generated projective Λ'-module with a commuting Λ-action from the right. Then there is an isomorphism of Λ'-modules:*

$$\begin{aligned} \nu : Y \otimes_\Lambda \operatorname{Hom}_{\Lambda^\circ}(\mathbb{T},\Lambda^\circ) &\to \operatorname{Hom}_{\Lambda'^\circ}(Y^* \otimes_{\Lambda^\circ} \mathbb{T},\Lambda'^\circ) \\ y \otimes f &\mapsto (\varphi \otimes t \mapsto \varphi(y\, f(t))) \end{aligned}$$

which is natural in both Y and $\mathbb{T}$. If a group G acts Λ-linearly on $\mathbb{T}$, then this isomorphism is also G-equivariant, if we let G act trivially on Y.
Combining the first two statements yields that duals commute with tensor products.

Proof. The fact that duals are self-inverse is clear on finitely generated free modules and hence on finitely generated projective modules, since they are direct summands of free modules and the direct sum of two homomorphisms is an isomorphism only if both are isomorphisms.
For the second part the most work has to be done to show that the map is well-defined. We omit these tedious computations. The map is an isomorphism if $\mathbb{T}$ is finitely generated free and by the same argument as above this implies it to be an isomorphism in the projective case, too. The equivariance and the naturality are immediate. $\square$

Lemma 1.3.18. *Let Λ, Λ', Y and $\mathbb{T}$ be as above, with Λ and Λ' being adic rings, the right-action of Λ on Y being continuous and $G = G_K$ for some finite extension K of $\mathbb{Q}_p$ acting continuously and Λ-linear on $\mathbb{T}$. Then the local Tate duality commutes with base change in the following way:*

$$\begin{tikzcd} Y \otimes_\Lambda R\Gamma(K,\mathbb{T}) \arrow[r, "\mathrm{id}_Y \otimes_\Lambda \psi(K,\mathbb{T})"] \arrow[ddd, "\omega"'] & Y \otimes_\Lambda R\operatorname{Hom}_{\Lambda^\circ}(R\Gamma(K,\mathbb{T}^*(1)),\Lambda^\circ)[-2] \arrow[d, "\nu"] \\ & R\operatorname{Hom}_{\Lambda'^\circ}(Y^* \otimes_{\Lambda^\circ} R\Gamma(K,\mathbb{T}^*(1)),\Lambda'^\circ)[-2] \arrow[d, "\omega"] \\ & R\operatorname{Hom}_{\Lambda'^\circ}(R\Gamma(K,Y^* \otimes_{\Lambda^\circ} \mathbb{T}^*(1)),\Lambda'^\circ)[-2] \arrow[d, "\nu"] \\ R\Gamma(K,Y \otimes_\Lambda \mathbb{T}) \arrow[r, "\psi(K,Y\otimes_\Lambda\mathbb{T})"] & R\operatorname{Hom}_{\Lambda'^\circ}(R\Gamma(K,(Y \otimes_\Lambda \mathbb{T})^*(1)),\Lambda'^\circ)[-2]. \end{tikzcd}$$

All involved maps are isomorphisms in the derived category.

Proof. This follows by unpacking the definitions of the involved maps. The local Tate duality, ω and ν are isomorphisms in the derived category by theorem 1.3.15, proposition 1.3.12 and lemma 1.3.17, respectively. $\square$

Local Tate duality is best known in the case where instead of being an adic ring, Λ is a finite extension of $\mathbb{Q}_p$ (for instance Theorem 1.4.1 in [Rub14]). We can obtain this version via base change (compare [FK06] 1.6.13):

Theorem 1.3.19. *Let L be a finite extension of $\mathbb{Q}_p$ and V a finite dimensional L-vector space with a continuous L-linear G_K-action where K is another finite extension of $\mathbb{Q}_p$. Then the cup-product*

$$C(K,V) \otimes C(K,V^*(1)) \to C(K,L(1))$$

induces a quasi-isomorphism

$$\psi(K,V) : R\Gamma(K,V) \to R\operatorname{Hom}_L(R\Gamma(K,V^*(1)),L)[-2].$$

By slightly extending the base change result from above, we can relate this version of the local Tate duality with the one for adic rings in theorem 1.3.15: Let Λ, $\mathbb{T}$, G_K be as in theorem 1.3.15. Further, let L a finite extension of $\mathbb{Q}_p$ and V a finite dimensional L-vector space on which Λ acts continuously and L-linearly from the right. Then, we have the commuting diagram of L-homomorphisms

$$\begin{tikzcd} V \otimes_\Lambda R\Gamma(K,\mathbb{T}) \arrow[r, "\mathrm{id}_V \otimes \psi(K,\mathbb{T})"] \arrow[d] & V \otimes_\Lambda R\operatorname{Hom}_{\Lambda^\circ}(R\Gamma(K,\mathbb{T}*(1)),\Lambda^\circ)[-2] \arrow[d] \\ R\Gamma(K,V\otimes_\Lambda \mathbb{T}) \arrow[r, "\psi(K,V\otimes_\Lambda\mathbb{T})"] & R\operatorname{Hom}_L(R\Gamma(K,(V\otimes_\Lambda \mathbb{T})^*(1)),L)[-2]. \end{tikzcd}$$

The G_K-action on $V \otimes_\Lambda \mathbb{T}$ is given by letting G_K act trivially on V. The vertical maps are given similarly as in the above lemma 1.3.18. Again, the commutativity follows from unpacking the definitions. We will show that the vertical maps are isomorphisms in the derived category. Let $\mathcal{O}_L$ be the ring of integers in L. Then there is a finitely generated $\mathcal{O}_L$-submodule T of V such that $L \otimes_{\mathcal{O}_L} T = V$ and which is invariant under the Λ-action, i.e. $T\Lambda \subset T$. To see this, let T' be any spanning $\mathcal{O}_L$-submodule of V, generated by $t_1, \ldots, t_n$. Then T' is an open neighbourhood of zero in V. Since Λ acts continuously, there is an open ideal $\mathfrak{a}_i$ of Λ such that $t_i\mathfrak{a}_i \subset T'$. Then, the intersection $\cap_i \mathfrak{a}_i$ is an open ideal of Λ which stabilises T' and contains J^n for some $n \in \mathbb{N}$ and J the Jacobson radical of Λ. Since Λ is an adic ring the quotient ring Λ/J^n is finite, so that the Λ-invariant $\mathcal{O}_L$-module $T = \sum_{\bar{\lambda}\in\Lambda/J^n} T'\lambda$ is finitely generated and still spanning.
$\mathcal{O}_L$ is an adic ring. We can split $V \otimes_\Lambda -$ up as $L \otimes_{\mathcal{O}_L} T \otimes_\Lambda -$. By proposition 1.3.12 the canonical map $T \otimes_\Lambda R\Gamma(K,\mathbb{T}) \to R\Gamma(K, T \otimes \mathbb{T})$ is a quasi-isomorphism. By a slight generalisation of proposition 2.7.11 in [NSW13], the map $L \otimes_{\mathcal{O}_L} R\Gamma(K, T \otimes_\Lambda \mathbb{T}) \to R\Gamma(K, L \otimes_{\mathcal{O}_L} T \otimes_\Lambda \mathbb{T})$ is a quasi-isomorphism, too. So the left vertical map is a quasi-isomorphism. Similar arguments apply to the right map. Here, it just remains to note that for any finitely generated $\mathcal{O}_L$-module S, the canonical map $L \otimes_{\mathcal{O}_L} \operatorname{Hom}_{\mathcal{O}_L}(S, \mathcal{O}_L) \to \operatorname{Hom}_L(L \otimes_{\mathcal{O}_L} S, L)$ has an inverse given by choosing the greatest denominator for images of the finite number of generators of S under an element of $\operatorname{Hom}_L(L \otimes_{\mathcal{O}_L} S, L)$. So the right map is also a quasi-isomorphism.

Remark 1.3.20. The local Tate duality in theorem 1.3.19 is natural in the same way as described in 1.3.16.

Remark 1.3.21. We have to be cautious about signs when dealing with local Tate duality. Both $\psi(K,V)$ and $\psi(K,V^*(1))^*[-2]$ are isomorphisms $R\Gamma(K,V) \to R\Gamma(K,V^*(1))^*[-2]$. While both of them are induced by the cup product $C(K,V) \otimes_L C(K,V^*(1)) \to C(K,L(1))$, they differ. The cup product is only skew commutative, i.e. if x is in $C^p(K,V)$ and y is in $C^q(K,V^*(1))$, then we only have $x \cup y = (-1)^{pq} y \cup x$ ([NSW13] proposition 1.4.4). The local Tate duality is non-trivial in degrees zero, one and two. In degrees zero and two the skew commutativity does not give rise to a sign. In degree one, however, we get a sign, so that $\psi^1(K,V) = -\psi^1(K,V^*(1))^*[-2]$ as maps $H^1(K,V) \cong H^1(K,V^*(1))^*$.

1.3.4 Finite parts of Galois cohomology

Let us recall the finite parts of Galois cohomology as introduced in [BK07] §3. We fix a finite extension L of $\mathbb{Q}_p$ and a finite dimensional L-vector space V with continuous L-linear $G_{\mathbb{Q}_p}$-action.

Definition 1.3.22. *We define the* finite parts of the Galois cohomology of V *as follows:*

$$\begin{aligned}
&H^0_f(\mathbb{Q}_p,V) := H^0(\mathbb{Q}_p,V)\\
&H^1_f(\mathbb{Q}_p,V) := \ker\left(H^1(\mathbb{Q}_p,V) \to H^1(\mathbb{Q}_p, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)\right)\\
&H^i_f(\mathbb{Q}_p,V) := 0 \text{ for } i \neq 0,1,
\end{aligned}$$

where the action of $G_{\mathbb{Q}_p}$ on $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$ is diagonally.
We can define a subcomplex $C_f(\mathbb{Q}_p,V)$ of $C(\mathbb{Q}_p,V)$ with cohomology $H^\bullet_f$ as in 2.4.2 of [FK06] by setting

$$\begin{aligned}
&C^0_f(\mathbb{Q}_p,V) := C^0(\mathbb{Q}_p,V)\\
&C^1_f(\mathbb{Q}_p,V) := \ker\left(\ker\left(C^1(\mathbb{Q}_p,V) \xrightarrow{d^1} C^2(\mathbb{Q}_p,V)\right) \to H^1(\mathbb{Q}_p,V)/H^1_f(\mathbb{Q}_p,V)\right)\\
&C^i_f(\mathbb{Q}_p,V) := 0 \text{ for } i \neq 0,1.
\end{aligned}$$

We denote the image of $C_f(\mathbb{Q}_p,V)$ in the derived category of L-vector spaces by $R\Gamma_f(\mathbb{Q}_p,V)$.

We can construct a more explicit complex with the same cohomology as done on page 612 of [BB05a]:

Lemma 1.3.23. *The complex*

$$C'_f(\mathbb{Q}_p,V) := \left[D_{\mathrm{cris}}(V) \xrightarrow{(1-\varphi,\bar{1})} D_{\mathrm{cris}}(V) \oplus t(V)\right]$$

which is concentrated in degrees 0 *and* 1 *is quasi-isomorphic to $C_f(\mathbb{Q}_p,V)$.*

Proof. Consider the short exact sequence of $G_{\mathbb{Q}_p}$-modules from lemma 1.2.17

$$0 \to \mathbb{Q}_p \to B_{\mathrm{cris}} \xrightarrow{(1-\varphi,\bar{1})} B_{\mathrm{cris}} \oplus B_{\mathrm{dR}}/B^0_{\mathrm{dR}} \to 0.$$

We can tensor this sequence over $\mathbb{Q}_p$ with V and obtain an exact sequence of L-linear, continuous $G_{\mathbb{Q}_p}$-representations

$$0 \to V \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V \oplus B_{\mathrm{dR}}/B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \to 0.$$

The beginning of the associated long exact cohomology sequence is

$$0 \to V^{G_{\mathbb{Q}_p}} \to D_{\mathrm{cris}}(V) \xrightarrow{(1-\varphi,\bar{1})} D_{\mathrm{cris}}(V) \oplus t(V) \to H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V). \quad (1.1)$$

So, the cohomology of $C'_f(\mathbb{Q}_p, V)$ is precisely the finite parts of the Galois cohomology of V. For complexes of vector spaces it is equivalent to have the same cohomology groups or to be quasi-isomorphic, since each short exact sequence of vector spaces splits and vector spaces over a fixed field form an abelian category. In particular, a complex of vector spaces admits a quasi-isomorphism to the complex of cohomology groups with zero derivatives and vice versa. Therefore, the long exact cohomology sequence above yields a canonical quasi-isomorphism $C_f(\mathbb{Q}_p, V) \xrightarrow{q-iso} C'_f(\mathbb{Q}_p, V)$. □

Remark 1.3.24. The arguments at the end of the proof of 1.3.23 show that $C(\mathbb{Q}_p, V)$ and $C_f(\mathbb{Q}_p, V)$ are perfect complexes.

Remark 1.3.25. The map $t(V) \to H^1_f(\mathbb{Q}_p, V)$ is sometimes called exponential map of Bloch Kato (see [BK07] definition 3.10).

Bloch and Kato showed that the finite parts of Galois cohomology behave nicely under the local Tate duality (see [BK07] proposition 3.8; they only considered V to be a $\mathbb{Q}_p$-vector space, but their arguments immediately carry over to V being a L-vector space.)

Proposition 1.3.26. *Assume that V is a de Rham representation. Then in the perfect pairing*

$$H^1(\mathbb{Q}_p, V) \otimes H^1(\mathbb{Q}_p, V^*(1)) \to H^2(\mathbb{Q}_p, L(1)) \cong L$$

the finite parts $H^1_f(\mathbb{Q}_p, V)$ and $H^1_f(\mathbb{Q}_p, V^*(1))$ are exact annihilators of each other. In particular, we get a quasi-isomorphism

$$\psi_f(\mathbb{Q}_p, V) : C_f(\mathbb{Q}_p, V) \to \mathrm{Hom}_L\left(C(\mathbb{Q}_p, V^*(1))/C_f(\mathbb{Q}_p, V^*(1)), L\right)[-2].$$

We sometimes denote $R\Gamma(\mathbb{Q}_p, V^*(1))/R\Gamma_f(\mathbb{Q}_p, V^*(1))$ by $R\Gamma_{/f}(\mathbb{Q}_p, V^*(1))$.

Chapter 2

ε-isomorphisms

2.1 Local constants

In this section, we collect some results on local constants as defined by Deligne in [Del73] §4 and §5.

Let E be a field of characteristic 0, which contains the p^n-th roots of unity for all $n \in \mathbb{N}$. Let D be a finite dimensional E-vector space endowed with the discrete topology and with a continuous E-linear W_K-action, where K is a finite extension of $\mathbb{Q}_p$. The local constants also depend on a non-trivial homomorphism $\psi : K \to E^\times$ with open kernel and a Haar measure μ on K. We will always choose μ such that $\mathcal{O}_K$ has measure 1 and omit it in the notation. With these choices, we have the following theorem:

Theorem 2.1.1. *Let m_a be the multiplication by a. For each D as above there is an element $\varepsilon(D,\psi)$ of $E^\times$ such that*

(1) If $0 \to D_1 \to D_2 \to D_3 \to 0$ is a short exact sequence of continuous W_K-representations over E, then
$$\varepsilon(D_2,\psi) = \varepsilon(D_1,\psi)\varepsilon(D_3,\psi).$$

(2) For σ in $I_K \subset W_K$ we have $\varepsilon(D, \psi \circ m_{\chi_{cycl}(\sigma)}) = \det_E(\sigma|D)\varepsilon(D,\psi)$.

(3) Let ω be the one-dimensional W_K-representation over E which is given by the map $W_K \twoheadrightarrow W_K^{ab} \xrightarrow[\sim]{rec_K^{-1}} K^\times \xrightarrow{\|\cdot\|_K} E^\times$. Here, we adopt the convention that a geometric Frobenius Fr maps to p under $rec_{\mathbb{Q}_p}^{-1}$ and hence to p^{-1} under ω (see [Del73] 2.3 and [Tat79] 1.4.1). Then we have $\varepsilon(D,\psi)\varepsilon(D^* \otimes_E \omega, \psi \circ m_{-1}) = 1$.

(4) Let L be a finite extension of K. There is a constant $\lambda(L/K,\psi) \in E$ such that for all W_L-representations D over E we have $\varepsilon(\mathrm{Ind}_{W_L}^{W_K}(D),\psi) = \lambda(L/K,\psi)^{\dim_E D}\varepsilon(D,\psi \circ Tr_{L/K})$.

(5) Let D' be an unramified W_K-representation over E. Then
$$\varepsilon(D \otimes_E D',\psi) = \det_E(\mathrm{Fr}^f|D')^{a(D)+\dim_E(D)n(\psi)}\varepsilon(D,\psi)^{\dim_E(D')},$$
where f is the residue degree of $K/\mathbb{Q}_p$, the number $n(\psi)$ is the largest integer such that $\pi_K^{-n}\mathcal{O}_K$ is in the kernel of ψ and $a(D)$ is the Artin conductor of D.

(6) Let τ be a ring automorphism of E and D^τ the E-linear W_K-representation $E \otimes_\tau D$. Then $\tau(\varepsilon(D,\psi)) = \varepsilon(D^\tau, \tau \circ \psi)$.

Proof. The existence of the local factors is theorem 4.1 in [Del73]. Its proof shows claim 6. The other claims claim follow directly from [Del73] 5.2, 5.4 and 5.7.1.. □

Remark 2.1.2. If K is $\mathbb{Q}_p$, we will only work with ψ's whose kernel is $\mathbb{Z}_p$. Such a homomorphism is determined by the values $\psi(p^{-n})$, which are all primitive p^n-th roots of unity, i.e. ψ corresponds to a $\mathbb{Z}_p$-basis of $\lim_n \mu(p^n) = \mathbb{Z}_p(1)$. So instead of ψ, we choose a basis ξ of $\mathbb{Z}_p(1)$, which we write additively so that property (2), (3) and (6) read in this case

(2) For σ in $I \subset W_{\mathbb{Q}_p}$ we have $\varepsilon(D, \chi_{cycl}(\sigma)\xi) = \det_E(\sigma|D)\varepsilon(D, \xi)$.

(3) We have $\varepsilon(D, \xi)\varepsilon(D^* \otimes_E \omega, -\xi) = 1$.

(6) Let τ be a ring automorphism of E and D^τ the E-linear $W_{\mathbb{Q}_p}$-representation $E \otimes_\tau D$. Then $\tau(\varepsilon(D, \xi)) = \varepsilon(D^\tau, \tau(\xi))$.

2.2 ε-isomorphisms of de Rham representations

In this section, we will construct ε-isomorphisms associated to de Rham representations, which are related to Deligne's ε-factors from the previous section, and prove their key properties. We will elaborate the approach in [FK06] section 3.3 following chapter VII of [Ven17]. In particular, we will include Nakamura's correction to Fukaya's and Kato's approach suggested in remark 3.6 in [Nak17] which ensures the multiplicativity of the ε-isomorphisms.

Let us fix some notation. In this chapter, L will be a finite extension of $\mathbb{Q}_p$, we denote by $\mathbb{Q}_p^{nr}$ the maximal unramified extension of $\mathbb{Q}_p$ and by $\mathbb{Z}_p^{nr}$ its ring of integers. Further, we denote the completion of $\mathbb{Q}_p^{nr}$ by $\widehat{\mathbb{Q}_p^{nr}}$ and its valuation ring by $\widehat{\mathbb{Z}_p^{nr}}$. Then we put $\widetilde{L} := \widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} L$. Finally, let V be an L-linear de Rham representation of $G_{\mathbb{Q}_p}$ and ξ a base of $\mathbb{Z}_p(1)$.

Lemma 2.2.1. *Taking the determinant yields a map*

$$\det_L(-|V) : G_{\mathbb{Q}_p}^{ab} = r\,\mathrm{Gal}(\mathbb{Q}_p^{ab}, \mathbb{Q}_p) = G_{\mathbb{Q}_p}/\overline{[G_{\mathbb{Q}_p}, G_{\mathbb{Q}_p}]} \to L^\times.$$

Proof. The map $\det_L(-|V) : G_{\mathbb{Q}_p} \to L^\times = K_1(L)$ factors over $G_{\mathbb{Q}_p}/[G_{\mathbb{Q}_p}, G_{\mathbb{Q}_p}]$ since $K_1(L)$ is abelian. Let T be a spanning, finitely generated $\mathcal{O}_L$-lattice in V, which is $G_{\mathbb{Q}_p}$-stable. This exists by similar arguments to those after theorem 1.3.19. Then $\det_{\mathcal{O}_L}(\sigma|T) \in \mathcal{O}_L^\times = \lim_n(\mathcal{O}_L/\pi_L^n\mathcal{O}_L)^\times$. Suppose $\det_{\mathcal{O}_L}(\sigma|T) \neq 1$. Then there is some n such that $\det_{\mathcal{O}_L/\pi_L^n\mathcal{O}_L}(\sigma|T/\pi_L.T) \neq 1$. By continuity, we have some open normal subgroup U_n of $G_{\mathbb{Q}_p}$ that acts trivially on the finite module T/π_L^nT. Hence, each element of the open set gU_n does not have determinant 1 and hence $\det_{\mathcal{O}_L}(-|T)$ and thus $\det_L(-|V)$ have closed kernels, so that they factor over $G_{\mathbb{Q}_p}^{ab}$. $\square$

The aim of the section is to prove the following proposition:

Proposition 2.2.2. *For each L-linear de Rham representation V of $G_{\mathbb{Q}_p}$, there is an isomorphism*

$$\varepsilon_{L,\xi}(V) : \mathbb{1}_{\widetilde{L}} \to \big(d_L(R\Gamma(\mathbb{Q}_p, V)) \cdot d_L(V)\big)_{\widetilde{L}}$$

in the category $V(\widetilde{L})$ which satisfies the following properties:

Multiplicativity:

Let $\Sigma : 0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of L-linear de Rham representations of $G_{\mathbb{Q}_p}$, then the ε-isomorphisms are multiplicative in the following way:

$$\big(d_L(C(\mathbb{Q}_p, \Sigma))d_L(\Sigma)\big)_{\widetilde{L}} \circ \varepsilon_{L,\xi}(V_2) = \varepsilon_{L,\xi}(V_1) \cdot \varepsilon_{L,\xi}(V_3).$$

Here, $d_L(C(\mathbb{Q}_p, \Sigma))$ is given by the sequence $C(\mathbb{Q}_p, \Sigma)$, which is exact by the arguments in remark 1.3.4.

Change of ξ:

Let $\sigma \in I(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \subset G_{\mathbb{Q}_p}^{ab}$. Then $\varepsilon_{L,\chi_{cycl}(\sigma)\xi}(V) = \det_L(\sigma|V)\varepsilon_{L,\xi}(V)$. Here, the element $\det_L(\sigma|V)$ of $K_1(L)$ acts on the ε-isomorphism is as in corollary 1.1.37 (3). Since $\mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \subset G_{\mathbb{Q}_p}^{ab}$ and $\chi_{cycl} : \mathrm{Gal}(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$, we have covered all possible choices of ξ.

Frobenius invariance:

Let $\phi \in G_{\mathbb{Q}_p}$ be a Frobeniuslift with $\chi_{cycl}(\phi) = 1$. Denote by $\varphi_p := \phi|_{\widehat{\mathbb{Z}_p^{nr}}} : \widehat{\mathbb{Z}_p^{nr}} \to \widehat{\mathbb{Z}_p^{nr}}$ the restriction of $\phi : B_{\mathrm{dR}} \to B_{\mathrm{dR}}$. The ring homomorphism $\varphi_p \otimes \mathrm{id}_L : \widetilde{L} \to \widetilde{L}$ induces a base change homomorphism $(\varphi_p \otimes \mathrm{id}_L)^* : K_1(\widetilde{L}) \to K_1(\widetilde{L})$ (see lemma 1.1.10). We put $K_1(\widetilde{L})_V = \{x \in K_1(\widetilde{L}) | (\varphi_p \otimes \mathrm{id})^*(x) = \det_L(\phi|V)^{-1}x\}$. Then the ε-isomorphism $\varepsilon_{L,\xi}(V)$ belongs to $V(L)\big(\mathbb{1}, d_L(R\Gamma(\mathbb{Q}_p, L)d_L(V)\big) \overset{K_1(L)}{\times} K_1(\widetilde{L})_V$.

Duality:

$\varepsilon_{L,\xi}(V) \cdot \varepsilon_{L,-\xi}(V^*(1))^* \cdot \overline{d_L(\psi(\mathbb{Q}_p, V))_{\widetilde{L}}}^{-1} = (-1)^{\dim_L H^0(\mathbb{Q}_p, V)} d_L\left(V(-1) \xrightarrow{\cdot\xi} V\right)_{\widetilde{L}}$ after appropriate trivialisations.

2.2.1 Construction of $\varepsilon_{L,\xi}(V)$

We will construct $\varepsilon_{L,\xi}(V)$ as the product of a rational number $\Gamma_L(V)$ and two isomorphisms

$$\begin{aligned} \theta(V) :& \mathbb{1}_L \to d_L(R\Gamma(\mathbb{Q}_p, V)) \cdot d_L(D_{\mathrm{dR}}(V)) \\ \varepsilon_{L,\xi}^{dR}(V) :& \mathbb{1}_{\widetilde{L}} \to \big(d_L(D_{\mathrm{dR}}(V))^{-1} \cdot d_L(V)\big)_{\widetilde{L}} \end{aligned}$$

using the trivialisation $d_L(D_{\mathrm{dR}}(V)) \cdot d_L(D_{\mathrm{dR}}(V))^{-1} \xrightarrow{\sim} \mathbb{1}_L$.

Let us start with the definition of $\Gamma_L(V)$. In 1.2.25, we defined $h(r) = \dim_L gr^r(D_{\mathrm{dR}}(V))$ and we put

$$\Gamma^*(r) = \begin{cases} (r-1)! & \text{for } r \geq 1 \\ (-1)^r(-r)!^{-1} & \text{for } r \leq 0. \end{cases}$$

Then we set $\Gamma_L(V) := \prod_{r \in \mathbb{Z}} \Gamma^*(r)^{-h(-r)}$.

Next, we will construct the morphism $\theta(V)$. In lemma 1.3.23 we established a quasi-isomorphism

$$C_f(\mathbb{Q}_p, V) \xrightarrow{q-iso} \left[D_{\mathrm{cris}}(V) \xrightarrow{(\mathrm{id}-\varphi, \bar{\mathrm{id}})} D_{\mathrm{cris}}(V) \oplus t(V)\right] = C_f'(\mathbb{Q}_p, V).$$

We can trivialise

$$\begin{aligned} d_L(C_f'(\mathbb{Q}_p, V)) &= d_L(D_{\mathrm{cris}}(V)) \cdot d_L(D_{\mathrm{cris}}(V) \oplus t(V))^{-1} \\ &= d_L(D_{\mathrm{cris}}(V)) \cdot d_L(D_{\mathrm{cris}}(V))^{-1} \cdot d_L(t(V))^{-1} \xrightarrow{\mu \cdot \mathrm{id}} d_L(t(V))^{-1} \end{aligned}$$

which induces a morphism

$$\eta(V) : \mathbb{1}_L \to d_L(R\Gamma_f(\mathbb{Q}_p, V)) d_L(C_f'(\mathbb{Q}_p, V))^{-1} \to d_L(R\Gamma_f(\mathbb{Q}_p, V)) d_L(t(V)).$$

Applying the same arguments to $V^*(1)$ gives us $\eta(V^*(1))$. We want to multiply $\eta(V)$ and $\eta(V^*(1))$ to get $\theta(V)$. To do this, we consider the exact sequence

$$\Sigma_{R\Gamma,V} : 0 \to C_f(\mathbb{Q}_p, V) \to C(\mathbb{Q}_p, V)) \to C(\mathbb{Q}_p, V)/C_f(\mathbb{Q}_p, V) \to 0.$$

Together with the local Tate duality for finite parts (proposition 1.3.26) it induces an isomorphism

$$\begin{aligned}&d_L(R\Gamma_f(\mathbb{Q}_p, V^*(1)))^* \cdot d_L(R\Gamma_f(\mathbb{Q}_p, V))\\&\xrightarrow{\overline{d_L(\psi_f(\mathbb{Q}_p,V^*(1)))^*}\cdot \mathrm{id}} d_L\big(R\Gamma(\mathbb{Q}_p, V)/R\Gamma_f(\mathbb{Q}_p, V)\big) \cdot d_L(R\Gamma_f(\mathbb{Q}_p, V)) \xrightarrow{\overline{d_L(\Sigma_{R\Gamma,V})}} d_L(R\Gamma(\mathbb{Q}_p, V)).\end{aligned}$$

We can deal with the factor $t(V^*(1))^*$ in a similar way. By corollary 1.2.29 we have an isomorphism $\psi_{\mathrm{dR},V} : D^0_{\mathrm{dR}}(V) \xrightarrow{\sim} t(V^*(1))^*$. So the exact sequence

$$\Sigma_{\mathrm{dR},V} : 0 \to D^0_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V) \to t(V) \to 0$$

yields an isomorphism

$$d_L(t(V^*(1))^* \cdot d_L(t(V)) \xrightarrow{\overline{d_L(\psi_{\mathrm{dR},V})}\cdot \mathrm{id}} d_L(D^0_{\mathrm{dR}}(V)) \cdot d_L(t(V)) \xrightarrow{\overline{d_L(\Sigma_{\mathrm{dR},V})}} d_L(D_{\mathrm{dR}}(V)).$$

We combine the above morphisms to define $\theta(V)$:

Definition 2.2.3. *$\theta(V)$ is defined as the morphism*

$$\begin{aligned}&\mathbb{1} \xrightarrow{\overline{\eta(V^*(1))^*}\cdot\eta(V)} d_L(R\Gamma_f(\mathbb{Q}_p, V^*(1)))^* \cdot d_L(t(V^*(1)))^* \cdot d_L(R\Gamma_f(\mathbb{Q}_p, V)) \cdot d_L(t(V))\\&\xrightarrow{\overline{d_L(\psi_f(\mathbb{Q}_p,V^*(1)))^*}\cdot\overline{d_L(\psi_{\mathrm{dR},V})}\cdot \mathrm{id}} d_L\left(R\Gamma(\mathbb{Q}_p, V)/R\Gamma_f(\mathbb{Q}_p, V)\right) d_L(D^0_{\mathrm{dR}}(V)) d_L(R\Gamma_f(\mathbb{Q}_p, V)) d_L(t(V))\\&\xrightarrow{\overline{d_L(\Sigma_{R\Gamma,V})}\cdot\overline{d_L(\Sigma_{\mathrm{dR},V})}} d_L(R\Gamma(\mathbb{Q}_p, V)) d_L(D_{\mathrm{dR}}(V)).\end{aligned}$$

Remark 2.2.4. Another way of defining $\theta(V)$, which is popular in the litarture (see [Nak17] before lemma 3.4, footnote 23 in [Ven05b] or (2.2) in [BB05a]), is via a long exact sequence. We recall this definition briefly and convince ourselves, that it in fact yields the same morphism $\theta(V)$. We simplify our notation by omitting the coefficients from the various cohomology groups and the local Tate dualities when they are understood to be $\mathbb{Q}_p$ and write, for instance, $H^0(V) = H^0(\mathbb{Q}_p, V)$ and $\psi^1_f(V) = \psi_f(\mathbb{Q}_p, V)$. We have the exact sequence (1.1):

$$0 \to H^0(V) \to D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V) \oplus t(V) \xrightarrow{\exp_V} H^1_f(V) \to 0 \tag{2.1}$$

and the dual of that sequence for $V^*(1)$:

$$0 \to H^1_f(V^*(1))^* \to D_{\mathrm{cris}}(V^*(1))^* \oplus t(V^*(1))^* \to D_{\mathrm{cris}}(V^*(1))^* \to H^0(V^*(1))^* \to 0. \tag{2.2}$$

We merge both of them via the short exact sequence

$$0 \to H^1_f(V) \xrightarrow{i} H^1(V) \xrightarrow{\psi^1_f(V^*(1))^*\circ p} H^1_f(\mathbb{Q}_p, V^*(1))^* \to 0$$

where $p : H^1(V) \twoheadrightarrow H^1(V)/H^1_f(V)$ and obtain a long exact sequence

$$\begin{aligned}\Sigma_{l,V} : 0 &\to H^0(V) \to D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V) \oplus t(V) \to H^1(V)\\&\to D_{\mathrm{cris}}(V^*(1))^* \oplus t(V^*(1))^* \to D_{\mathrm{cris}}(V^*(1))^* \to H^0(V^*(1))^* \to 0.\end{aligned} \tag{2.3}$$

This sequence induces a morphism in the determinant category

$$\begin{aligned}&d_L(D_{\mathrm{cris}}(V))d_L(D_{\mathrm{cris}}(V))^{-1}d_L(D_{\mathrm{cris}}(V^*(1))^*)d_L(D_{\mathrm{cris}}(V^*(1))^*)^{-1}\\&\to d_L(H^0(V))d_L(H^1(V))^{-1}d_L(H^0(V^*(1))^*)d_L(t(V))d_L(t(V^*(1)^*).\end{aligned}$$

One can construct it by breaking the long exact sequence up into short ones, apply the determinant functor to them and multiply the resulting isomorphisms with alternating exponent. We obtain a morphism between the same objects as $\theta(V)$ if we trivialise $d_L(D_{\mathrm{cris}}(V))$ and $d_L(D_{\mathrm{cris}}(V^*(1))^*)$ and use the local Tate duality $\psi^0(V^*(1))^* : H^0(V^*(1))^* \cong H^2(V)$, the duality $\psi^{-1}_{\mathrm{dR},V} : t(V^*(1))^* \cong D^0_{\mathrm{dR}}(V)$ and the short exact sequence $\Sigma_{\mathrm{dR},V}$. Finally, since all the appearing cohomology groups have finite L-dimension, we have a canonical isomorphism $\bigotimes_{i\in\mathbb{Z}} d_L(H^i(V)) \cong d_L(R\Gamma(V))$ (see proposition 1.1.42 (5), compare also [BB05b] Proposition 3.1).
The resulting isomorphism $\mathbb{1} \to d_L(R\Gamma(V))d_L(D_{\mathrm{dR}}(V))$ is indeed $\theta(V)$. The isomorphism $\mathbb{1} \to d_L(H^0(V))d_L(H^1_f(V))^{-1}d_L(t(V))$ induced by the sequence (2.1) is up to the canonical identification $d_L(R\Gamma_f(V))) \cong d_L(H^0(V))d_L(H^1_f(V))^{-1}$ the same as $\eta(V)$. Similarly for the dual sequence (2.2). The sequence that we used to merge (2.1) and (2.2) contains $\psi^1_f(V^*(1))^*$ and we used $\psi^0(V^*(1))^*$ as well, which corresponds to the application of $\psi_f(V^*(1))^*$ in the first definition of $\theta(V)$. By [BB05b] theorem 3.3, the canonical identifications between $R\Gamma_f(V)$, $R\Gamma(V)$, $R\Gamma_f(V^*(1))^*$ and their respective cohomology groups are compatible with the exact sequence

$$0 \to R\Gamma_f(V) \to R\Gamma(V) \xrightarrow{\psi_f(V^*(1))^* \circ p} R\Gamma_f(V^*(1))^* \to 0$$

and the induced long exact cohomology sequence. This shows that both definitions of $\theta(V)$ agree.

It remains to construct $\varepsilon^{\mathrm{dR}}_{L,\xi}(V)$. This will be the part related to Deligne's local ε-factors. We recall that for a L-linear de Rham representation V and $\sigma : L \hookrightarrow \overline{\mathbb{Q}_p}$, we have the Weil-Deligne representation $D_{\mathrm{pst}}(V)_\sigma = \overline{\mathbb{Q}_p} \otimes_{A,\sigma} D_{\mathrm{pst}}(V)$ of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}_p}$ (see 1.2.37).

Definition 2.2.5. *For each $\sigma : L \hookrightarrow \overline{\mathbb{Q}_p}$ and ξ a basis of $\mathbb{Z}_p(1)$, we define a factor*

$$\varepsilon_L(D_{\mathrm{pst}}(V), \xi)_\sigma = \varepsilon(D_{\mathrm{pst}}(V)_\sigma, \xi) \cdot \det_{\overline{\mathbb{Q}_p}}\left(-\mathrm{Fr}\,\Big|(D_{\mathrm{pst}}(V)_\sigma)^I \,/\, \big(D_{\mathrm{pst}}(V)_\sigma)^I\big)^{N=0}\right),$$

in $\overline{\mathbb{Q}_p}^\times$, where Fr *is any geometric Frobenius and collect all of these ε-factors into one element*

$$\varepsilon_L(D_{\mathrm{pst}}(V), \xi) \in (\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L)^\times \cong \prod_\sigma \overline{\mathbb{Q}_p}^\times$$
$$x \otimes y \mapsto (x\sigma(y))_\sigma.$$

Remark 2.2.6. Note that this definition is the one given on page 35 of [Nak17] and different from the one in [FK06] 3.3.4. Only with this definition, the ε-isomorphisms for de Rahm representations will be multiplicative in general. Compare also remark 3.6 in [Nak17], where this problem is identified. A more conceptual reason than the fact that multiplicativity should hold is that $D_{\mathrm{pst}}(V)$ is a Weil-Deligne representation and for those it is common to include the correction factor (see for instance [Roh94] §11 "delta-factor" or 4.1.6 of [Tat79] and in slightly different form 8.12 in [Del73]).

Lemma 2.2.7. *The element* $\det_{\overline{\mathbb{Q}_p}}\big(-\mathrm{Fr}\,\big|D_{\mathrm{pst}}(V)^I_\sigma/(D_{\mathrm{pst}}(V)^I_\sigma)^{N=0}\big)_\sigma$ *of* $\prod_\sigma \overline{\mathbb{Q}_p}^\times$ *is the same as* $1 \otimes \det_L(-\varphi|\, D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V)) \in \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$ *under the isomorphism* $\prod_\sigma \overline{\mathbb{Q}_p} \cong \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$ *in lemma 1.2.4.*

Proof. The I-action commutes with N on D_{pst}. Together with lemma 1.2.38 this yields

$$D_{\mathrm{pst}}(V)^I_\sigma/({D_{\mathrm{pst}}}_\sigma(V)^I)^{N=0} = \left(D_{\mathrm{pst}}(V)^I/(D_{\mathrm{pst}}(V)^I)^{N=0}\right)_\sigma \cong \overline{\mathbb{Q}_p} \otimes_{A,\sigma} \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V).$$

So we get

$$\begin{aligned}\det_{\overline{\mathbb{Q}_p}}(-\mathrm{Fr}|\, D_{\mathrm{pst}}(V)^I_\sigma/(D_{\mathrm{pst}}(V)^I_\sigma)^{N=0}) &= \det_{\overline{\mathbb{Q}_p}}\left(-\mathrm{Fr}\,|\overline{\mathbb{Q}_p} \otimes_{A,\sigma} \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V)\right) \\ &= \sigma(\det_L(-\mathrm{Fr}|\, D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V))).\end{aligned}$$

The last equality stems from the fact that tensoring along a ring homomorphism changes a determinant by that ring homomorphism. But the element $(\sigma(\det_L(-\mathrm{Fr}|\, D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V))))_\sigma$ in $\prod_\sigma \overline{\mathbb{Q}_p}^\times$ corresponds to the element $1 \otimes \det_L(-\mathrm{Fr}|\, D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V))$ in $(\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L)^\times$. Finally, by lemma 1.2.38 the linearised action of Fr on $D_{\mathrm{st}}(V)$ and $D_{\mathrm{cris}}(V)$ is the same as φ. □

So far, we only have defined a factor, but we want an isomorphism

$$\varepsilon^{\mathrm{dR}}_{L,\xi}(V) : \mathbb{1}_{\widetilde{L}} \to \left(d_L(D_{\mathrm{dR}}(V))^{-1} \cdot d_L(V)\right)_{\widetilde{L}}.$$

We relate $D_{\mathrm{dR}}(V)$ and V via the canonical isomorphism for de Rham representations.

Definition 2.2.8. *For an L-linear de Rham representation V of $G_{\mathbb{Q}_p}$, the canonical isomorphism (denoted α in theorem 1.2.3)*

$$\begin{array}{lcl}\mathrm{can}: & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L \otimes_L D_{\mathrm{dR}}(V) \;\cong\; B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \;\cong\; B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L \otimes_L V \\ & b \otimes l \otimes \sum_i b_i \otimes v_i \longmapsto \sum_i bb_i \otimes l \otimes v_i\end{array}$$

is an isomorphism of $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-modules. Thus, it induces a morphism

$$\mathrm{can} : d_L(D_{\mathrm{dR}}(V))_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L} \to d_L(V)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$$

in the determinant category $V(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)$.

Definition 2.2.9. *Let t be the uniformiser of B^+_{dR} corresponding to the chosen base ξ. Then we define $\varepsilon^{\mathrm{dR}}_{L,\xi}(V) \in V(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)\left(\mathbb{1}, \left(d_L(D_{\mathrm{dR}}(V))^{-1} \cdot d_L(V)\right)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}\right)$ as the element*

$$t^{-t_H(V)}\varepsilon_L(D_{\mathrm{pst}}(V), \xi) \cdot \mathrm{can}$$

multiplied with the identity of $d_L(D_{\mathrm{dR}}(V))^{-1}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$ and pre-composed with $\overline{\mu_{d_L(D_{\mathrm{dR}}(V))}}$. Here, $t \in B^\times_{\mathrm{dR}}$ and $\varepsilon_L(D_{\mathrm{pst}}(V), \xi) \in (\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L)^\times$ act as elements of $(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)^\times = K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)$ (see 1.1.8) on can.

Remark 2.2.10. Let $S := V(L)\big(d_L(D_{\mathrm{dR}}(V)), d_L(V)\big)$. By lemma 1.1.25, we know that

$$S \overset{K_1(L)}{\times} K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) \cong V(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)\left(d_L(D_{\mathrm{dR}}(V))_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}, d_L(V)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}\right),$$

where (f, α) is sent to $\alpha \cdot f_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$.

Now, we have defined all the ingredients for $\varepsilon_{L,\xi}(V)$. But $\varepsilon^{\mathrm{dR}}_{L,\xi}(V)$ only lies in $V(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)\left(d_L(D_{\mathrm{dR}}(V)))_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}, (d_L(V))_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}\right)$. However, we want $\varepsilon_{L,\xi}(V)$ to live in the determinant category of $\widetilde{L}$. This will be ensured by the next proposition.

Proposition 2.2.11. *Using the inclusion $\widehat{\mathbb{Q}_p^{nr}} \subset B_{\mathrm{dR}}$ we have that $t^{-t_H(V)}\varepsilon_L(D_{\mathrm{pst}}(V),\xi)\cdot \mathrm{can}$ lies in $S \overset{K_1(L)}{\times} K_1(\widetilde{L}) \subset S \overset{K_1(L)}{\times} K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)$. In other words, $\varepsilon^{\mathrm{dR}}_{L,\xi}(V)$ is a morphism in $V(\widetilde{L})\left(\mathbb{1}, d_L(D_{\mathrm{dR}}(V)^{-1}\cdot d_L(V)_{\widetilde{L}}\right)$.*

Proof. We have that $K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)^\times$ and $K_1(\widetilde{L}) = \widetilde{L}^\times$ by example 1.1.7 and lemma 1.1.8. Consider the following action of $G_{\mathbb{Q}_p}$ on

$$S \overset{K_1(L)}{\times} K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) = S \overset{K_1(L)}{\times} (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)^\times : \sigma((x,y)) = (x, (\sigma \otimes \mathrm{id})(y)).$$

Recall that $B_{\mathrm{dR}}^{G_{\mathbb{Q}_p^{nr}}} = \widehat{\mathbb{Q}_p^{nr}}$ (1.2.11 (5)) and hence $(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)^{G_{\mathbb{Q}_p^{nr}}} = \widetilde{L}$. So, it suffices to prove that $\varepsilon^{\mathrm{dR}}_{L,\xi}(V)$ is $G_{\mathbb{Q}_p^{nr}}$-invariant.

Claim 1: $\tau(\mathrm{can}) = \det_L(\tau|V)^{-1}\cdot \mathrm{can}$ for any $\tau \in G_{\mathbb{Q}_p}$.
Let $f : D_{\mathrm{dR}}(V) \to V$ be an L-isomorphism, which exists since V is de Rham and hence both sides have the same L-dimension. Under the isomorphism in 2.2.10 can corresponds to $(d_L(f),\alpha)$ for some $\alpha \in K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)$. The image of the $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-linear isomorphism

$$g \coloneqq \alpha \otimes f : (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) \otimes_L D_{\mathrm{dR}}(V) \to (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) \otimes_L V, 1 \otimes d \mapsto \alpha \otimes f(d)$$

under the determinant functor also corresponds to $(d_L(f),\alpha)$. This is because the determinant of multiplication by α is just α and the action of $K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)$ on $(d_L(f),1)$ is described in lemma 1.1.24. Hence, the image of $\mathrm{can}\circ g^{-1}$ under $d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$ is the identity of $d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. In other words, this $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-isomorphism has determinant 1 in $K_1(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L)$. But then the determinant of the $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-isomorphism $(\tau \otimes \mathrm{id} \circ \mathrm{can} \circ \tau^{-1} \otimes \mathrm{id}) \circ (\tau \otimes \mathrm{id} \circ g \circ \tau^{-1} \otimes \mathrm{id})^{-1}$ is one as well by lemma 1.1.11. Here, τ acts just on B_{dR}. But that means that the two $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-isomorphisms $\tau \otimes \mathrm{id} \circ \mathrm{can} \circ \tau^{-1} \otimes \mathrm{id}$ and $\tau \otimes \mathrm{id} \circ g \circ \tau^{-1} \otimes \mathrm{id}$ have the same image under the determinant functor $d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$. This is not changed by post-composing the $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L$-isomorphism $\mathrm{id} \otimes \tau$, where τ acts on V. But since can is equivariant with respect to the action of $G_{\mathbb{Q}_p}$ as $\tau \otimes \mathrm{id}$ on the left and diagonally on the right, we have that $\tau \otimes \tau \circ \mathrm{can} \circ \tau^{-1} \otimes \mathrm{id} = \mathrm{can}$. Hence, the determinants of can and $\tau \otimes \tau \circ g \circ \tau^{-1} \otimes \mathrm{id}$ are the same. We compute the latter. The action of τ on V adds the factor $\det_L(\tau|V)$. The remaining map $\tau \otimes \mathrm{id} \circ g \circ \tau^{-1} \otimes \mathrm{id}$ is given by

$$(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) \otimes_L D_{\mathrm{dR}}(V) \to (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L) \otimes_L V, b \otimes d \mapsto b\tau(\alpha) \otimes f(d).$$

So its determinant is $(d_L(f), \tau(\alpha))$, which is just $\tau(\mathrm{can})$.

Claim 2: Let $\tau \in G_{\mathbb{Q}_p}$ and $\tau' \in G_{\mathbb{Q}_p^{nr}} = I$ such that $\chi_{cycl}(\tau) = \chi_{cycl}(\tau')$, then

$$(\tau \otimes 1)\varepsilon_L(D_{\mathrm{pst}}(V),\xi) = \det_A(\tau' |\, D_{\mathrm{pst}}(V)^\tau)\varepsilon_L(D_{\mathrm{pst}}(V)^\tau, \xi).$$

Here $\tau \otimes 1$ acts on $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$ and $D_{\mathrm{pst}}(V)^\tau \coloneqq A \otimes_{A,\tau\otimes 1} D_{\mathrm{pst}}(V)$, where

$$\tau \otimes 1 = \tau|_{\mathbb{Q}_p^{nr}} \otimes 1 : A = \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L \to \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L = A.$$

In particular, if τ already lies in I, such that its restriction to $\mathbb{Q}_p^{nr}$ is trivial, we get that

$$(\tau \otimes 1)\varepsilon_L(D_{\mathrm{pst}}(V),\xi) = \det_A(\tau |\, D_{\mathrm{pst}}(V))\varepsilon_L(D_{\mathrm{pst}}(V), \xi).$$

In order to prove this claim, we recall the superscript notation from theorem 2.1.1 (6) according to which for a $\overline{\mathbb{Q}_p}$-linear $W_{\mathbb{Q}_p}$-representation D the representation $\overline{\mathbb{Q}_p} \otimes_\tau D$ is denoted D^τ. Now, let $\sigma : L \hookrightarrow \overline{\mathbb{Q}_p}$ be a $\mathbb{Q}_p$-linear embedding. As above, such an embedding yields an embedding $A = \mathbb{Q}_p^{nr} \otimes_{\overline{\mathbb{Q}_p}} L \hookrightarrow \overline{\mathbb{Q}_p}$ by acting on L. We observe that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\tau^{-1}\sigma"] \arrow[d, "\tau\otimes\mathrm{id}"'] & \overline{\mathbb{Q}_p} \arrow[d, "\tau"] \\ A \arrow[r, "\sigma"] & \overline{\mathbb{Q}_p} \end{tikzcd} \qquad \begin{tikzcd} x \otimes y \arrow[r, mapsto] \arrow[d, mapsto] & x\tau^{-1}\sigma(y) \arrow[d, mapsto] \\ \tau(x) \otimes y \arrow[r, mapsto] & \tau(x)\sigma(y). \end{tikzcd}$$

As a result, we get the relation

$$(D_{\mathrm{pst}}(V)_{\tau^{-1}\sigma})^\tau = \overline{\mathbb{Q}_p} \otimes_{\tau,\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\tau^{-1}\sigma,A} D_{\mathrm{pst}}(V) \cong \overline{\mathbb{Q}_p} \otimes_{\sigma,A} A \otimes_{\tau\otimes 1,A} D_{\mathrm{pst}}(V) = (D_{\mathrm{pst}}(V)^\tau)_\sigma$$

between the two superscript notations and the subscript notation from corollary 1.2.37. First, we take care of the ε-factors in the definition of $\varepsilon_L(D_{\mathrm{pst}}(V), \xi)$. By the equivariance statement in lemma 1.2.4 (3) we get the first equality in

$$\begin{aligned} (\tau \otimes 1)\varepsilon(D_{\mathrm{pst}}(V)_\sigma, \xi) &= \tau\big(\varepsilon(D_{\mathrm{pst}}(V)_{\tau^{-1}\sigma}, \xi)\big) \\ &= \varepsilon\big((D_{\mathrm{pst}}(V)_{\tau^{-1}\sigma})^\tau, \tau(\xi)\big) \\ &= \varepsilon\big((D_{\mathrm{pst}}(V)^\tau)_\sigma, \tau(\xi)\big) \\ &= \varepsilon\big((D_{\mathrm{pst}}(V)^\tau)_\sigma, \chi_{cycl}(\tau')\xi\big) \\ &= \sigma\big(\det_A(\tau' |\, D_{\mathrm{pst}}(V)^\tau)\big)\varepsilon\big((D_{\mathrm{pst}}(V)^\tau)_\sigma, \xi\big). \end{aligned}$$

From the first to the second line, we used property (6) of theorem 2.1.1, from the second to the third the relation of the two super-/subscript notations and from the third to the fourth line the fact that τ and τ' act in the same way on $\mathbb{Z}_p(1)$. The final equality sign stems from part (2) of theorem 2.1.1. The factors $\sigma\big(\det_A(\tau' |\, D_{\mathrm{pst}}(V)^\tau)\big) \in \overline{\mathbb{Q}_p}$ for all σ yield the element $\det_A(\tau' |\, D_{\mathrm{pst}}(V)^\tau) \in A \subset \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$ as claimed.
Second, we consider the correction factor. By lemma 2.2.7, we can consider the correction factor as an element of L inside $\overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$. Therefore, $(\tau \otimes \mathrm{id})$ is trivial on the correction factor.
This finishes the proof of claim 2.

Claim 3: $\det_A(\tau |\, D_{\mathrm{pst}}(V)) = \chi_{cycl}(\tau)^{t_H(V)} \det_L(\tau | V)$ for all $\tau \in I$.
Multiplication with $\det_A(\tau |\, D_{\mathrm{pst}}(V))$ (or $\det_L(\tau|V)$) is the operation of τ on the maximal exterior product $\bigwedge_A D_{\mathrm{pst}}(V) \cong D_{\mathrm{pst}}(\bigwedge_L V)$ (or $\bigwedge_L V$), see the compatibility of $D_{\mathrm{st},K}$ with tensor products and quotients for semi-stable G_K-representations in theorem 1.2.3. $t_H(V)$ is the Hodge-Tate weight of $\bigwedge_L D_{\mathrm{dR}}(V) \cong D_{\mathrm{dR}}(\bigwedge_L V)$ (filtrations align due to proposition 1.2.24). So, we can assume that V is of L-dimension one. We now argue similarly as in [FO18] 7.16 and 7.17. Since $G_{\mathbb{Q}_p}$ acts L-linearly on V, the action comes from a character $\eta : G_{\mathbb{Q}_p} \to L^\times$. Each $D^i_{\mathrm{dR}}(V)$ is an L-vector space. Since V is de Rham and one-dimensional, we have that

$$D_{\mathrm{dR}}^{t_H(V)}(V)/D_{\mathrm{dR}}^{t_H(V)+1}(V) \cong (\mathbb{C}_p(t_H(V)) \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}} \cong (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V(t_H(V)))^{G_{\mathbb{Q}_p}}$$

has L-dimension 1. In other words $V(t_H(V))$ is $\mathbb{C}_p$-admissible as $\mathbb{Q}_p$-representation. But by [FO18] proposition 3.56, this means that I acts discretely on $V(t_H(V))$, i.e. $\left(\chi_{cycl}^{t_H(V)} \cdot \eta\right)(I)$ is finite. Therefore, we can write $\eta = \chi_{cycl}^{-t_H(V)} \cdot \chi^{nr} \cdot \chi$, where χ factors over some finite extension $E/\mathbb{Q}_p$ and χ^{nr} is unramified. Hence, $G_{E^{nr}}$ acts trivially on $V(t_H(V))$. The map

$$\vartheta^{-t_H(V)} : B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V(t_H(V)), b \otimes v \mapsto xt^{-t_H(V)} \otimes v \otimes \xi^{t_H(V)}$$

is $G_{\mathbb{Q}_p}$-equivariant by lemma 1.2.28. For a finite extension K/E, Galois over $\mathbb{Q}_p$ this yields a $\mathrm{Gal}(K^{nr}, \mathbb{Q}_p)$-equivariant isomorphism:

$$(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_{K^{nr}}} \cong \big(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V(t_H(V))\big)^{G_{K^{nr}}} = B_{\mathrm{st}}^{G_{K^{nr}}} \otimes_{\mathbb{Q}_p} V(t_H(V)) = \widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} V(t_H(V)),$$

where we have used 1.2.15 (4) at the last and the fact that $G_{K^{nr}}$ acts trivially on $V(t_H(V))$ at the second to last equality sign.

Using that E is finite over $\mathbb{Q}_p$, that χ is trivial on G_E and writing $V(t_H(V)) = L(\chi\chi^{nr})$, we conclude that there is a $G_{\mathbb{Q}_p}$-isomorphism

$$\begin{aligned} D_{\mathrm{pst}}(V) &= \bigcup_{\substack{\mathbb{Q}_p \subset K \subset \overline{\mathbb{Q}_p}, \\ K/\mathbb{Q}_p \text{ finite}}} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \bigcup_{\substack{E \subset K \subset \overline{\mathbb{Q}_p}, \\ K/\mathbb{Q}_p \text{ finite} \\ \text{Galois}}} (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ &= \bigcup_{\substack{E \subset K \subset \overline{\mathbb{Q}_p}, \\ K/\mathbb{Q}_p \text{ finite} \\ \text{Galois}}} \big((B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_{K^{nr}}}\big)^{\mathrm{Gal}(K^{nr},K)} = \bigcup_{\substack{E \subset K \subset \overline{\mathbb{Q}_p}, \\ K/\mathbb{Q}_p \text{ finite} \\ \text{Galois}}} \Big(\widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} L(\chi^{nr}\chi)\Big)^{\mathrm{Gal}(K^{nr},K)} \\ &\cong \bigcup_{\substack{E \subset K \subset \overline{\mathbb{Q}_p}, \\ K/\mathbb{Q}_p \text{ finite} \\ \text{Galois}}} \Big(\widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} L(\chi^{nr})\Big)^{\mathrm{Gal}(K^{nr},K)} (\chi). \end{aligned}$$

This description of $D_{\mathrm{pst}}(V)$ shows that $\tau \in I = G_{\mathbb{Q}_p^{nr}}$ acts on $D_{\mathrm{pst}}(V)$ via the character χ. On V such a τ acts via $\chi\chi_{cycl}^{-t_H(V)}$. This establishes the claimed formula.

Combining the three claims, we see that an element $\tau \in G_{\mathbb{Q}_p^{nr}}$ acts on $\varepsilon_L(D_{\mathrm{pst}}(V), \xi)$ can by multiplication with $\chi_{cycl}(\tau)^{t_H(V)}$. The factor $t^{-t_H(V)}$ in the definition of $\varepsilon_{L,\xi}^{\mathrm{dR}}(V)$ corrects this, so that $\varepsilon_{L,\xi}^{\mathrm{dR}}(V)$ is fixed by $G_{\mathbb{Q}_p^{nr}}$ and thus exists already in the determinant category over $\widetilde{L}$. $\square$

2.2.2 Properties of ε-isomorphisms of de Rham representations

Let us now verify the properties stated for $\varepsilon_{L,\xi}(L)$ in proposition 2.2.2. The first two properties follow without much work from the proof of proposition 2.2.11.

Proposition 2.2.12 (Change of ξ). *Let $\tau \in I(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$. Then $\varepsilon_{L,\chi_{cycl}(\tau)\xi}(V) = \det_L(\tau|V)\varepsilon_{L,\xi}(V)$.*

Proof. The only part of $\varepsilon_{L,\xi}(V)$ that depends on ξ is the choice of t and the local factors $\varepsilon(D_{\mathrm{pst}}(V)_\sigma, \xi)$ for each $\sigma : L \hookrightarrow \overline{\mathbb{Q}_p}$. By theorem 2.1.1 (2), we have

$$\varepsilon(D_{\mathrm{pst}}(V)_\sigma, \chi_{cycl}(\tau)\xi) = \det_{\overline{\mathbb{Q}_p}}(\tau|\, D_{\mathrm{pst}}(V)_\sigma)\varepsilon(D_{\mathrm{pst}}(V)_\sigma, \xi).$$

We can write $\det_{\overline{\mathbb{Q}_p}}(\tau|\, D_{\mathrm{pst}}(V)_\sigma) = \sigma(\det_A(\tau|\, D_{\mathrm{pst}}(V)))$, where σ stands for the map $\mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L \to \overline{\mathbb{Q}_p}, x \otimes y \mapsto x\sigma(y)$. Hence, as element of $\prod_\sigma \overline{\mathbb{Q}_p} \cong \overline{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} L$, we have $\det_{\overline{\mathbb{Q}_p}}(\tau|\, D_{\mathrm{pst}}(V)_\sigma)_\sigma = \det_A(\tau|\, D_{\mathrm{pst}}(V)$. Together with the factor from $t^{-t_H(V)}$, we have

$$\varepsilon_{L,\chi_{cycl}(\tau)}(V) = \chi_{cycl}(\tau)^{-t_H(V)} \cdot \det_A(\tau|\, D_{\mathrm{pst}}(V)) \cdot \varepsilon_{L,\xi}(V).$$

The appearing factor is nothing but $\det_L(\tau|V)$ by claim 3 in the proof of proposition 2.2.11. $\square$

Lemma 2.2.13. *Let ϕ be a Frobeniuslift and V an L-linear de Rham representation. Then there is an isomorphism of A-linear $W_{\mathbb{Q}_p}$-representations between $D_{\mathrm{pst}}(V)^\phi = A \otimes_{\phi\otimes \mathrm{id},A} D_{\mathrm{pst}}(V)$ and $D_{\mathrm{pst}}(V)$. In particular, one has $(D_{\mathrm{pst}}(V)^\phi)^I \cong D_{\mathrm{pst}}(V)^I$ and in addition $\big((D_{\mathrm{pst}}(V)^\phi)^I\big)^{N=0} \cong (D_{\mathrm{pst}}(V)^I)^{N=0}$.*

Proof. The isomorphism $D_{\mathrm{pst}}(V)^\phi \xrightarrow{\sim} D_{\mathrm{pst}}(V)$ is given by

$$\begin{aligned} D_{\mathrm{pst}}(V)^\phi = A \otimes_{\phi\otimes\mathrm{id},A} D_{\mathrm{pst}}(V) \cong \mathbb{Q}_p^{nr} \otimes_{\phi,\mathbb{Q}_p^{nr}} D_{\mathrm{pst}}(V) &\to D_{\mathrm{pst}}(V) \\ q\otimes d &\mapsto q\varphi(d). \end{aligned}$$

This is clearly $\mathbb{Q}_p^{nr}$-linear and since φ is L-linear, it is an A-homomorphism. The last map sends $\phi(q)\otimes d$ and $1\otimes qd$ to $\phi(q)\varphi(d)$ and $\varphi(qd)$ respectively. As φ is ϕ-semi-linear, these elements are equal and the map well-defined. It is a bijection because φ is one on each $D_{\mathrm{st},K}(V)$. The map clearly commutes with application of φ and also with the naive $W_{\mathbb{Q}_p}$-action since φ does. Hence, it commutes with the linearised action of $W_{\mathbb{Q}_p}$. Finally, the relation $N\varphi = p\varphi N$ yields the commutativity of the following diagram

$$\begin{tikzcd} D_{\mathrm{pst}}(V)^\phi \arrow[r, "\sim"] \arrow[d, "N"'] & D_{\mathrm{pst}}(V) \arrow[d, "p^{-1}N"] \\ D_{\mathrm{pst}}(V)^\phi \arrow[r, "\sim"] & D_{\mathrm{pst}}(V). \end{tikzcd}$$

Multiplication with p is an isomorphism on $D_{\mathrm{pst}}(V)$, so we get $\big((D_{\mathrm{pst}}(V)^\phi)^I\big)^{N=0} = (D_{\mathrm{pst}}(V)^I)^{N=0}$. $\square$

Proposition 2.2.14 (Frobenius invariance)**.** *Let $\phi \in G_{\mathbb{Q}_p}$ be a Frobeniuslift with $\chi_{cycl}(\phi) = 1$. Denote by $\varphi_p := \phi|_{\widehat{\mathbb{Z}_p^{nr}}} : \widehat{\mathbb{Z}_p^{nr}} \to \widehat{\mathbb{Z}_p^{nr}}$ the restriction of $\phi : B_{\mathrm{dR}} \to B_{\mathrm{dR}}$. The ring homomorphism $\varphi_p \otimes \mathrm{id}_L : \widetilde{L} \to \widetilde{L}$ induces a base change homomorphism $(\varphi_p\otimes\mathrm{id}_L)^* : K_1(\widetilde{L}) \to K_1(\widetilde{L})$ (see lemma 1.1.10). We put $K_1(\widetilde{L})_V = \{x \in K_1(\widetilde{L}) | (\varphi_p\otimes\mathrm{id})^*(x) = \det_L(\phi|V)^{-1}x\}$. Then the ε-isomorphism $\varepsilon_{L,\xi}(V)$ belongs to $V(L)\big(\mathbb{1}, d_L(R\Gamma(\mathbb{Q}_p,L)d_L(V)\big) \overset{K_1(L)}{\times} K_1(\widetilde{L})_V$.*

Proof. The factor $\Gamma_L(V)$ is a rational number and hence not influenced by $(\varphi_p\otimes\mathrm{id})^*$. The isomorphism $\theta(V)$ is defined in the determinant category over L, so that after base change to $\widetilde{L}$ its contribution to $K_1(\widetilde{L})$ is an element of $K_1(L)$ on which $(\varphi_p\otimes\mathrm{id})^*$ is also trivial. So it remains to prove that $\varepsilon^{\mathrm{dR}}_{L,\xi}(V)$ lies in $V(L)\big(\mathbb{1}, d_L(D_{\mathrm{dR}}(V))^{-1}d_L(V)\big) \overset{K_1(L)}{\times} K_1(\widetilde{L})_V$. In other words, using the action defined in the proof of proposition 2.2.11, we have to show that $\phi(\varepsilon^{\mathrm{dR}}_{L,\xi}(V)) = \det_L(\phi|V)^{-1}\varepsilon^{\mathrm{dR}}_{L,\xi}(V)$. By claim 1 of the proof of proposition 2.2.11, we have $\phi(\mathrm{can}) = \det_L(\phi|V)^{-1}\,\mathrm{can}$. By claim 2, we have $(\phi\otimes\mathrm{id})\varepsilon_L(D_{\mathrm{pst}}(V),\xi) = \varepsilon_L(D_{\mathrm{pst}}(V)^\phi,\xi)$ since $\chi_{cycl}(\phi) = 1$. The A-linear $W_{\mathbb{Q}_p}$-isomorphism $D_{\mathrm{pst}}(V)^\phi \cong D_{\mathrm{pst}}(V)$ in lemma 2.2.13 ensures that the factors $\varepsilon(D_{\mathrm{pst}}(V)^\phi,\xi)_\sigma$ and $\varepsilon(D_{\mathrm{pst}}(V),\xi)_\sigma$ agree for every embedding $L \hookrightarrow \overline{\mathbb{Q}_p}$, as local constants are insensitive to isomorphic representations (2.1.1 (1)). Since the isomorphism respects the kernel of the monodromy operator, we get that the correction factor is also the same for $D_{\mathrm{pst}}(V)$ and $D_{\mathrm{pst}}(V)^\phi$. Finally, since $\chi_{cycl}(\phi) = 1$, ϕ is trivial on t. $\square$

We have to work a little harder for the multiplicativity and the duality. The next lemma remedies the missing exactness of D_{cris} on short exact sequences of de Rham representations.

Lemma 2.2.15. *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of de Rham representations. Then we have an exact sequence*

$$0 \to D_{\mathrm{cris}}(V_1) \to D_{\mathrm{cris}}(V_2) \to D_{\mathrm{cris}}(V_3) \xrightarrow{\delta} D_{\mathrm{cris}}(V_1^*(1))^* \to D_{\mathrm{cris}}(V_2^*(1))^* \to D_{\mathrm{cris}}(V_3^*(1))^* \to 0,$$

denoted Σ_{cris}, of L-vector spaces (all duals are L-duals). Moreover, the diagram

$$\begin{tikzcd} D_{\mathrm{cris}}(V_3) \arrow[r, "\delta"] \arrow[d, "\varphi"'] & D_{\mathrm{cris}}(V_1^*(1))^* \arrow[d, "(\varphi^{-1})^*"] \\ D_{\mathrm{cris}}(V_3) \arrow[r, "\delta"] & D_{\mathrm{cris}}(V_1^*(1))^* \end{tikzcd}$$

commutes.

Proof. Since all the V_i's are de Rham the first part of lemma 1.2.39 ensures the exactness of the rows of the following diagram of L-vector spaces:

$$\begin{tikzcd} 0 \arrow[r] & D_{\mathrm{st}}(V_1) \arrow[r] \arrow[d, "N"'] & D_{\mathrm{st}}(V_2) \arrow[r] \arrow[d, "N"'] & D_{\mathrm{st}}(V_3) \arrow[r] \arrow[d, "N"'] & 0 \\ 0 \arrow[r] & D_{\mathrm{st}}(V_1(-1)) \arrow[r] & D_{\mathrm{st}}(V_2(-1)) \arrow[r] & D_{\mathrm{st}}(V_3(-1)) \arrow[r] & 0. \end{tikzcd}$$

The vertical maps really are $\vartheta \circ N = N \circ \vartheta$, which is φ-equivariant by lemma 1.2.28 (3). The fourth part of that lemma lets us extend this diagram vertically by the exact sequences

$$0 \to D_{\mathrm{cris}}(V_i) \to D_{\mathrm{st}}(V_i) \to D_{\mathrm{st}}(V_i(-1)) \to D_{\mathrm{st}}(V_i(-1))/N\, D_{\mathrm{st}}(V_i(-1)) \to 0. \tag{$\star$}$$

By the snake lemma, we get an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & D_{\mathrm{cris}}(V_1) \arrow[r] & D_{\mathrm{cris}}(V_2) \arrow[r] & D_{\mathrm{cris}}(V_3) \arrow[dll, "\delta'"] \\ & D_{\mathrm{st}}(V_1(-1))/N \arrow[r] & D_{\mathrm{st}}(V_2(-1))/N \arrow[r] & D_{\mathrm{st}}(V_3(-1))/N \arrow[r] & 0 \end{tikzcd}$$

The functor $\mathrm{Hom}_L(-, L)$ is left exact, so that

$$\mathrm{Hom}_L\big(D_{\mathrm{st}}(V_i(-1))/N\, D_{\mathrm{st}}(V_i(-1), L)\big) \cong \mathrm{Hom}_L(D_{\mathrm{st}}(V_i(-1)), L)^{N^*=0} = (D_{\mathrm{st}}(V_i(-1))^*)^{N^*=0}.$$

Now, we use part (2) of lemma 1.2.39 to pull in the dual and conclude that $(D_{\mathrm{st}}(V_i(-1))^*)^{N^*=0} \cong D_{\mathrm{st}}(V_i(-1)^*)^{N=0} = D_{\mathrm{cris}}(V_i^*(1))$. Dualising all of this, we get a natural isomorphism $\delta''_{V_i} : D_{\mathrm{st}}(V_i(-1))/N\, D_{\mathrm{st}}(V_i(-1)) \to D_{\mathrm{cris}}(V_i^*(1))^*$. We apply it to the lower part of the result of the snake lemma to obtain the desired exact sequence. We put $\delta = \delta''_{V_1} \circ \delta'$. Lastly, we investigate the action of φ. The sequences ($\star$) for $i = 1, 2, 3$ are compatible with φ. Hence, δ' commutes with φ. According to part (2) of lemma 1.2.39, pulling the dual into $D_{\mathrm{st}}(-)$ as in δ''_{V_1} does not only transpose but also inverts φ. Hence, we get the desired relation $(\varphi^{-1})^* \circ \delta = \delta \circ \varphi$. □

Corollary 2.2.16. *The morphism $\theta(V) \cdot \det_L(-\varphi|V^*(1))^{-1}$ is multiplicative in short exact sequences of L-linear de Rham representations. That is to say that for $\Sigma : 0 \to V_1 \to V_2 \to V_3 \to 0$ a short exact sequence of L-linear de Rham representations and for the induced exact sequences $\Sigma_C : 0 \to C(\mathbb{Q}_p, V_1) \to C(\mathbb{Q}_p, V_2) \to C(\mathbb{Q}_p, V_3) \to 0$ (lemma 1.3.14) as well as $\Sigma_{\mathrm{dR}} : 0 \to D_{\mathrm{dR}}(V_1) \to D_{\mathrm{dR}}(V_2) \to D_{\mathrm{dR}}(V_3) \to 0$, we have*

$$d_L(\Sigma_C) d_L(\Sigma_{\mathrm{dR}}) \circ \theta(V_2) \cdot \det_L(-\varphi|V_2^*(1)) = \theta(V_1)\theta(V_3)\det_L(-\varphi|V_1^*(1))\det_L(-\varphi|V_3^*(1)).$$

Proof. Since the V_i's are de Rham, applying the functor D_{dR} is an exact functor to the category of filtered $\mathbb{Q}_p$-vector spaces (1.2.24). Hence, proceeding from Σ_{dR} to the associated graded rings is exact as well and we get that $\Sigma_t : 0 \to t(V_1) \to t(V_2) \to t(V_3) \to 0$ is exact. Since Kummer (L-) duals of de Rham representations are de Rham (1.2.3 (4) and 1.2.28), we also get the exactness of the sequence $\Sigma'_t : 0 \to t(V_1^*(1))^* \to t(V_2^*(1))^* \to t(V_3^*(1))^* \to 0$. Adding these exact sequences to two copies of the exact sequence Σ_{cris} in lemma 2.2.15, we get by the compatibility of the latter with φ an exact sequence of complexes $\widetilde{\Sigma}$:

$$0 \longrightarrow C'_f(\mathbb{Q}_p, V_1) \longrightarrow C'_f(\mathbb{Q}_p, V_2) \longrightarrow C'_f(\mathbb{Q}_p, V_3) \longrightarrow$$
$$\longrightarrow \widetilde{C_f}(\mathbb{Q}_p, V_1^*(1))^*[1] \longrightarrow \widetilde{C_f}(\mathbb{Q}_p, V_2^*(1))^*[1] \longrightarrow \widetilde{C_f}(\mathbb{Q}_p, V_3^*(1))^*[1] \longrightarrow 0,$$

with $\widetilde{C_f}(\mathbb{Q}_p, V)$ being the complex $\left[D_{\mathrm{cris}}(V) \xrightarrow{(1-\varphi^{-1},\bar{1})} D_{\mathrm{cris}}(V) \oplus t(V)\right]$ concentrated in degree 0 and 1. The complexes $\widetilde{C_f}(\mathbb{Q}_p, V^*(1))^*$ and $C'_f(\mathbb{Q}_p, V^*(1))^*$ are isomorphic via the complex homomorphism $(-\varphi^*, \mathrm{id})$. The images of $\widetilde{C_f}(\mathbb{Q}_p, V^*(1))^*$ and $C'_f(\mathbb{Q}_p, V^*(1)))^*$ are the same under d_L. The isomorphism $d_L((-\varphi^*, \mathrm{id}))$ is hence an automorphism of this object and as such the element $\det_L(-\varphi|\, D_{\mathrm{cris}}(V^*(1))) \in K_1(L) = L^\times$. If we denote by Σ' the sequence $\widetilde{\Sigma}$ with the $\widetilde{C_f}(\mathbb{Q}_p, V^*(1))^*$ replaced by $C'_f(\mathbb{Q}_p, V^*(1)))^*$, we get that

$$d_L(\widetilde{\Sigma}) = d_L(\Sigma') \prod_{i=1,2,3} \det_L(-\varphi|\, D_{\mathrm{cris}}(V_i^*(1)))^{(-1)^i}.$$

Here, we associated morphisms in the determinant category to the exact sequences $\widetilde{\Sigma}$ and Σ' by breaking up the exact sequence into short exact sequences, multiplying the images of those short exact sequences in the determinant category and trivialising the auxiliary kernels and cokernels. Moreover, we multiply with (inverses) of identities to ensure that both $d_L(\widetilde{\Sigma})$ and $d_L(\Sigma')$ are morphisms

$$\begin{aligned} &d_L(C'_f(\mathbb{Q}_p, V_2)) \cdot d_L(C'_f(\mathbb{Q}_p, V_2^*(1))^*) \to \\ &d_L(C'_f(\mathbb{Q}_p, V_1)) \cdot d_L(C'_f(\mathbb{Q}_p, V_1^*(1))^*) \cdot d_L(C'_f(\mathbb{Q}_p, V_3)) \cdot d_L(C'_f(\mathbb{Q}_p, V_3^*(1))^*), \end{aligned} \tag{†}$$

where we have also used that $d_L(C[1]) = d_L(C)^{-1}$. Together with the natural quasi isomorphism $C_f(\mathbb{Q}_p, V) \xrightarrow{q-iso} C'_f(\mathbb{Q}_p, V)$ from lemma 1.3.23, we get from $d_L(\Sigma')$ a morphism

$$\begin{aligned} f : &d_L(C_f(\mathbb{Q}_p, V_2)) \cdot d_L(C_f(\mathbb{Q}_p, V_2^*(1))^*) \to \\ &d_L(C_f(\mathbb{Q}_p, V_1)) \cdot d_L(C_f(\mathbb{Q}_p, V_1^*(1))^*) \cdot d_L(C_f(\mathbb{Q}_p, V_3)) \cdot d_L(C_f(\mathbb{Q}_p, V_3^*(1))^*). \end{aligned}$$

Now that we have related $d_L(\widetilde{\Sigma})$ with the finite parts of Galois cohomology, we turn our attention to its relation with $\eta(V)$. The morphism $d_L(\widetilde{\Sigma})$ (as written as in (†)) is nothing but $d_L(\Sigma_{\mathrm{cris}})d_L(\Sigma_t)^{-1}d_L(\Sigma_{\mathrm{cris}})^{-1}d_L(\Sigma'_t)^{-1}$ by part (2) of remark 1.1.43. So after multiplying it with $d_L(\Sigma_t)$ and $d_L(\Sigma'_t)$ it is just an instance of a morphism in $V(L)$ multiplied with its inverse. Hence, $d_L(\widetilde{\Sigma})d_L(\Sigma_t)d_L(\Sigma'_t)$ is compatible with trivialising the D_{cris}'s and the tangent spaces t (compare part (3) of 1.1.20). The morphisms $\eta(V)$ and $\overline{\eta(V^*(1))^*}$ are just these trivialisations composed with the quasi-isomorphisms from lemma 1.3.23, which are incorporated in f. As a result, we get that

$$\begin{aligned}
&f \cdot d_L(\Sigma_t) d_L(\Sigma_t') \circ \eta(V_2)\overline{\eta(V_2^*(1))^*}\det_L(-\varphi|\, D_{\mathrm{cris}}(V_2^*(1))^{-1} \\
=&\eta(V_1)\overline{\eta(V_1^*(1))^*}\det_L(-\varphi|\, D_{\mathrm{cris}}(V_1^*(1))^{-1} \cdot \eta(V_3)\overline{\eta(V_3^*(1))^*}\det_L(-\varphi|\, D_{\mathrm{cris}}(V_3^*(1)))^{-1}.
\end{aligned}$$

We have the commutative nine term diagram

$$\begin{array}{cccccc}
 & D^0_{\mathrm{dR}}(\Sigma): & & \Sigma_{\mathrm{dR}}: & & \Sigma_t: \\
\Sigma_{\mathrm{dR},V_1}: & D^0_{\mathrm{dR}}(V_1) & \longrightarrow & D_{\mathrm{dR}}(V_1) & \longrightarrow & t(V_1) \\
 & \downarrow & & \downarrow & & \downarrow \\
\Sigma_{\mathrm{dR},V_2}: & D^0_{\mathrm{dR}}(V_2) & \longrightarrow & D_{\mathrm{dR}}(V_2) & \longrightarrow & t(V_2) \\
 & \downarrow & & \downarrow & & \downarrow \\
\Sigma_{\mathrm{dR},V_3}: & D^0_{\mathrm{dR}}(V_3) & \longrightarrow & D_{\mathrm{dR}}(V_3) & \longrightarrow & t(V_3),
\end{array}$$

which has short exact rows and columns. By part (4) of proposition 1.1.42, determinant functors behave well with respect to such diagrams and we get that

$$d_L(\Sigma_{\mathrm{dR},V_1}) \cdot d_L(\Sigma_{\mathrm{dR},V_3}) \circ d_L(\Sigma_{\mathrm{dR}}) = d_L(D^0_{\mathrm{dR}}(\Sigma)) \cdot d_L(\Sigma_t) \circ d_L(\Sigma_{\mathrm{dR},V_2})$$

and hence by the naturality of $\psi_{dR,V} : D^0_{\mathrm{dR}}(V) \to t(V^*(1))^*$, we get that

$$d_L(\Sigma_t') \circ d_L(\psi_{\mathrm{dR},V_2}) = d_L(\psi_{\mathrm{dR},V_1}) \cdot d_L(\psi_{\mathrm{dR},V_3}) \circ d_L(D^0_{\mathrm{dR}}(\Sigma)).$$

This shows that the part of $\theta(V)$ that relates to $d_L(\Sigma_t)d_L(\Sigma_t')$ is just $d_L(\Sigma_{\mathrm{dR}})$.
A similar argument does not quite work for the Galois cohomology, since the functors $C_f(\mathbb{Q}_p, -)$ are not right-exact, so that there is no proper nine term diagram, just the following diagram with exact rows and columns (we omitted "$\mathbb{Q}_p$" for better readability):

$$\begin{array}{cccccccccc}
 & & & & & \Sigma_C: & & & & \\
 & & & 0 & & 0 & & & & \\
 & & & \downarrow & & \downarrow & & & & \\
\Sigma_1: & 0 & \longrightarrow & C_f(V_1) & \longrightarrow & C(V_1) & \longrightarrow & C_f(V_1^*(1))^*[-2] & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma_2: & 0 & \longrightarrow & C_f(V_2) & \longrightarrow & C(V_2) & \longrightarrow & C_f(V_2^*(1))^*[-2] & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma_3: & 0 & \longrightarrow & C_f(V_3) & \longrightarrow & C(V_3) & \longrightarrow & C_f(V_3^*(1))^*[-2] & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & & .
\end{array}$$

It remains to prove that f is compatible with the morphisms obtained from Σ_C and the Σ_i's. In other words, the commutativity of

$$\begin{tikzcd}
d_L(C(V_2)) \arrow[r, "d_L(\Sigma_C)"] \arrow[d, "d_L(\Sigma_2)"'] & d_L(C(V_1)) \cdot d_L(C(V_3)) \arrow[d, "d_L(\Sigma_1)\cdot d_L(\Sigma_3)"] \\
d_L(C_f(V_2)) \cdot d_L(C_f(V_2^*(1))^*) \arrow[d] & \begin{array}{c} d_L(C_f(V_1)) \cdot d_L(C_f(V_1^*(1))^*) \\ \cdot d_L(C_f(V_3)) \cdot d_L(C_f(V_3^*(1))^*) \end{array} \arrow[d] \\
d_L(C'_f(V_2)) \cdot d_L(C'_f(V_2^*(1))^*) \arrow[r, "f"] & \begin{array}{c} d_L(C'_f(V_1)) \cdot d_L(C'_f(V_1^*(1))^*) \\ \cdot d_L(C'_f(V_3)) \cdot d_L(C'_f(V_3^*(1))^*) \end{array}
\end{tikzcd},$$

where the unlabelled arrows come from the quasi-isomorphisms in 1.3.23, remains to be proven. □

Remark 2.2.17. We strongly believe that the above diagram is commutative and we will assume this in the following. We suspect that the commutativity might be established by a large diagram obtained from breaking $\widetilde{\Sigma}$ up in short exact sequences and using the associativity property of the determinant functor. Alternatively, one could use the canonical identification of the determinant of a complex with the determinants of its cohomology and then break up the long exact cohomology sequences and use the associativity of the determinant functor. This is possible in our case by the results of chapter 3 in [BB05b] since the category of finite dimensional L-vector spaces is not only exact but abelian.
Instead of relying on the associativity of the determinant functor, one could use its value on quasi-isomorphism. By remark 1.1.43 (5), we can assume that $d_L(C_f(V)) = d_L(C'_f(V))$ without loss of generality. Then by the horseshoe lemma applied to the short exact sequences Σ_i, we can construct a projective resolution of $C(V_i)$ that is level wise the same as $C'_f(V_i) \oplus C'_f(V_i^*(1))^*$ and assume again without loss of generality that $d_L(C_f(V_i)) = d_L(C'_f(V_i)) \cdot d_L(C'_f(V_i^*(1))^*)$. Applying the horseshoe lemma to Σ_C, we get a projective resolution of $C(V_2)$ that is levelwise the same as the complex $C'_f(V_1) \oplus C'_f(V_1^*(1))^* \oplus C'_f(V_3) \oplus C'_f(V_3^*(1))^*$. The two projective resolutions of $C(V_2)$ are quasi-isomorphic. The question now is whether the determinant functor applied to this quasi-isomorphism is precisely f. Finally, we would like to point the reader's attention to the anticommutativity of the connecting homomorphism of the long cohomomolgy sequences described in proposition 1.3.4 of [NSW13], which might result in sign issues.

Proposition 2.2.18 (Multiplicativity)**.** *Let $\Sigma : 0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of L-linear de Rham representations of $G_{\mathbb{Q}_p}$, then the ε-isomorphisms are multiplicative:*

$$(d_L(C(\mathbb{Q}_p, \Sigma)) d_L(\Sigma))_{\widetilde{L}} \circ \varepsilon_{L,\xi}(V_2) = \varepsilon_{L,\xi}(V_1) \cdot \varepsilon_{L,\xi}(V_3).$$

Proof.

Claim 1: $\Gamma_L(V_2) = \Gamma_L(V_1)\Gamma_L(V_3)$.
Since Σ is a short exact sequence of L-linear de Rham representations, the functor $D_{\mathrm{dR}}(-)$ maps it to a short exact sequence of filtered L-modules, Σ_{dR}. Putting $h(r)_i = \dim_L gr^r\, D_{\mathrm{dR}}(V_i)$, we thus get $h(r)_2 = h(r)_1 + h(r)_3$. Since $\Gamma_L(V_i) = \prod_{r \in \mathbb{Z}} \Gamma^*(r)^{-h(-r)_i}$ the claim follows.

Claim 2: $t^{-t_H(V)}$ is multiplicative in V.
Since $h(r)$ is additive in short exact sequences of de Rham representations, so is

$t_H(V) = \sum_{r\in\mathbb{Z}} rh(r)_V$. As a result, we get that $t^{-t_H(V)}$ is multiplicative in V.

Claim 3: $\varepsilon_L(V,\xi)\cdot \det_L(-\varphi|\, D_{\mathrm{cris}}(V^*(1)))$ is multiplicative in V.
Since the V_i are de Rham and hence potentially semi-stable the sequence

$$0 \to D_{\mathrm{pst}}(V_1) \to D_{\mathrm{pst}}(V_2) \to D_{\mathrm{pst}}(V_3) \to 0$$

is exact as sequence of linearised $W_{\mathbb{Q}_p}$-representations (see lemma 1.2.39 part (1)). So the multiplicativity of the local constants (2.1.1 (1)) implies the multiplicativity of $\varepsilon(D_{\mathrm{pst}}(V)_\sigma,\xi)$. Furthermore, the sequence

$$0 \to D_{\mathrm{st}}(V_1) \to D_{\mathrm{st}}(V_2) \to D_{\mathrm{st}}(V_3) \to 0$$

is exact by part (1) of lemma 1.2.39. This shows that $\det_L(-\varphi|\, D_{\mathrm{st}}(V))$ is multiplicative. Finally, the exact sequence in lemma 2.2.15 yields the multiplicativity of the term $\det_L(-\varphi|\, D_{\mathrm{cris}}(V))\det_L(-\varphi|\, D_{\mathrm{cris}}(V^*(1))$. The claim follows since

$$\varepsilon_L(D_{\mathrm{pst}}(V),\xi) = (\varepsilon(D_{\mathrm{pst}}(V)_\sigma,\xi))_\sigma \det_L(-\varphi|\, D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V)).$$

Claim 4: can is multiplicative.
We have the following isomorphism of short exact sequences of B_{dR}-vector spaces:

$$\begin{tikzcd}
0 \arrow[r] & B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V_1) \arrow[r] \arrow[d, "\mathrm{can}_1"] & B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V_2) \arrow[r] \arrow[d, "\mathrm{can}_2"] & B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V_3) \arrow[r] \arrow[d, "\mathrm{can}_3"] & 0 \\
0 \arrow[r] & B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V_1 \arrow[r] & B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V_2 \arrow[r] & B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} V_3 \arrow[r] & 0.
\end{tikzcd}$$

Since the determinant functor is natural in isomorphisms of short exact sequences, we obtain

$$d_L(\Sigma)_{B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} L} \circ \mathrm{can}_2 = \mathrm{can}_1\cdot \mathrm{can}_3 \circ d_L(\Sigma_{\mathrm{dR}})_{B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} L}.$$

Claim 5: $\varepsilon^{\mathrm{dR}}_{L,\xi}(V)\cdot \det_L(-\varphi|V^*(1))$ is multiplicative in V.
Consider the diagram

$$\begin{tikzcd}
 & d_L(D_{\mathrm{dR}}(V_2))d_L(D_{\mathrm{dR}}(V_2))^{-1} \arrow[r, "\mathrm{can}_2\cdot \mathrm{id}_{2^{-1}}"] \arrow[dr, "(\mathrm{can}_1\,\mathrm{can}_3\circ d_L(\Sigma_{\mathrm{dR}})\cdot \mathrm{id}_{2^{-1}}"'] \arrow[dd, "d_L(\Sigma_{\mathrm{dR}})\cdot d_L(\Sigma_{\mathrm{dR}})^{-1}"] & d_L(V_2)d_L(D_{\mathrm{dR}}(V_2))^{-1} \arrow[d, "d_L(\Sigma)\cdot \mathrm{id}_{2^{-1}}"'] \arrow[dd, bend left, "d_L(\Sigma)\cdot d_L(\Sigma_{\mathrm{dR}})^{-1}"] \\
\mathbb{1} \arrow[ur, "\overline{\mu_{2^{-1}}}"] \arrow[dr, "\overline{\mu_{1^{-1}}\mu_{3^{-1}}}"'] & & \begin{array}{c} d_L(V_1)d_L(V_3) \\ d_L(D_{\mathrm{dR}}(V_2))^{-1} \end{array} \arrow[d, "\mathrm{id}_{d_L(V_1)d_L(V_3)}\cdot d_L(\Sigma_{\mathrm{dR}})^{-1}"'] \\
 & \begin{array}{c} d_L(D_{\mathrm{dR}}(V_1))d_L(D_{\mathrm{dR}}(V_3)) \\ d_L(D_{\mathrm{dR}}(V_1))^{-1}d_L(D_{\mathrm{dR}}(V_3))^{-1} \end{array} \arrow[r, "\mathrm{can}_1\,\mathrm{can}_3\,\mathrm{id}_{1^{-1}.3^{-1}}"'] & \begin{array}{c} d_L(V_1)d_L(V_3) \\ d_L(D_{\mathrm{dR}}(V_1))^{-1}d_L(D_{\mathrm{dR}}(V_3))^{-1}. \end{array}
\end{tikzcd}$$

Any subscript of the form i^{-1} is a shorthand for $d_L(D_{\mathrm{dR}}(V_i))^{-1}$. Moreover, we left out the necessary base changes to $V(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} L)$ for better readability. The left triangle is commutative by the definition of inverses of morphisms (see remark 1.1.20 (3)). The right triangle commutes obviously and the upper triangle by claim 4. The inner square commutes since "$-\cdot-$" is a functor and is thus compatible with composition.

Next, we multiply the top and bottom arrow with the terms in claims 2 and 3, the top one with the term associated to V_2, the bottom one with the product of the terms associated to V_1 and V_3. These terms are the same by the multiplicativity established in claims 2 and 3. The resulting diagram will actually live in $V(\widetilde{L})$ by proposition 2.2.11 and is precisely the claim.

Claim 6: $\theta(V)\det_L(-\varphi|\, D_{\mathrm{cris}}(V^*(1))^{-1}$ is multiplicative in V.
This is just corollary 2.2.16.

Multiplying the results of claims 1, 5 and 6 yields the statement of the proposition since $d_L(\Sigma_{\mathrm{dR}})$ and $\det_L(-\varphi|\, D_{\mathrm{cris}}(V^*(1))$ cancel with their respective inverses. □

Remark 2.2.19. As Nakamura points out in his remark 3.6 of [Nak17], the correction factor $\det_L(\varphi|\, D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V))$ is necessary for the proof of the multiplicativity of ε-isomorphisms of de Rham representations. Fütterer comments on page 66 of [Füt18] that the correction factor vanishes for one-dimensional L-linear de Rham representations since in this case being semi-stable and crystalline coincide. To illustrate its importance, we sketch a situation in which the correction factor matters. Let L be $\mathbb{Q}_p$. Colmez and Fontaine [CF00] proved a category equivalence between the category of semi-stable (crystalline) representations and certain "linear"-algebra categories via the functor D_{st} (D_{cris}). In the light of this equivalence, proposition 8.3.8 in [BC09] implies for each $\lambda \in \mathbb{Z}_p^\times$ the existence of a two-dimensional, semi-stable but non-crystalline representation V with a one-dimensional semi-stable and hence crystalline subrepresentation V'. The quotient V'' is again one-dimensional, semi-stable and thus crystalline. So the correction terms for V' and V'' are trivial, while $D_{\mathrm{st}}(V)/\, D_{\mathrm{cris}}(V)$ is one-dimensional and the Frobenius φ acts as multiplication by $p\lambda$. Hence, without the correction terms the ε-isomorphisms would not be multiplicative in the exact sequence $0 \to V' \to V \to V'' \to 0$ of de Rham representations.

Proposition 2.2.20 (Duality)**.** *Let V be an L-linear de Rham representation of $G_{\mathbb{Q}_p}$. Then the morphism*

$$
\begin{aligned}
d_L(V(-1))_{\widetilde{L}} &\xrightarrow{\overline{\mu}} \Big(d_L\big(R\Gamma(\mathbb{Q}_p, V^*(1))\big)^* d_L\big(V^*(1)\big)^* \big(d_L(R\Gamma(\mathbb{Q}_p, V^*(1)))^*\big)^{-1}\Big)_{\widetilde{L}} \\
&\xrightarrow{\varepsilon_{L,\xi}(V)\cdot\varepsilon_{L,-\xi}(V^*(1))^*\cdot\overline{d_L(\psi(\mathbb{Q}_p,V))_{\widetilde{L}}}^{-1}} \Big(d_L(R\Gamma(\mathbb{Q}_p, V)d_L(V)d_L(R\Gamma(\mathbb{Q}_p, V))^{-1}\Big)_{\widetilde{L}} \\
&\xrightarrow{\mu} d_L(V)_{\widetilde{L}}
\end{aligned}
$$

is the same as the morphism $d_L(V(-1) \xrightarrow{\cdot\xi} V)_{\widetilde{L}}$ up to the sign $(-1)^{\dim_L H^0(\mathbb{Q}_p, V^*(1))}$.

Proof.

Claim 1: $h(-r)_{V^*(1)} = h(-(1-r))_V$.
By proposition 1.2.24 the morphism $\nu : D_{\mathrm{dR}}(V^*) \to D_{\mathrm{dR}}(V)^*$ is an isomorphism of filtered L-vector spaces. So, $\dim_L D^r_{\mathrm{dR}}(V^*) = \dim_L(\mathrm{Fil}^{1-r}\, D_{\mathrm{dR}}(V))^\perp = \dim_L D_{\mathrm{dR}}(V) - \dim_L D^{1-r}_{\mathrm{dR}}(V)$. This implies $h(r)_{V^*} = h(-r)_V$. The isomorphism $\vartheta^{-1} : D_{\mathrm{dR}}(V^*) \cong D_{\mathrm{dR}}(V^*(1))$ shifts the filtration down by one (1.2.28), which yields $h(r)_{V^*} = h(r-1)_{V^*(1)}$. Together, we obtain the relation $h(-r)_{V^*(1)} = h(-r+1)_{V^*} = h(-(1-r))_V$.

Claim 2: $\Gamma_L(V)\cdot\Gamma_L(V^*(1)) = (-1)^{t_H(V)+\dim_L t(V)}$.
By definition, we have
$$\Gamma^*(r)\Gamma^*(1-r) = \begin{cases}(-1)^{r-1}, & \text{for } r\geq 1\\ (-1)^r, & \text{for } r\leq 0.\end{cases}$$
Again by definition, we have $\Gamma_L(V) = \prod_{r\in\mathbb{Z}}\Gamma^*(r)^{-h(-r)_V}$. Using claim 1, we also get
$$\Gamma_L(V^*(1)) = \prod_{r\in\mathbb{Z}}\Gamma^*(r)^{-h(-r)_{V^*(1)}} = \prod_{r\in\mathbb{Z}}\Gamma^*(1-r)^{-h(-r)_V}.$$
So, raising $\Gamma^*(r)\Gamma^*(1-r)$ the power $-h(-r)_V$ and multiplying these expressions for all $r\in\mathbb{Z}$ results in
$$\Gamma_L(V)\Gamma_L(V^*(1)) = (-1)^{\sum_r -rh(-r)_V+\sum_{r\geq 1}h(-r)_V}.$$
The claim follows from the identities $\sum_{r\in\mathbb{Z}} -rh(-r)_V = t_H(V)$ and $\sum_{r\geq 1}h(-r)_V = \dim_L t(V)$.

Claim 3: $t_H(V^*(1)) = -\dim_L V - t_H(V)$.
This is an easy calculation using claim 1 and the fact that V is de Rham:
$$\begin{aligned} t_H(V^*(1)) &= \sum_{r\in\mathbb{Z}} -rh(-r)_{V^*(1)} = \sum_{r\in\mathbb{Z}} -(1-r)h(-(1-r))_{V^*(1)} = \sum_{r\in\mathbb{Z}} -(1-r)h(-r)_V\\ &= -\sum_{r\in\mathbb{Z}} h(-r)_V - \sum_{r\in\mathbb{Z}} -rh(-r)_V = -\dim_L V - t_H(V).\end{aligned}$$

Claim 4: $\varepsilon(D_{\mathrm{pst}}(V)_\sigma,\xi)\cdot\varepsilon(D_{\mathrm{pst}}(V^*(1))_\sigma,-\xi) = 1$ for all $\sigma : L\hookrightarrow\overline{\mathbb{Q}_p}$.
We want to use the duality statement for local factors, $\varepsilon(D,\xi)\varepsilon(D^*\otimes_{\overline{\mathbb{Q}_p}}\omega,-\xi) = 1$, from theorem 2.1.1 (3). There, instead of a Tate twist, a twist by the one-dimensional $\overline{\mathbb{Q}_p}$-representation ω appears. So, we need to show that for all $\sigma : L\hookrightarrow\overline{\mathbb{Q}_p}$ the $W_{\mathbb{Q}_p}$-representation $\overline{\mathbb{Q}_p}\otimes_{A,\sigma}D_{\mathrm{pst}}(V^*(1))$ is isomorphic to $(\overline{\mathbb{Q}_p}\otimes_{A,\sigma}D_{\mathrm{pst}}(V))^*\otimes_{\overline{\mathbb{Q}_p}}\omega$. Since D_{pst} is compatible with duals by lemma 1.2.39 (2), we only need to consider the twists. For notational convenience, let ω also denote the rank one A-representation of $W_{\mathbb{Q}_p}$ defined by the same character as the original ω, whose image lies in $\mathbb{Q}^\times$. It suffices to show that $D_{\mathrm{pst}}(V(-1))$ and $D_{\mathrm{pst}}(V)\otimes_A\omega$ are isomorphic as $W_{\mathbb{Q}_p}$-representations over A. Let e be a basis of ω. Then
$$\vartheta^{-1}\otimes e : D_{\mathrm{pst}}(V(-1))\to D_{\mathrm{pst}}(V)\otimes\omega, \sum_i b_i\otimes v_i\otimes\xi^{-1}\mapsto\sum_i b_it^{-1}\otimes v_i\otimes e$$
is clearly an isomorphism of A-modules. By lemma 1.2.28, we know that ϑ respects the naive $W_{\mathbb{Q}_p}$-action and satisfies $p^{-1}\varphi^{-1}\circ\vartheta^{-1} = \vartheta^{-1}\circ\varphi^{-1}$. Let τ be in $W_{\mathbb{Q}_p}$. With the linearised $W_{\mathbb{Q}_p}$-action the power of p coming from $\omega(\tau)$ and the one coming from the interchanging of φ and ϑ cancels: Indeed, we have:
$$\begin{aligned}\vartheta^{-1}\otimes e\circ\tau\circ\varphi^{-v(\tau)} &= \omega(\tau)\tau\circ\vartheta^{-1}\otimes e\circ\varphi^{-v(\tau)}\\ &= p^{v(\tau)}\tau\circ p^{-v(\tau)}\varphi^{-v(\tau)}\circ\vartheta\otimes e = \tau\circ\varphi^{-v(\tau)}\circ\vartheta\otimes e.\end{aligned}$$
So $\vartheta^{-1}\otimes e$ is equivariant with respect to the linearised $W_{\mathbb{Q}_p}$-action, which proves the claim.

Claim 5: $\det_L\big(-\varphi\big|\,D_{\mathrm{st}}(V)/\,D_{\mathrm{cris}}(V)\big)\cdot\det_L\big(-\varphi\big|\,D_{\mathrm{st}}(V^*(1))/\,D_{\mathrm{cris}}(V^*(1))\big) = 1$.
Recall the exact sequence from lemma 1.2.39 part (4):
$$0\to D_{\mathrm{cris}}(V^*(1))\to D_{\mathrm{st}}(V^*(1))\xrightarrow{N} D_{\mathrm{st}}(V^*)\to D_{\mathrm{st}}(V^*)/N\,D_{\mathrm{st}}(V^*)\to 0,$$

where the middle arrow is given by the φ-equivariant map $N \circ \vartheta = \vartheta \circ N$. The multiplicativity of the determinant in the above exact sequence shows

$$\det_L\big(-\varphi \big| D_{\mathrm{st}}(V^*(1))/ D_{\mathrm{cris}}(V^*(1))\big) = \det_L\big(-\varphi\big| D_{\mathrm{st}}(V^*)\big) \cdot \det_L\big(-\varphi\big| D_{\mathrm{st}}(V^*)/N\, D_{\mathrm{st}}(V^*)\big)^{-1}.$$

By lemma 1.2.39 (2), we have that $(-N)^* \circ \nu = \nu \circ N$. Thus, ν induces an isomorphism $\delta'' : D_{\mathrm{st}}(V^*)/N\, D_{\mathrm{st}}(V^*) \to D_{\mathrm{cris}}(V)^*$ as in the proof of lemma 2.2.15. Further, the relation $\nu \circ \varphi = (\varphi^{-1})^* \circ \nu$ implies the same on the quotient δ''. As a result of these relations, we get

$$\begin{aligned} \det_L(-\varphi|\, D_{\mathrm{st}}(V^*)) &= \det_L((-\varphi^{-1})^*|\, D_{\mathrm{st}}(V)^*) = \det_L(-\varphi|\, D_{\mathrm{st}}(V))^{-1} \text{ and} \\ \det_L(-\varphi|\, D_{\mathrm{st}}(V^*)/N\, D_{\mathrm{st}}(V^*)) &= \det_L((-\varphi^{-1})^*|\, D_{\mathrm{cris}}(V)^*) = \det_L(-\varphi|\, D_{\mathrm{cris}}(V))^{-1}. \end{aligned}$$

This proves claim 5. Together with claim 4, we have:

Claim 6: $\varepsilon_L(D_{\mathrm{pst}}(V), \xi) \cdot \varepsilon_L(D_{\mathrm{pst}}(V^*(1)), -\xi) = 1$.

Claim 7: $\varepsilon^{\mathrm{dR}}_{L,\xi}(V) \cdot \varepsilon^{\mathrm{dR}}_{L,-\xi}(V^*(1))^* \cdot \overline{d_L(\vartheta)_{\widetilde{L}}} = (-1)^{t_H(V^*(1))} d_L(V(-1) \xrightarrow{\cdot\xi} V)_{\widetilde{L}}$.
The commutativity of the diagram

$$\begin{array}{ccc} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V(-1)) & \xrightarrow{\mathrm{can}_{V(-1)}} & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1) \\ {\scriptstyle \mathrm{id}_{B_{\mathrm{dR}}} \otimes \vartheta^{-1}} \downarrow & & \downarrow {\scriptstyle \cdot(t^{-1}\otimes\xi)} \\ B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V) & \xrightarrow{\mathrm{can}_V} & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \end{array}$$

is clear. After applying $d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$, the map on the right hand side becomes

$$\begin{aligned} & d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}\left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \xrightarrow{\cdot(t^{-1}\otimes 1)} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \right) \circ d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}\left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1) \xrightarrow{\cdot(1\otimes\xi)} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \right) \\ =& \det_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}(t^{-1} \otimes 1|\, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \cdot d_L(V(-1) \xrightarrow{\cdot\xi} V)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L} \\ =& t^{-\dim_L V} \cdot d_L(V(-1) \xrightarrow{\cdot\xi} V)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}. \end{aligned}$$

The diagram

$$\begin{array}{ccc} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V(-1)) & \xrightarrow{\mathrm{can}_{V(-1)}} & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V(-1) \\ \| & & \| \\ B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V^*(1))^* & & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*(1)^* \\ \| & & \| \\ (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V^*(1)))^{*_{B_{\mathrm{dR}} \otimes L}} & \xleftarrow[\mathrm{can}^*_{V^*(1)}]{} & (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V^*(1))^{*_{B_{\mathrm{dR}} \otimes L}} \end{array}$$

also commutes. We glue the two diagrams together along $\mathrm{can}_{V(-1)}$ and apply the determinant functor $d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L}$. Then we multiply with signed powers of t and local factors and obtain the commutative diagram

$$\begin{array}{ccc} d_L(D_{\mathrm{dR}}(V^*(1)))^*_{\widetilde{L}} & \xleftarrow{(-1)^{t_H(V^*(1))}\ (-t)^{-t_H(V^*(1))} \varepsilon_L(D_{\mathrm{pst}}(V^*(1),-\xi)\, \mathrm{can}_{V^*(1))^*}} & d_L(V(-1))_{\widetilde{L}} \\ {\scriptstyle \overline{d_L(\vartheta)_{\widetilde{L}}}} \downarrow & & \downarrow {\scriptstyle d_L(\cdot\xi)_{\widetilde{L}}} \\ d_L(D_{\mathrm{dR}}(V))_{\widetilde{L}} & \xrightarrow[t^{-t_H(V)} \varepsilon_L(D_{\mathrm{pst}}(V),\xi)\, \mathrm{can}_V]{} & d_L(V)_{\widetilde{L}}. \end{array}$$

The appearing powers of t cancel by claim 3 the factor $t^{-\dim_L V}$ coming from the determinant of $\cdot(t^{-1}\otimes\xi)$. Using claim 6 we see that the ε_L-factors cancel. Finally, we use remark 1.1.27 (2) to write the composition as a product and obtain

$$\varepsilon^{\mathrm{dR}}_{L,\xi}(V)\cdot\varepsilon^{\mathrm{dR}}_{L,-\xi}(V^*(1))^*\cdot\overline{d_L(\vartheta)}_{\widetilde{L}}=(-1)^{t_H(V^*(1))}d_L(\cdot\xi)_{\widetilde{L}}.$$

Claim 8: $\theta(V^*(1))^*\cdot\overline{d_L(\psi(\mathbb{Q}_p,V))}^{-1}\cdot\overline{d_L(\vartheta)}^{-1}\cdot\theta(V)=(-1)^{\dim_L H^1_f(V^*(1))}\in L^\times=\mathrm{Aut}_{V(L)}(\mathbb{1})$. In the notation of the definition of $\theta(V)$ (2.2.3), we have

$$\theta(V)=\overline{d_L(\Sigma_{R\Gamma,V})}\cdot\overline{d_L(\Sigma_{\mathrm{dR},V})\circ d_L(\psi_f(\mathbb{Q}_p,V^*(1)))^*}\cdot\overline{d_L(\psi_{\mathrm{dR},V})\circ\eta(V^*(1))^*}\cdot\eta(V)\text{ and}$$
$$\theta(V^*(1))=\overline{d_L(\Sigma_{R\Gamma,V^*(1)})}\cdot\overline{d_L(\Sigma_{\mathrm{dR},V^*(1)})\circ d_L(\psi_f(\mathbb{Q}_p,V))^*}\cdot\overline{d_L(\psi_{\mathrm{dR},V^*(1)})\circ\eta(V)^*}\cdot\eta(V^*(1)).$$

To relate $\theta(V)$ and $\theta(V^*(1))$, we use the isomorphism of short exact sequences from corollary 1.2.29

$$\begin{tikzcd}
\Sigma_{\mathrm{dR},V}: & 0 \arrow[r] & D^0_{\mathrm{dR}}(V) \arrow[r] \arrow[dd, "\psi_{\mathrm{dR},V}"'] & D_{\mathrm{dR}}(V) \arrow[r] \arrow[d, "\vartheta"] & t(V) \arrow[r] \arrow[dd, "\psi^*_{\mathrm{dR},V^*(1)}"] & 0 \\
 & & & D_{\mathrm{dR}}(V(-1)) \arrow[d, "\nu"] & & \\
\Sigma^*_{\mathrm{dR},V^*(1)}: & 0 \arrow[r] & t(V^*(1))^* \arrow[r] & D_{\mathrm{dR}}(V^*(1))^* \arrow[r] & D^0_{\mathrm{dR}}(V^*(1))^* \arrow[r] & 0.
\end{tikzcd}$$

Omitting ν and the isomorphism $(-)^*(1)\cong(-)(-1)^*$ (1.2.8), we get that

$$d_L(\Sigma^*_{\mathrm{dR},V^*(1)})\circ d_L(\vartheta)=d_L(\psi_{\mathrm{dR},V})d_L(\psi_{\mathrm{dR},V^*(1)})^*\circ d_L(\Sigma_{\mathrm{dR},V}).$$

Next, we treat the corresponding short exact Galois cohomology sequence. The skew commutativity of the cup product (see remark 1.3.21) shows that $\psi^i(\mathbb{Q}_p,V)=(-1)^i\psi^i(\mathbb{Q}_p,V^*(1))^*[-2]$ for $i=0,1$ or 2. This implies $d_L(\psi(\mathbb{Q}_p,V))=(-1)^{\dim_L H^1(\mathbb{Q}_p,V)}d_L(\psi(\mathbb{Q}_p,V^*(1))^*)$. The Tate duality on the finite parts of Galois cohomology are given by restriction of the usual Tate duality. In other words, the diagrams

$$\begin{tikzcd}
R\Gamma_f(\mathbb{Q}_p,V) \arrow[r, hook] \arrow[d, "\psi_f(\mathbb{Q}_p,V)"'] & R\Gamma(\mathbb{Q}_p,V) \arrow[d, "\psi(\mathbb{Q}_p,V)"] \\
R\Gamma_{/f}(\mathbb{Q}_p,V^*(1))^*[-2] \arrow[r, hook] & R\Gamma(\mathbb{Q}_p,V^*(1))^*[-2]
\end{tikzcd}$$

and

$$\begin{tikzcd}
R\Gamma(\mathbb{Q}_p,V) \arrow[r, two heads] \arrow[d, "\psi(\mathbb{Q}_p,V^*(1))^*[-2]"'] & R\Gamma_{/f}(\mathbb{Q}_p,V) \arrow[d, "\psi_f(\mathbb{Q}_p,V^*(1))^*[-2]"] \\
R\Gamma(\mathbb{Q}_p,V^*(1))^*[-2] \arrow[r, two heads] & R\Gamma_f(\mathbb{Q}_p,V^*(1))^*[-2]
\end{tikzcd}$$

commute. Using the above relation between $\psi(\mathbb{Q}_p,V)$ and $\psi(\mathbb{Q}_p,V^*(1))^*[-2]$, we get an isomorphism of short exact sequences

$$\begin{tikzcd}
\Sigma_{R\Gamma,V}:0 \arrow[r] & R\Gamma_f(\mathbb{Q}_p,V) \arrow[r] \arrow[d, "\psi_f(\mathbb{Q}_p,V)"'] & R\Gamma(\mathbb{Q}_p,V) \arrow[r] \arrow[d, "\psi(\mathbb{Q}_p,V)"'] & R\Gamma_{/f}(\mathbb{Q}_p,V) \arrow[r] \arrow[d, "{[(-1)^i\psi^i_f(\mathbb{Q}_p,V^*(1))^*[-2]]}"'] & 0 \\
\Sigma^*_{R\Gamma,V^*(1)}[-2]:0 \arrow[r] & R\Gamma_{/f}(\mathbb{Q}_p,V^*(1))^*[-2] \arrow[r] & R\Gamma(\mathbb{Q}_p,V^*(1))^*[-2] \arrow[r] & R\Gamma_f(\mathbb{Q}_p,V^*(1))^*[-2] \arrow[r] & 0
\end{tikzcd}$$

which induces

$$\begin{aligned}&d_L(\Sigma^*_{R\Gamma,V^*(1)})\circ d_L(\psi(\mathbb{Q}_p,V))\\=&(-1)^{\dim_L H^1_f(\mathbb{Q}_p,V^*(1))}d_L(\psi_f(\mathbb{Q}_p,V))\cdot d_L(\psi_f(\mathbb{Q}_p,V^*(1))^*)\circ d_L(\Sigma_{R\Gamma,V})\end{aligned}$$

and similarly if one uses $\psi(\mathbb{Q}_p,V^*(1))^*[-2]$ instead of $\psi(\mathbb{Q}_p,V)$

$$\begin{aligned}&d_L(\Sigma^*_{R\Gamma,V^*(1)})\circ d_L(\psi(\mathbb{Q}_p,V^*(1))^*)\\=&(-1)^{\dim_L H^1_f(\mathbb{Q}_p,V)}d_L(\psi_f(\mathbb{Q}_p,V))\cdot d_L(\psi_f(\mathbb{Q}_p,V^*(1))^*)\circ d_L(\Sigma_{R\Gamma,V}).\end{aligned}$$

Before putting everything together, we recall that for a short exact sequence Σ, we have $d_L(\Sigma^*)=\overline{d_L(\Sigma)^*}$ up to commutativity (see lemma 1.1.35 (2)). Now, we can compute

$$\begin{aligned}&d_L(\psi(\mathbb{Q}_p,V))\cdot d_L(\vartheta)\circ\theta(V)\\=&d_L(\psi(\mathbb{Q}_p,V))\cdot d_L(\vartheta)\\&\circ\overline{d_L(\Sigma_{R\Gamma,V})}\cdot\overline{d_L(\Sigma_{\mathrm{dR},V})}\circ\overline{d_L(\psi_f(\mathbb{Q}_p,V^*(1)))^*}\cdot\overline{d_L(\psi_{\mathrm{dR},V})}\circ\overline{\eta(V^*(1))^*}\cdot\eta(V)\\=&\pm\,\overline{d_L(\Sigma^*_{R\Gamma,V^*(1)})}\cdot\overline{d_L(\Sigma^*_{\mathrm{dR},V^*(1)})}\circ d_L(\psi_f(\mathbb{Q}_p,V))\cdot d_L(\psi_{\mathrm{dR},V^*(1)})^*\circ\overline{\eta(V^*(1))^*}\cdot\eta(V)\\=&\pm\,d_L(\Sigma_{R\Gamma,V^*(1)})^*\cdot d_L(\Sigma_{\mathrm{dR},V^*(1)})^*\circ d_L(\psi_f(\mathbb{Q}_p,V))\cdot d_L(\psi_{\mathrm{dR},V^*(1)})^*\circ\overline{\eta(V^*(1))^*}\cdot\eta(V).\end{aligned}$$

Here and in the following "$\pm$" stands for $(-1)^{\dim_L H^1_f(\mathbb{Q}_p,V^*(1))}$. Thus

$$\begin{aligned}&\theta(V^*(1))^*\circ d_L(\psi(\mathbb{Q}_p,V))\cdot d_L(\vartheta)\circ\theta(V)\\=&\pm\,\overline{\eta(V)}\cdot\eta(V^*(1))^*\circ\overline{d_L(\psi_f(\mathbb{Q}_p,V))}\cdot\overline{d_L(\psi_{\mathrm{dR},V^*(1)})^*}\circ\overline{d_L(\Sigma_{R\Gamma,V^*(1)})^*}\cdot\overline{d_L(\Sigma_{\mathrm{dR},V^*(1)})^*}\circ\\&d_L(\Sigma_{R\Gamma,V^*(1)})^*\cdot d_L(\Sigma_{\mathrm{dR},V^*(1)})^*\circ d_L(\psi_f(\mathbb{Q}_p,V))\cdot d_L(\psi_{\mathrm{dR},V^*(1)})^*\circ\overline{\eta(V^*(1))^*}\cdot\eta(V)\\=&(-1)^{\dim_L H^1_f(\mathbb{Q}_p,V^*(1))}.\end{aligned}$$

Put $X:=d_L(R\Gamma(\mathbb{Q}_p,V))\cdot d_L(D_{\mathrm{dR}}(V))\cdot d_L(R\Gamma(\mathbb{Q}_p,V^*(1))^*)\cdot d_L(D_{\mathrm{dR}}(V^*(1))^*)$. Again, we write the above identity as a product of three morphisms rather than a composition (see remark 1.1.27 (2)). The element $(-1)^{\dim_L H^1_f(V^*(1))}$ in $K_1(L)\subset K_1(\widetilde{L})=\mathrm{Aut}_{V(L)}(\mathbb{1})$ is the same as the map $\mu_X\circ\theta(V)\cdot d_L(\psi(\mathbb{Q}_p,V))\cdot d_L(\vartheta)\cdot\theta(V^*(1))^*\cdot\mathrm{id}_{X^{-1}}\circ\overline{\mu_X}$:

$$\begin{aligned}\mathbb{1}&\to\mathbb{1}\cdot d_L(R\Gamma(\mathbb{Q}_p,V))\cdot d_L(D_{\mathrm{dR}}(V))\cdot d_L(R\Gamma(\mathbb{Q}_p,V^*(1))^*)\cdot d_L(D_{\mathrm{dR}}(V^*(1))^*)\cdot X^{-1}\\&\to d_L(R\Gamma(\mathbb{Q}_p,V))\cdot d_L(D_{\mathrm{dR}}(V))\cdot d_L(R\Gamma(\mathbb{Q}_p,V^*(1))^*)\cdot d_L(D_{\mathrm{dR}}(V^*(1))^*)\cdot\mathbb{1}\cdot X^{-1}\\&\to\mathbb{1}.\end{aligned}$$

As a last step, we replace $d_L(\psi(\mathbb{Q}_p,V))\cdot d_L(\vartheta)$ by $\overline{d_L(\psi(\mathbb{Q}_p,V))}^{-1}\cdot\overline{d_L(\vartheta)}^{-1}$. By the definition of inverses of morphisms, we have

$$\begin{aligned}\mu_{d_L(R\Gamma(\mathbb{Q}_p,V))}&=\mu_{d_L(R\Gamma(\mathbb{Q}_p,V^*(1)))^*}\circ d_L(\psi(\mathbb{Q}_p,V))\cdot d_L(\psi(\mathbb{Q}_p,V)^{-1}\text{ and}\\\mu_{d_L(D_{\mathrm{dR}}(V))}&=\mu_{d_L(D_{\mathrm{dR}}(V(-1)))}\circ d_L(\vartheta)\cdot d_L(\vartheta)^{-1}.\end{aligned}$$

Rearranging yields

$$\begin{aligned}\mu_{d_L(R\Gamma(\mathbb{Q}_p,V))}\circ\mathrm{id}\cdot\overline{d_L(\psi(\mathbb{Q}_p,V)^{-1}}&=\mu_{d_L(R\Gamma(\mathbb{Q}_p,V^*(1)))^*}\circ d_L(\psi(\mathbb{Q}_p,V))\cdot\mathrm{id}\;\text{ and}\\\mu_{d_L(D_{\mathrm{dR}}(V))}\circ\mathrm{id}_{d_L(D_{\mathrm{dR}}(V))}\cdot\overline{d_L(\vartheta)}^{-1}&=\mu_{d_L(D_{\mathrm{dR}}(V^*(1))^*)}\circ d_L(\vartheta)\cdot\mathrm{id}_{d_L(D_{\mathrm{dR}}(V(-1)))^{-1}}.\end{aligned}$$

We amend the trivialisations accordingly and obtain that $(-1)^{\dim_L H^1_f(\mathbb{Q}_p, V^*(1))}$ in $\mathrm{Aut}_{V(L)}(\mathbb{1})$ is the same as

$$\mu_{d_L(R\Gamma(\mathbb{Q}_p,V))\cdot d_L(D_{\mathrm{dR}}(V))} \circ \theta(V) \cdot \theta(V^*(1))^* \cdot \overline{d_L(\psi(\mathbb{Q}_p,V))}^{-1} \cdot \overline{d_L(\vartheta)}^{-1}$$
$$\circ\, \mu_{d_L(R\Gamma(\mathbb{Q}_p,V^*(1)))^*\cdot d_L(D_{\mathrm{dR}}(V(-1)))}.$$

Claim 9: $\varepsilon_{L,\xi}(V) \cdot \varepsilon_{L,-\xi}(V^*(1))^* \cdot \overline{d_L(\psi(\mathbb{Q}_p,V))_{\widetilde{L}}}^{-1} = (-1)^{\dim_L H^0(V^*(1))}$.
We multiply the identities in claims 2, 7 and 8, trivialise $\overline{d_L(\vartheta)}$ and get

$$\varepsilon_{L,\xi}(V) \cdot \varepsilon_{L,-\xi}(V^*(1))^* \overline{d_L(\psi(\mathbb{Q}_p,V))_{\widetilde{L}}}^{-1} = (-1)^{t_H(V)+\dim_L t(V)+t_H(V^*(1))+\dim_L H^1_f(V^*(1))} d_L(\cdot\xi)_{\widetilde{L}}.$$

By claim 3 and the identities $\dim_L H^1_f(V^*(1)) = \dim_L H^0(V^*(1)) + \dim_L t(V^*(1))$ and $\dim_L V = \dim_L t(V) + \dim_L t(V^*(1))$, the sign is $(-1)^{\dim_L H^0(V^*(1))}$. $\square$

Remark 2.2.21. The last part of the proof of claim 8 above explains the relation between the duality formulation in proposition 3.3.8 [FK06], where $d_L(\psi(\mathbb{Q}_p,V))$ appears, and ours with $\overline{d_L(\psi(\mathbb{Q}_p,V))^{-1}}$. We differed from the formulation in [FK06], since our version of the duality is easier to write down explicitly as morphism, which will be useful in the equivariant version in conjecture 2.3.5 (6).

Remark 2.2.22.
(1) The duality in proposition 3.3.8 of [FK06] differs from our proposition 2.2.20 by the sign $(-1)^{\dim_L H^0(V^*(1))}$, which does not vanish in general (consider for instance $L(-1)^*$ as V). The difference stems from the relation between $\psi(\mathbb{Q}_p,V)$ and $\psi(\mathbb{Q}_p,V^*(1))^*[-2]$. Fukaya and Kato claim the relations

$$d_L(\psi(\mathbb{Q}_p,V)) = (-1)^{\dim_L D^0_{\mathrm{dR}}(V)} d_L(\psi_f(\mathbb{Q}_p,V)) \cdot d_L(\psi_f(\mathbb{Q}_p,V^*(1)))^* \text{ and}$$
$$d_L(\psi(\mathbb{Q}_p,V)) = (-1)^{\dim_L V} \cdot d_L(\psi(\mathbb{Q}_p,V^*(1))^*$$

instead of our relations

$$d_L(\psi(\mathbb{Q}_p,V)) = (-1)^{\dim_L H^1_f(\mathbb{Q}_p,V^*(1))} d_L(\psi_f(\mathbb{Q}_p,V)) \cdot d_L(\psi_f(\mathbb{Q}_p,V^*(1)))^* \text{ and}$$
$$d_L(\psi(\mathbb{Q}_p,V)) = (-1)^{\dim_L H^1(\mathbb{Q}_p,V)} \cdot d_L(\psi(\mathbb{Q}_p,V^*(1))^*$$

Since we have $\dim_L H^1_f(\mathbb{Q}_p,V^*(1)) = \dim_L H^0(\mathbb{Q}_p,V^*(1)) + \dim_L t(V^*(1))$ and $\dim_L D^0_{\mathrm{dR}}(V) = \dim_L t(V^*(1))$, this results in the sign difference.
(2) Fukaya and Kato assert a self-duality for their duality statement. Our version is self-dual as well, if one uses our relation between $\psi(\mathbb{Q}_p,V)$ and $\psi(\mathbb{Q}_p,V^*(1))^*$: Interchanging V and $V^*(1)$ and taking the L-dual introduces the sign $(-1)^{t_H(V^*(1))+t_H(V)+\dim_L H^1(\mathbb{Q}_p,V)}$ on the left-hand side and $(-1)^{\dim_L H^0(\mathbb{Q}_p,V^*(1))+\dim_L H^0(\mathbb{Q}_p,V)}$ on the right-hand side. Using claim 3, the local Tate duality and Tate's local Euler-Poincaré characteristic formular $\dim_L V = \sum_{i=0}^2 (-1)^i \dim_L H^i(\mathbb{Q}_p,V)$, ([NSW13] theorem 7.3.1 plus limiting process), one sees that both signs agree.
(3) Nakamura reproduces the proof of the proposition 3.3.8 [FK06] in lemma 3.7 (2) of [Nak17], gets by without the extra sign and hence supports the result of Fukaya and Kato. He argues with the more explicit definition of $\theta(V)$ via the long exact sequence. Unfortunately, we cannot follow his reasoning and our own attempts of using the explicit definition of $\theta(V)$

(see remark 2.2.4) back the duality result with extra sign factor: Nakamura considers the following commutative diagram, in which the upper row appears in the long exact sequence (2.3) for V and the lower row belongs to the dual of the long exact sequence (2.3) for $V^*(1)$. We present it in our notation (omitting the "$\mathbb{Q}_p$" in all cohomology related expressions) and fill in some arrows to the best of our knowledge, so that checking commutativity is easier:

$$\begin{tikzcd}
t(V) \arrow[rr, "-\exp_V"] \arrow[dr, two heads, "-\exp_V"'] \arrow[ddd, equal] & & H^1(V) \arrow[rr] \arrow[dr, two heads, "\psi^1_f(V^*(1))^*\circ p"] \arrow[ddd, "\psi^1(V^*(1))^*"] & & t(V^*(1))^* \arrow[ddd, equal] \\
& H^1_f(V) \arrow[ur, hook] \arrow[d, "(-1)"] & & H^1_f(V^*(1))^* \arrow[ur, hook] \arrow[d, equal] & \\
& H^1_f(V) \arrow[dr, hook, "p^*\circ\psi^1_f(V)"] & & H^1_f(V^*(1))^* \arrow[dr, hook] & \\
t(V) \arrow[rr] \arrow[ur, two heads, "\exp_V"] & & H^1(V^*(1))^* \arrow[rr, "\exp^*_{V^*(1)}"] \arrow[ur, two heads, "i^*"] & & t(V^*(1))^*.
\end{tikzcd}$$

Here, i (p) stands for an inclusion (projection) of the form $H^1_f(V) \hookrightarrow H^1(V)$ ($H^1(V) \twoheadrightarrow H^1_{/f}(V)$). Nakamura seems to claim that the diagram explains, why a factor of $(-1)^{\dim_L t(V)}$ appears. However, in our opinion this diagram rather explains a factor of $(-1)^{\dim_L H^1_f(V)}$, which matches our results since here $\psi^1(V^*(1))^*$ is used instead of $\psi^1(V)$ which we use. The corresponding diagram in our situation would be

$$\begin{tikzcd}
D_{\mathrm{cris}}(V)\oplus t(V) \arrow[rr, "\exp_V"] \arrow[dr, two heads, "\exp_V"'] \arrow[ddd, equal] & & H^1(V) \arrow[rr] \arrow[dr, two heads, "\psi^1_f(V^*(1))^*\circ p"] \arrow[ddd, "\psi^1(V)"] & & D_{\mathrm{cris}}(V^*(1))^*\oplus t(V^*(1))^* \arrow[ddd, "(-1)"] \\
& H^1_f(V) \arrow[ur, hook, "i"] \arrow[d, equal] & & H^1_f(V^*(1))^* \arrow[ur, hook, "\exp^*_{V^*(1)}"'] \arrow[d, "(-1)"] & \\
& H^1_f(V) \arrow[dr, hook, "p^*\circ\psi^1_f(V)"'] & & H^1_f(V^*(1))^* \arrow[dr, hook, "\exp^*_{V^*(1)}"] & \\
D_{\mathrm{cris}}(V)\oplus t(V) \arrow[rr] \arrow[ur, two heads, "\exp_V"] & & H^1(V^*(1))^* \arrow[rr, "\exp^*_{V^*(1)}\circ i^*"'] \arrow[ur, two heads, "i^*"] & & D_{\mathrm{cris}}(V^*(1))^*\oplus t(V^*(1))^*
\end{tikzcd}$$

where again, the upper row is part of the long exact sequence $\Sigma_{l,V}$ (see (2.3)) for V and the lower one is part of the dual of the corresponding sequence for $V^*(1)$. One sees that (the dual of) the short exact sequence merging (the duals of) the sequences (2.1) and (2.2) are isomorphic via the isomorphism $(\mathrm{id}_{H^1_f(V)}, \psi^1(V), (-1))$, this yields the sign $(-1)^{\dim_L H^1_f(V^*(1))}$ in the comparison of $\theta(V)$ and $\theta(V^*(1))$. Alternatively, one could consider the induced morphism between $\Sigma_{l,V}$ and $\Sigma^*_{l,V^*(1)}$, which is given by $(\mathrm{id}, \mathrm{id}, \mathrm{id}, \psi^1(V), (-1), (-1), (-1))$. The induced sign is again

$$(-1)^{\dim_L D_{\mathrm{cris}}(V^*(1))+\dim_L t(V^*(1))+\dim_L D_{\mathrm{cris}}(V^*(1))+\dim_L H^0(V^*(1))} = (-1)^{\dim_L H^1_f(V^*(1))}.$$

2.3 Equivariant ε-isomorphisms

In this section we will state the equivariant ε-isomorphism conjecture of Fukaya and Kato ([FK06] 3.4.3).

Definition 2.3.1. *For an adic ring Λ, we denote by $\widetilde{\Lambda}$ the ring $\lim_n \left(\widehat{\mathbb{Z}_p^{nr}} \otimes_{\mathbb{Z}_p} \Lambda/J(\Lambda)^n\right)$, where $J(\Lambda)$ is the Jacobson radical of Λ.*

Let Λ be an adic ring and $\mathbb{T}$ a finitely generated projective Λ-module with a continuous Λ-linear $G_{\mathbb{Q}_p}$-action. The topology of $\mathbb{T}$ is induced by Λ.
The following proposition is a first indicator for the possible existence of equivariant ε-isomorphisms.

Proposition 2.3.2. *The classes of $R\Gamma(\mathbb{Q}_p, \mathbb{T})$ and $\mathbb{T}$ add to zero in $K_0(\Lambda)$. So there is an isomorphism $\mathbb{1} \to d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}))d_\Lambda(\mathbb{T})$.*

Proof. We consider the perfect complex $R\Gamma(\mathbb{Q}_p, \mathbb{T})$ as an element of $K_0(\Lambda)$ via the isomorphism $K_0(\Lambda) \cong \pi_0(V(\Lambda))$ or equivalently by replacing it with a strictly perfect complex which admits a quasi-isomorphism to $R\Gamma(\mathbb{Q}_p, \mathbb{T})$. The proposition is proven in 3.1.3 of [FK06]. The argument is to use 1.3.12 and that Λ is adic to reduce to the case of Λ being a finite field. Then the first claim is the local Euler-Poincaré characteristic formula ([Ser07] II 5.7 theorem 5.). The existence of the isomorphism is due to $K_0(\Lambda) \cong \pi_0(V(\Lambda))$ (lemma 1.1.37 (2)) $\square$

The equivariant ε-isomorphism conjecture states that isomorphisms such as in proposition 2.3.2 exist in a compatible way and with some additional properties for different pairs $(\Lambda, \mathbb{T})$. In order to state the conjecture precisely, we need to introduce some more notation.

Lemma 2.3.3. *The map $[\mathbb{T}, -] : G_{\mathbb{Q}_p} \to K_1(\Lambda), \sigma \mapsto [\mathbb{T}, \sigma]$ factors over $G_{\mathbb{Q}_p}^{ab}$.*

Proof. The proof is a more general version of the proof of lemma 2.2.1. The map $G_{\mathbb{Q}_p} \to K_1(\Lambda), \sigma \mapsto [\mathbb{T}, \sigma]$ factors over $G_{\mathbb{Q}_p}/[G_{\mathbb{Q}_p}, G_{\mathbb{Q}_p}]$ since $K_1(\Lambda)$ is commutative. Moreover, for an adic ring Λ, we have by Proposition 1.5.1 of [FK06] that $K_1(\Lambda) \cong \lim_n K_1(\Lambda/J(\Lambda)^n)$. Thus, $[\mathbb{T}, \sigma] = 1$ if and only if $[\mathbb{T}/J^nT, \sigma] = 1$ for all $n \geq 1$. Let $\sigma \in G_{\mathbb{Q}_p}$ such that $[\mathbb{T}, \sigma] \neq 1$. Then there is some $n \geq 1$ such that $[\mathbb{T}/J^n\mathbb{T}, \sigma] \neq 1$. Since $\mathbb{T}/J^n\mathbb{T}$ is finitely generated over the finite ring Λ/J^n (Λ is adic), it is a finite module. Since $G_{\mathbb{Q}_p}$ acts continuously on $\mathbb{T}$, we find an open subgroup N of $G_{\mathbb{Q}_p}$ that operates trivially on $\mathbb{T}/J^n\mathbb{T}$. Hence, the open set $N\sigma$ is not in the kernel of the above map. So the kernel is closed and hence it induces a map $[\mathbb{T}, -] : G_{\mathbb{Q}_p}^{ab} = \mathrm{Gal}(\mathbb{Q}_p^{ab}, \mathbb{Q}_p) = G_{\mathbb{Q}_p}/\overline{[G_{\mathbb{Q}_p}, G_{\mathbb{Q}_p}]} \to K_1(\Lambda)$. $\square$

Remark 2.3.4. Let Λ act continuously from the right on some finite dimensional L-vector space V for $L/\mathbb{Q}_p$ finite. The map $[\mathbb{T}, -] : G_{\mathbb{Q}_p}^{ab} \to K_1(\Lambda)$ corresponds after base change to the determinant $\det_L(-|V \otimes_\Lambda \mathbb{T}) : G_{\mathbb{Q}_p}^{ab} \to L^\times = K_1(L)$ from lemma 2.2.1.

Conjecture 2.3.5 (Equivariant ε-isomorphism conjecture)**.** *There is a unique way of assigning an isomorphism*

$$\varepsilon_{\Lambda,\xi}(\mathbb{T}) : \mathbb{1} \to \left(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}))d_\Lambda(\mathbb{T})\right)_{\widetilde{\Lambda}}$$

in $V(\widetilde{\Lambda})$ to every triple $(\Lambda, \mathbb{T}, \xi)$ with Λ and $\mathbb{T}$ as above and ξ a basis of $\mathbb{Z}_p(1)$ such that the following properties hold:

Multiplicativity:

For fixed Λ and ξ let $\Sigma : 0 \to \mathbb{T}' \to \mathbb{T} \to \mathbb{T}'' \to 0$ be a short exact sequence of finitely generated projective Λ-modules with continuous $G_{\mathbb{Q}_p}$-actions. Recall that in this case we have the isomorphism $d_\Lambda(C(\mathbb{Q}_p, \Sigma))$ from lemma 1.3.14. Then the ε-isomorphism is multiplicative in the following way:

$$\big(d_\Lambda(C(\mathbb{Q}_p, \Sigma))d_\Lambda(\Sigma)\big)_{\widetilde{\Lambda}} \circ \varepsilon_{\Lambda,\xi}(\mathbb{T}) = \varepsilon_{\Lambda,\xi}(\mathbb{T}')\varepsilon_{\Lambda,\xi}(\mathbb{T}'').$$

Base change:

Let Λ' be another adic ring and Y a finitely generated projective Λ'-module with commuting continuous Λ-right-action. Let $\mathbb{T}'$ be $Y \otimes_\Lambda \mathbb{T}$. Then the ε-isomorphism for the triple $(\Lambda', \mathbb{T}', \xi)$ is given by $\varepsilon_{\Lambda',\xi}(\mathbb{T}') = Y \otimes_\Lambda \varepsilon_{\Lambda,\xi}(\mathbb{T})$. Strictly speaking, the right hand side should read $(\widetilde{\Lambda}' \otimes_{\Lambda'} Y) \otimes_{\widetilde{\Lambda}} \varepsilon_{\Lambda,\xi}(\mathbb{T})$ and needs to be post-composed by the natural map $d_\Lambda(\omega) : d_\Lambda(Y \otimes_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T})) \to d_\Lambda(R\Gamma(\mathbb{Q}_p, Y \otimes_\Lambda \mathbb{T}))$ from proposition 1.3.12 both of which we will often omit.

Change of ξ:

Let $\sigma \in I(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \subset G_{\mathbb{Q}_p}^{ab}$. Then we have $\varepsilon_{\Lambda,\chi_{cycl}(\sigma)\xi}(\mathbb{T}) = [\mathbb{T}, \sigma]\varepsilon_{\Lambda,\xi}(\mathbb{T})$. The action of $[\mathbb{T}, \sigma] \in K_1(\Lambda)$ on the ε-isomorphism is as in corollary 1.1.37 (3)

Frobenius invariance:

Let $\phi \in G_{\mathbb{Q}_p}$ be a Frobeniuslift with $\chi_{cycl}(\phi) = 1$. Denote by $\varphi_p := \phi|_{\widehat{\mathbb{Z}_p^{nr}}} : \widehat{\mathbb{Z}_p^{nr}} \to \widehat{\mathbb{Z}_p^{nr}}$ the restriction of $\phi : B_{\mathrm{dR}} \to B_{\mathrm{dR}}$. The ring homomorphism $\varphi_p \otimes \mathrm{id}_\Lambda : \widetilde{\Lambda} \to \widetilde{\Lambda}$ induces a base change homomorphism $(\varphi_p \otimes \mathrm{id}_\Lambda)^* : K_1(\widetilde{\Lambda}) \to K_1(\widetilde{\Lambda})$ (see lemma 1.1.10). We put $K_1(\widetilde{\Lambda})_{\mathbb{T}} = \{x \in K_1(\widetilde{\Lambda}) | (\varphi_p \otimes \mathrm{id})^*(x) = [\mathbb{T}, \phi]^{-1}x\}$. Then the ε-isomorphism $\varepsilon_{\Lambda,\xi}(\mathbb{T})$ belongs to $V(\Lambda)\big(\mathbb{1}, d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T})d_\Lambda(\mathbb{T})\big) \overset{K_1(\Lambda)}{\times} K_1(\widetilde{\Lambda})_{\mathbb{T}}$.

Specialisation:

Suppose we have $(\Lambda, \mathbb{T}, \xi) = (\mathcal{O}_L, T, \xi)$ such that $V = L \otimes_{\mathcal{O}_L} T$ is an L-linear de Rham representation. Then, base changing to $\widetilde{L}$ yields the ε-isomorphism of the de Rham representation V (see proposition 2.2.2): $L \otimes_{\mathcal{O}_L} \varepsilon_{\mathcal{O}_L,\xi}(T) = \varepsilon_{L,\xi}(L \otimes_{\mathcal{O}_L} T)$. More precisely the correct base change is $\widetilde{L} \otimes_{\widetilde{\mathcal{O}_L}} \varepsilon_{\mathcal{O}_L,\xi}(T) = \varepsilon_{L,\xi}(L \otimes_{\mathcal{O}_L} T)$ post-composed with map from $d_L\big(L \otimes_{\mathcal{O}_L} R\Gamma(\mathbb{Q}_p, T) \xrightarrow{\omega} R\Gamma(\mathbb{Q}_p, L \otimes_{\mathcal{O}_L} T\big)$.

Duality:

$\varepsilon_{\Lambda,\xi}(\mathbb{T})\varepsilon_{\Lambda^\circ,-\xi}(\mathbb{T}^*(1))^*\overline{d_\Lambda(\psi(\mathbb{Q}_p, \mathbb{T}))}^{-1} = d_\Lambda(\mathbb{T}(-1) \xrightarrow{\cdot\xi} \mathbb{T})$ up to appropriate trivialisations and a sign, see remark 2.3.7 below.

Remark 2.3.6. Combining the base change and the specialisation property yields the following specialisation property. Let L be a finite extension of $\mathbb{Q}_p$ and $\alpha : \Lambda \to M_n(L)$ a continuous ring homomorphism such that $L^n \otimes_{\alpha,\Lambda} \mathbb{T}$ is an L-linear de Rham representation of $G_{\mathbb{Q}_p}$. Then $L^n \otimes_{\alpha,\Lambda} \varepsilon_{\Lambda,\xi}(\mathbb{T}) = \varepsilon_{L,\xi}(L^n \otimes_{\alpha,\Lambda} \mathbb{T})$.

To be precise the left hand side should be post-composed by the determinant of the quasi-isomorphism $L^n \otimes_{\alpha,\Lambda} R\Gamma(\mathbb{Q}_p, \mathbb{T})) \to R\Gamma(\mathbb{Q}_p, L^n \otimes_{\alpha,\Lambda} \mathbb{T})$, which can be constructed by splitting $L^n \otimes_{\alpha,\Lambda} -$ up as $L \otimes_{\mathcal{O}_L} \mathcal{O}_L^n \otimes_{\alpha,\Lambda} -$ as after theorem 1.3.19 and using proposition 1.3.12 and proposition 2.7.11 of [NSW13].

Remark 2.3.7. The correct "sign" in the duality statement of conjecture 2.3.5 that makes the equivariant version compatible with the duality in 2.2.2, would be $[H^0(\mathbb{Q}_p, \mathbb{T}), -1]$. Indeed, if $\alpha : \Lambda \to M_n(L)$ is as in remark 2.3.6 we have $L^n \otimes_{\alpha,\Lambda} H^0(\mathbb{Q}_p, \mathbb{T}) \cong H^0(\mathbb{Q}_p, L^n \otimes_{\alpha,\Lambda} \mathbb{T})$ and

hence $[H^0(\mathbb{Q}_p, \mathbb{T}), -1]$ would be mapped to $(-1)^{\dim_L H^0(L^n \otimes_{\alpha,\Lambda} \mathbb{T})}$ under the map $K_1(\Lambda) \xrightarrow{L^n \otimes_{\alpha,\Lambda} -} K_1(L) = L^\times$. The Λ-module $H^0(\mathbb{Q}_p, \mathbb{T})$ is finitely generated since by proposition 1.3.12, $R\Gamma(\mathbb{Q}_p, \mathbb{T})$ is a perfect complex. However, we do not see why it should be projective. Therefore, $[H^0(\mathbb{Q}_p, \mathbb{T}), -1]$ might not define an element of $K_1(\Lambda)$.
We have not found a way to describe an element x of $K_1(\Lambda)$, depending on $\mathbb{T}$, such that for all α as in remark 2.3.6, we have $(L^n \otimes_{\alpha,\Lambda} -)(x) = (-1)^{\dim_L H^0(\mathbb{Q}_p, L^n \otimes_{\alpha,\Lambda} \mathbb{T})}$ for a general adic ring Λ. In the light of lemma 5.1.12 in [Wit08] the isomorphism $K_1(\Lambda) \cong \lim_n K_1(\Lambda/J^n)$ will not be helpful to define x. While Λ/J is semisimple, so that $\Lambda/J \otimes_\Lambda H^0(\mathbb{Q}_p, \mathbb{T}) \cong H^0(\mathbb{Q}_p, \Lambda/J \otimes_\Lambda \mathbb{T})$ is a projective Λ/J-module, the exact sequence

$$0 \to K_1(\Lambda, J) \to K_1(\Lambda) \to K_1(\Lambda/J) \to 0$$

only defines x up to $K_1(\Lambda, J)$. The most promising approach is to work with projective resolutions, which was suggested to us by Prof. Venjakob. We need Λ to have finite global dimension. Therefore, we assume that Λ is $\mathcal{O}[\![G]\!]$, where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ and G a profinte group that satsifies the condition in 1.4.2 of [FK06], so that Λ is an adic ring, and of finite cohomological p-dimension. Then the global dimension of Λ is finite by [Bru66] theorem 4.1 (see also [Ven02] theorem 3.26 for more information on the case where G is a compact p-adic Lie group without p-torsion[1]). Let $P^\bullet \to H^0(\mathbb{Q}_p, \mathbb{T})$ be a projective resolution of finite length, such that all P^i are finitely generated. Such a resolution exists since $H^0(\mathbb{Q}_p, \mathbb{T})$ is finitely generated. Then we define x to be the element $\prod_{i \in \mathbb{N}_0} [P^i, -1]$ in $K_1(\Lambda)$. This is well-defined, i.e. independent of the choice of $P^\bullet$, by the last part of the proof of the resolution theorem 3.1.14 in [Ros94]. Since $L^n \otimes_{\alpha,\Lambda} -$ is right-exact and $P^\bullet$ consists of projective modules, $L^n \otimes_{\alpha,\Lambda} P^\bullet \to L^n \otimes_{\alpha,\Lambda} H^0(\mathbb{Q}_p, \mathbb{T}) = H^0(\mathbb{Q}_p, L^n \otimes_{\alpha,\Lambda} \mathbb{T})$ is an exact sequence of finitely generated L-vector spaces. The identity $(L^n \otimes_{\alpha,\Lambda} -)(x) = (-1)^{\dim_L H^0(\mathbb{Q}_p, L^n \otimes_{\alpha,\Lambda} \mathbb{T})}$ is now clear by the additivity of the dimension in exact sequences of finite dimensional vector spaces.

[1] We are thankful to Oliver Thomas who pointed us to [Ven02].

Chapter 3

ε-isomorphisms and twists

In this section, we explore how the equivariant ε-isomorphism changes when the Galois representation $\mathbb{T}$ is twisted.

3.1 Problem setting

We will consider ε-isomorphisms for special triples $(\Lambda, \mathbb{T}, \xi)$. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$. Let $F/M/\mathbb{Q}_p$ be a tower of finite extensions. F is Galois over K with group $G = \mathrm{Gal}(F, K)$. Let T be a finitely generated projective $\mathcal{O}_L$-module (hence a free $\mathcal{O}_L$-module) with continuous, $\mathcal{O}_L$-linear G_K-action. Let $\chi : G_K \to \mathcal{O}_L^\times$ be a continuous character, which factors over $G = G_K/G_F$. We denote the twist $T \otimes_{\mathcal{O}_L} \mathcal{O}_L(\chi)$ by $T(\chi)$. Let Λ be the group ring $\mathcal{O}_L[G]$. In this case $\widetilde{\Lambda}$ is $\widehat{\mathbb{Z}_p^{nr}} \otimes_{\mathbb{Z}_p} \Lambda$ since Λ a profinite, finitely generated $\mathbb{Z}_p$-module, so that we have $-\widehat{\otimes}_{\mathbb{Z}_p}\Lambda = - \otimes_{\mathbb{Z}_p} \Lambda$ by proposition 5.5.3 (d) of [RZ00]. We define $\Lambda^\natural \otimes_{\mathcal{O}_L} T$ to be the G_K-representation with underlying module structure $\Lambda \otimes_{\mathcal{O}_L} T$ and Galois action given by $\sigma(\bar{g} \otimes t) = \overline{g\sigma^{-1}} \otimes \sigma(t)$. Note that the Λ-left-action and the Galois action commute. We set $\mathbb{T}(T) \coloneqq \mathrm{Ind}_K^{\mathbb{Q}_p}(\Lambda^\natural \otimes_{\mathcal{O}_L} T)$. Let ξ be a fixed basis of $\mathbb{Z}_p(1)$.
Our aim is to show that if an ε-isomorphism for the triple $(\Lambda, \mathbb{T}(T), \xi)$ exists then there is also an ε-isomorphism for the triple $(\Lambda, \mathbb{T}(T(\chi)), \xi)$. The uniqueness in conjecture 2.3.5 is only claimed for a system of ε-isomorphism for all possible triples $(\Lambda, \mathbb{T}, \xi)$ that satisfies all the listed properties, several of which concern the relationship of ε-isomorphisms for different triples. At the same time proposition 2.3.2 shows that a mere isomorphism between the correct objects in the determinant category always exists. Therefore, we have to be precise by what we mean with the existence of an ε-isomorphism. In particular, we will clearly state which properties of the untwisted ε-isomorphism are needed to ensure the corresponding property of the ε-isomorphism in the twisted situation. Another strategy to circumvent this issue is that we use the same construction to get from an isomorphism

$$\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) : \mathbb{1} \to \big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))d_\Lambda(\mathbb{T}(T))\big)_{\widetilde{\Lambda}}$$

to an isomorphism

$$\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))) : \mathbb{1} \to \big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi)))d_\Lambda(\mathbb{T}(T(\chi)))\big)_{\widetilde{\Lambda}}$$

for all choices of T and ξ. For this fixed candidate $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$, we will check each property of an ε-isomorphism assuming that $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ satisfies this property.

3.2 Twist as a base change

The key ingredient for constructing the ε-isomorphism in the twisted situation is to shift the twist from the $\mathcal{O}_L$-module T to a base change from the ring Λ to itself. Then the latter induces a twist functor $V(\Lambda) \to V(\Lambda)$. The idea for such a twist functor can be found in [LVZ15] p27 f and in [BV16] p33 ff.

Construction of the twist functor There is a continuous $\mathcal{O}_L$-algebra homomorphism $Tw_\chi : \Lambda \to \Lambda$ which is given on $\bar{g} \in G$ by $\bar{g} \mapsto \chi(\bar{g})^{-1}\bar{g}$. The continuity can be easily deduced from the fact that Tw_χ is $\mathcal{O}_L$-linear and the π_L-adic and the $J(\Lambda)$-adic topology on Λ agree (see 1.3.10 and 1.3.11). This homomorphism induces a functor $Tw_\chi^* := \Lambda \otimes_{Tw_\chi, \Lambda} - = \Lambda \otimes_{Tw_\chi} \Lambda \otimes_\Lambda -$ from the category of Λ-modules to itself. The Λ-left-module $\Lambda \otimes_{Tw_\chi} \Lambda$ is free of rank 1 via the map

$$\Lambda \otimes_{Tw_\chi} \Lambda \to \Lambda, \overline{g'} \otimes \bar{g} = \overline{g'}\chi(\bar{g})^{-1}\bar{g} \otimes 1 = 1 \otimes \chi(g')\overline{g'}\bar{g} \mapsto \overline{g'}\chi^{-1}(g)\bar{g}$$

(it also is a free Λ-right-module of rank 1 via $\overline{g'} \otimes \bar{g} \mapsto \chi(g')\overline{g'}\bar{g}$) and the left and right actions of Λ commute. We can apply lemma 1.1.40 and proposition 1.1.42 (3) to see that Tw_χ^* induces compatible (up to isomorphism) functors $\Lambda \otimes_{Tw_\chi} - : (D^p(\Lambda), is) \to (D^p(\Lambda), is)$ and $Tw_\chi^* : V(\Lambda) \to V(\Lambda)$. Note that $Tw_\chi^* : V(\Lambda) \to V(\Lambda)$ is a monoidal functor which commutes with d_Λ up to isomorphism of determinant functors and that we do not need the derived tensor product, since we tensor with a free module.
We can lift this construction to $\widetilde{\Lambda}$. As above, $\widehat{\mathbb{Z}_p^{nr}} \otimes_{\mathbb{Z}_p} \Lambda \otimes_{Tw_\chi, \Lambda} \Lambda$ is free of rank 1 as $\widetilde{\Lambda}$-left- or -right-module. So the exact functor $\left(\widehat{\mathbb{Z}_p^{nr}} \otimes_{\mathbb{Z}_p} \Lambda \otimes_{Tw_\chi, \Lambda} \Lambda\right) \otimes_{\widetilde{\Lambda}} - : \mathrm{PMod}(\widetilde{\Lambda}) \to \mathrm{PMod}(\widetilde{\Lambda})$ induces a functor $Tw_\chi^* : V(\widetilde{\Lambda}) \to V(\widetilde{\Lambda})$ with similar properties. Using lemma 1.1.35 part (3) we see that the following diagram commutes up to isomorphism of monoidal functors:

$$\begin{array}{ccc} V(\Lambda) & \xrightarrow{Tw_\chi^*} & V(\Lambda) \\ {\scriptstyle \widetilde{\Lambda}\otimes_\Lambda -}\downarrow & & \downarrow{\scriptstyle \widetilde{\Lambda}\otimes_\Lambda -} \\ V(\widetilde{\Lambda}) & \xrightarrow{Tw_\chi^*} & V(\widetilde{\Lambda}). \end{array}$$

The next lemma shows how base change of Λ along Tw_χ translates to the normal notion of twisting.

Lemma 3.2.1.

(1) Let M be a Λ-module. Then the $\mathcal{O}_L$-linear map

$$\rho : Tw_\chi^*(M) \to M \otimes_{\mathcal{O}_L} \mathcal{O}_L(\chi), \bar{g} \otimes m \mapsto \chi(g)\bar{g}m \otimes e_\chi,$$

where e_χ denotes a basis of $\mathcal{O}_L(\chi)$ on which Λ acts through χ, is a natural isomorphism of topological Λ-modules.

(2) Let N be an $\mathcal{O}_L$-linear G_K-module. The $\mathcal{O}_L$-linear map

$$\rho' : Tw_\chi^*(\Lambda^\natural \otimes_{\mathcal{O}_L} N) \to \Lambda^\natural \otimes_{\mathcal{O}_L} N(\chi), \overline{g'} \otimes \bar{g} \otimes n \mapsto \overline{g'}\chi(\bar{g})^{-1}\bar{g} \otimes n \otimes e_\chi$$

is a natural isomorphism of topological (Λ, G_K)-modules. Here, G_K acts on $\mathcal{O}_L(\chi)$ via χ and Λ only be left-multiplication on $\Lambda^\natural \otimes_{\mathcal{O}_L} N(\chi)$.

(3) Let N be as above. Then the map

$$Tw_\chi^*(\mathbb{T}(N)) \to \mathbb{T}(N(\chi)), \overline{g'} \otimes \sigma \otimes \overline{g} \otimes n \mapsto \sigma \otimes \overline{g'}\chi(\overline{g})^{-1}\overline{g} \otimes n \otimes e_\chi$$

is a natural isomorphism of topological $(\Lambda, G_{\mathbb{Q}_p})$-modules.

Proof.

(1) The map is well defined since $\overline{g} \otimes m = \overline{1} \otimes \chi(g)\overline{g}m$ both map to $\chi(g)\overline{g}m \otimes e_\chi$. The inverse of the map is given by $m \otimes e_\chi \mapsto \overline{1} \otimes m$. ρ is Λ-linear since

$$\rho(\overline{g}(\overline{1} \otimes m)) = \rho(\overline{g} \otimes m) = \chi(g)\overline{g}m \otimes e_\chi = \overline{g}m \otimes \chi(g)e_\chi = \overline{g}(m \otimes e_\chi) = \overline{g}\rho(\overline{1} \otimes m).$$

Since the topology on both modules is induced by Λ and ρ as well as ρ^{-1} are Λ-linear, they are both continuous.

(2) The map is well-defined since $\overline{g'} \otimes \overline{g} \otimes n = \overline{g'}\chi(g)^{-1}\overline{g} \otimes \overline{1} \otimes n$ both map to $\overline{g'}\chi(g)^{-1}\overline{g} \otimes n \otimes e_\chi$. The inverse map is given by $\overline{g} \otimes n \otimes e_\chi \mapsto \overline{g} \otimes \overline{1} \otimes n$. The map is clearly Λ-linear. For $\sigma \in G_K$ we have

$$\begin{aligned}\sigma(\rho'(\overline{g'} \otimes \overline{g} \otimes n)) &= \sigma(\overline{g'}\chi(g)^{-1}\overline{g} \otimes n \otimes e_\chi) = \overline{g'}\chi(g)^{-1}\overline{g\sigma^{-1}} \otimes \sigma(n) \otimes \chi(\sigma)e_\chi \\ &= \overline{g'}\chi(g\sigma^{-1})^{-1}\overline{g\sigma^{-1}} \otimes \sigma(n) \otimes e_\chi = \rho'(\overline{g'} \otimes \overline{g\sigma^{-1}} \otimes \sigma(n)) \\ &= \rho'(\sigma(\overline{g'} \otimes \overline{g} \otimes n)).\end{aligned}$$

The continuity is again due to the Λ-linearity.

(3) Since the Λ- and the G_K-action commute on $\Lambda^\natural \otimes N$, the isomorphism in (3) comes from the one in (2) after tensoring with $\mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_K]} -$. The topology of the induction is the product topology and hence continuity is again clear.

$\square$

Applying the above lemma to our setting yields:

Corollary 3.2.2.

(1) There is a natural Λ-isomorphism $\phi_0(T) : Tw_\chi^* R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)) \to R\Gamma(\mathbb{Q}_p, \mathbb{T})(\chi)$.

(2) There is a natural isomorphism $\phi_2(T) : Tw_\chi^*(\mathbb{T}(T)) \to \mathbb{T}(T(\chi))$ of topological $(\Lambda, G_{\mathbb{Q}_p})$-modules.

(3) The four different ways to pull out the twist make the following diagram commute:

$$\begin{tikzcd} Tw_\chi^*(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))) \arrow[r, "\omega"] \arrow[d, "\phi_0(T)"'] & R\Gamma(\mathbb{Q}_p, Tw_\chi^*(\mathbb{T}(T))) \arrow[d, "\phi_2(T)_*"] \\ R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))(\chi) \arrow[r, "\tilde{\vartheta}"] & R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))), \end{tikzcd}$$

where ω is the natural quasi-isomorphism from proposition 1.3.12 (3) and $\tilde{\vartheta}$ is the quasi-isomorphism that comes from Shapiro's lemma at the end of subsection 1.3.1. We call the natural morphism from the upper left to lower right corner $\phi_1(T)$.

Proof. The first two assertions are just (1) and (3) of lemma 3.2.1. The third one follows from unpacking the definitions. $\square$

Lemma 3.2.3. *The twist functor commutes with the $K_1(\Lambda)$-torseur structure in the following way: Let α be an element of $K_1(\Lambda)$ and $f : A \to B$ a morphism in $V(\Lambda)$. Then $Tw_\chi^*(\alpha f) = Tw_\chi^*(\alpha)Tw_\chi^*(\phi)$, where $Tw_\chi^* : K_1(\Lambda) \to K_1(\Lambda)$ is the base change homomorphism along the map $Tw_\chi : \Lambda \to \Lambda$ from lemma 1.1.10 sending $[P, \sigma]$ to $[Tw_\chi^*(P), Tw_\chi^*(\sigma)]$. The same holds for the twist functor $Tw_\chi^* : V(\widetilde{\Lambda}) \to V(\widetilde{\Lambda})$.*

Proof. Let $\alpha = [P, \sigma]$. By lemma 1.1.35 (4) and lemma 1.1.24, αf is given by

$$A \to d_\Lambda(P)d_\Lambda(P)^{-1}A \xrightarrow{d_\Lambda(\sigma)\cdot \mathrm{id}\cdot f} d_\Lambda(P)d_\Lambda(P)^{-1}B \to B.$$

The twist functor Tw_χ^* commutes with d_Λ and since it is a monoidal functor, it also commutes with taking inverses $(-)^{-1}$ (lemma 1.1.30). Therefore, we get that $Tw_\chi^*(\alpha f)$ is

$$\begin{aligned} Tw_\chi^*(A) &\to d_\Lambda(Tw_\chi^*(P))d_\Lambda(Tw_\chi^*(P))^{-1}Tw_\chi^*(A) \\ &\xrightarrow{d_\Lambda(Tw_\chi^*(\sigma))\,\mathrm{id}\,Tw_\chi^*(f)} d_\Lambda(Tw_\chi^*(P))d_\Lambda(Tw_\chi^*(P))^{-1}Tw_\chi^*(B) \to Tw_\chi^*(B). \end{aligned}$$

Employing the lemmata 1.1.35 and 1.1.24 once more, we see that this is just $[Tw_\chi^*(P), Tw_\chi^*(\sigma)]Tw_\chi^*(f)$. □

3.3 ε-isomorphism of the twisted representation

We can now define the candidate for an ε-isomorphism for $\mathbb{T}(T(\chi))$ given $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$.

Definition 3.3.1. *Let Λ, χ and T be as in the problem setting in section (3.1). Given an isomorphism $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) : \mathbb{1}_{\widetilde{\Lambda}} \to \big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)))d_\Lambda(\mathbb{T}(T))\big)_{\widetilde{\Lambda}}$, we define*

$$\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))) : \mathbb{1}_{\widetilde{\Lambda}} \to \big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))))d_\Lambda(\mathbb{T}(T(\chi)))\big)_{\widetilde{\Lambda}}$$

as the map

$$\begin{aligned} \mathbb{1} \xrightarrow{d_\Lambda(u_\chi)_{\widetilde{\Lambda}}} Tw_\chi^*(\mathbb{1}) &\xrightarrow{Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)))} Tw_\chi^*\Big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)))d_\Lambda(\mathbb{T}(T))\Big)_{\widetilde{\Lambda}} \\ &= Tw_\chi^*\big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)))\big)_{\widetilde{\Lambda}} \cdot Tw_\chi^*\big(d_\Lambda(\mathbb{T}(T))\big)_{\widetilde{\Lambda}} \\ &\xrightarrow{\left(d_\Lambda(\phi_1(T))\cdot d_\Lambda(\phi_2(T))\right)_{\widetilde{\Lambda}}} (d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))))d_\Lambda(\mathbb{T}(T(\chi))))_{\widetilde{\Lambda}}, \end{aligned}$$

where $u_\chi : 0 \xrightarrow{\sim} \Lambda \otimes_{Tw_\chi} 0$.

Remark 3.3.2. For the duality statement, we consider the twist homomorphism as a ring homomorphism $Tw : \Lambda^\circ \to \Lambda^\circ, \bar{g} \mapsto \chi(\bar{g})^{-1}\bar{g}$ and define a twist functor Tw_χ^* as the base change $\Lambda^\circ \otimes_{Tw_\chi^*} -$. Given an isomorphism

$$\varepsilon_{\Lambda^\circ,\xi}\big(\mathbb{T}(T)^*(1)\big) : \mathbb{1}_{\widetilde{\Lambda^\circ}} \to \Big(d_{\Lambda^\circ}\big(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)^*(1))\big)d_{\Lambda^\circ}\big(\mathbb{T}(T)^*(1)\big)\Big)_{\widetilde{\Lambda^\circ}},$$

we can define

$$\varepsilon_{\Lambda^\circ,\xi}\big(\mathbb{T}(T(\chi))^*(1)\big) : \mathbb{1}_{\widetilde{\Lambda^\circ}} \to \Big(d_{\Lambda^\circ}\big(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))^*(1))\big)d_{\Lambda^\circ}\big(\mathbb{T}(T(\chi))^*(1)\big)\Big)_{\widetilde{\Lambda^\circ}}$$

as before using the isomorphism

$$\phi_2' : \Lambda^\circ \otimes_{Tw_\chi} \mathbb{T}(T)^* \xrightarrow{\nu} (\Lambda \otimes_{Tw_\chi} \mathbb{T}(T))^* \xrightarrow{\phi_2(T)^{*-1}} \mathbb{T}(T(\chi))^*$$

with ν as in lemma 1.3.17 instead of $\phi_2(T)$ and instead of $\phi_1(T)$ the isomorphism

$$\phi_1'(T) : \Lambda^\circ \otimes_{Tw_\chi} R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)^*(1)) \xrightarrow{\omega} R\Gamma(\mathbb{Q}_p, \Lambda^\circ \otimes_{Tw_\chi} \mathbb{T}(T)^*(1)) \xrightarrow{\phi_{2*}'} R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))^*(1)).$$

The morphism ω is from proposition 1.3.12. Consider Λ as a (Λ, Λ)-module, where the right action is via Tw_χ. Then Λ^* is just Λ° with the usual left-action of Λ° and a right-action of Λ° via Tw_χ. This explains that the isomorphism ν in the definition of ϕ_2' is well-defined.

In order to ease notation, we have omitted all natural isomorphisms that arise from composing monoidal functors in the determinant category (1.1.35 (3)) or those connected to monoidal functors. We will continue doing this in the following. Also, the isomorphisms attached to units and inverses in the determinant category will be left out. Since units are unique up to unique isomorphism in a Picard category, we will sometimes also omit the isomorphism of units $d_\Lambda(u_\chi)$.

$\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ satisfies each of the properties of an equivariant ε-isomorphism that $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ satisfies.We will verify this in the following. Since we consider ε-isomorphism for a fixed adic ring Λ, we will consider the specialisation property from remark 2.3.6.

Proposition 3.3.3 (Specialisation). *Suppose that for each finite extension L' of $\mathbb{Q}_p$ and each continuous ring homomorphism $\alpha : \Lambda \to M_n(L')$, such that $L'^n \otimes_{\alpha,\Lambda} \mathbb{T}(T)$ is an L'-linear de Rham representation of $G_{\mathbb{Q}_p}$, we have $L'^n \otimes_{\alpha,\Lambda} \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) = \varepsilon_{L',\xi}(L'^n \otimes_{\alpha,\Lambda} \mathbb{T}(T))$. Then for each choice L' and α, such that $L'^n \otimes_{\alpha,\Lambda} \mathbb{T}(T(\chi))$ is an L'-linear de Rham representation of $G_{\mathbb{Q}_p}$, we also have $L'^n \otimes_{\alpha,\Lambda} \varepsilon_{\Lambda,\xi}(\mathbb{T}) = \varepsilon_{L',\xi}(L'^n \otimes_{\alpha,\Lambda} \mathbb{T})$.*

Proof. Let L' be a finite extension of $\mathbb{Q}_p$ and let Λ act continuously on L'^n from the right, via a continuous ring homomorphism $\alpha : \Lambda \to M_n(L')$, such that the L'-linear continuous $G_{\mathbb{Q}_p}$-representation $L'^n \otimes_\alpha \mathbb{T}(T(\chi))$ is de Rham. Then we have an $(L', G_{\mathbb{Q}_p})$-isomorphism

$$L'^n \otimes_\alpha \mathbb{T}(T(\chi)) \xrightarrow[\sim]{\mathrm{id}_{L'^n} \otimes \phi_2(T)^{-1}} L'^n \otimes_\alpha Tw_\chi^*(\mathbb{T}(T)) = L'^n \otimes_{\alpha \circ Tw_\chi} \mathbb{T}(T)$$

so that the specialisation of $\mathbb{T}(T)$ via the continuous ring homomorphism $\Lambda \xrightarrow{Tw_\chi} \Lambda \xrightarrow{\alpha} M_n(L')$ is also de Rham. Hence, we can use the specialisation property of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ along $\alpha \circ Tw_\chi$ and get that $d_{\widetilde{L'}}(\omega) \circ L'^n \otimes_{\alpha \circ Tw_\chi} \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) = \varepsilon_{L',\xi}(L'^n \otimes_{\alpha \circ Tw_\chi} \mathbb{T}(T))$. As a result, we deduce

$$\begin{aligned}
& d_{\widetilde{L'}}(\omega) \circ L'^n \otimes_\alpha \varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))) \\
=& d_{\widetilde{L'}}(\omega) \circ L'^n \otimes_\alpha \Big((d_\Lambda(\phi_1) d_\Lambda(\phi_2))_{\widetilde{\Lambda}} \circ Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}} \Big) \\
=& d_{\widetilde{L'}}(\omega) \circ \big(d_{L'}(L'^n \otimes_\alpha \phi_1) d_{L'}(L'^n \otimes_\alpha \phi_2) \big)_{\widetilde{L'}} \circ L'^n \otimes_{\alpha \circ Tw_\chi} \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) \\
=& \big(d_{L'}(R\Gamma(\mathbb{Q}_p, L'^n \otimes_\alpha \phi_2)) d_{L'}(L'^n \otimes_\alpha \phi_2) \big)_{\widetilde{L'}} \circ \varepsilon_{L',\xi}(L'^n \otimes_{\alpha \circ Tw_\chi} \mathbb{T}(T)) \\
=& \varepsilon_{L',\xi}(L' \otimes_\alpha \mathbb{T}(T(\chi))).
\end{aligned}$$

The first equality is the definition of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$. We used the commutativity of base change with d_Λ to get from the second to the third line. From the third to the fourth line, we used that, by definition of ϕ_1, we have $\omega_{L^n} \circ L'^n \otimes_\alpha \phi_1 = R\Gamma(\mathbb{Q}_p, L'^n \otimes_\alpha \phi_2) \circ \omega_{Tw}$ and the specialisation property of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$. The final equality is just the multiplicativity of $\varepsilon_{L',\xi}$ with respect to the exact sequence

$$0 \to L'^n \otimes_{\alpha\circ Tw_\chi} \mathbb{T}(T) \xrightarrow{L'^n \otimes \phi_2} L'^n \otimes_\alpha \mathbb{T}(T(\chi)) \to 0 \to 0.$$

Recall that the value of a determinant functor on such a short exact sequence is compatible with its value on the isomorphism by the third requirement in definition 1.1.32. □

Proposition 3.3.4 (Multiplicativity)**.** *Let $\Sigma : 0 \to T' \to T \to T'' \to 0$ be a short exact sequence of $\mathbb{O}_L$ -linear G_K-representations and let $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)$, $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T'))$ and $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T'')$ be isomorphisms such that the multiplicativity of conjecture 2.3.5 holds with respect to the induced short exact sequence of $G_{\mathbb{Q}_p}$-modules $\mathbb{T}(\Sigma) : 0 \to \mathbb{T}(T') \to \mathbb{T}(T) \to \mathbb{T}(T'') \to 0$. Then the multiplicativity holds for the isomorphisms $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))))$, $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T'(\chi))))$ and $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T''(\chi))))$ with respect to the short exact sequence obtained by twisting Σ with χ and applying $\mathbb{T}(-)$.*

Proof. Since ϕ_2 is natural, we have an isomorphism

$$\begin{array}{cccccccccc}
Tw_\chi^*(\mathbb{T}(\Sigma)) : & 0 & \longrightarrow & Tw_\chi^*(\mathbb{T}(T')) & \longrightarrow & Tw_\chi^*(\mathbb{T}(T)) & \longrightarrow & Tw_\chi^*(\mathbb{T}(T'')) & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \phi_2(T')} & & \downarrow{\scriptstyle \phi_2(T)} & & \downarrow{\scriptstyle \phi_2(T'')} & & \\
\mathbb{T}(\Sigma(\chi)) : & 0 & \longrightarrow & \mathbb{T}(T'(\chi)) & \longrightarrow & \mathbb{T}(T(\chi)) & \longrightarrow & \mathbb{T}(T''(\chi)) & \longrightarrow & 0
\end{array}$$

of short exact sequences of Λ-modules which induces the identity

$$d_\Lambda(\phi_2(T'))d_\Lambda(\phi_2(T'')) \circ d_\Lambda\big(\, Tw_\chi^*(\mathbb{T}(\Sigma))\big) = d_\Lambda\big(\,\mathbb{T}(\Sigma(\chi))\big) \circ d_\Lambda(\phi_2(T)).$$

Moreover, we have $d_\Lambda(\,Tw_\chi^*(\mathbb{T}(\Sigma))) = Tw_\chi^*(d_\Lambda(\mathbb{T}(\Sigma)))$ since d_Λ commutes with base change (proposition 1.1.42 (3)). Similarly, we get

$$d_\Lambda(\phi_1(T'))d_\Lambda(\phi_1(T'')) \circ Tw_\chi^*\, d_\Lambda(C(\mathbb{Q}_p, \mathbb{T}(\Sigma))) = d_\Lambda(C(\mathbb{Q}_p, \mathbb{T}(\Sigma(\chi)))) \circ d_\Lambda(\phi_1(T)).$$

Putting this together yields:

$$\begin{aligned}
&\Big(d_\Lambda\big(C(\mathbb{Q}_p, \mathbb{T}(\Sigma(\chi)))\big)d_\Lambda\big(\,\mathbb{T}(\Sigma(\chi))\big)\Big)_{\widetilde{\Lambda}} \circ \varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T(\chi))\big) \\
=&\Big(d_\Lambda\big(C(\mathbb{Q}_p, \mathbb{T}(\Sigma(\chi)))\big)d_\Lambda\big(\,\mathbb{T}(\Sigma(\chi))\big)\Big)_{\widetilde{\Lambda}} \circ \Big(d_\Lambda\big(\phi_1(T)\big)d_\Lambda\big(\phi_2(T)\big)\Big)_{\widetilde{\Lambda}} \circ Tw_\chi^*\,\big(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))\big) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}} \\
=&\Big(d_\Lambda\big(\phi_1(T')\big)d_\Lambda\big(\phi_1(T'')\big)d_\Lambda\big(\phi_2(T')\big)d_\Lambda\big(\phi_2(T'')\big)\Big)_{\widetilde{\Lambda}} \\
&\circ Tw_\chi^*\,\bigg(\Big(d_\Lambda\big(\,\mathbb{T}(\Sigma)\big)d_\Lambda\big(C(\mathbb{Q}_p, \mathbb{T}(\Sigma))\big)\Big)_{\widetilde{\Lambda}} \circ \varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T)\big)\bigg) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}} \\
=&\Big(d_\Lambda\big(\phi_1(T')\big)d_\Lambda\big(\phi_1(T'')\big)d_\Lambda\big(\phi_2(T')\big)d_\Lambda\big(\phi_2(T'')\big)\Big)_{\widetilde{\Lambda}} \circ Tw_\chi^*\,\Big(\,\varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T')\big)\,\varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T'')\big)\Big) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}} \\
=&\bigg(\Big(d_\Lambda\big(\phi_1(T')\big)d_\Lambda\big(\phi_2(T')\big)\Big)_{\widetilde{\Lambda}} \circ Tw_\chi^*\,\Big(\,\varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T')\big)\Big) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}}\bigg) \cdot \\
&\bigg(\Big(d_\Lambda\big(\phi_1(T'')\big)d_\Lambda\big(\phi_2(T'')\big)\Big)_{\widetilde{\Lambda}} \circ Tw_\chi^*\,\Big(\,\varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T'')\big)\Big) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}}\bigg) \\
=&\,\varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T'(\chi))\big)\,\varepsilon_{\Lambda,\xi}\,\big(\,\mathbb{T}(T''(\chi))\big).
\end{aligned}$$

where at the first and last equality sign, we used definition 3.3.1, at the second we used the naturality of the determinant functor on the short exact sequences from above, at the third we used the multiplicativity of the untwisted ε-isomorphisms and at the fourth we just rearranged. □

Proposition 3.3.5 (Base Change)**.** *Let $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ be compatible with base change. In other words, let Λ' be any adic ring and Y a finitely generated projective Λ'-module with commuting continuous Λ-right-action via a ring homomorphism $\beta : \Lambda \to \mathrm{End}(Y)$. If the ε-isomorphism for the triple $(\Lambda', Y \otimes_{\beta\circ Tw} \mathbb{T}(T), \xi)$ exists, then it is given by $\varepsilon_{\Lambda',\xi}(Y \otimes_{\beta\circ Tw} \mathbb{T}(T)) = Y \otimes_{\beta\circ Tw} \varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$. Assume further, that the we have*

$$\varepsilon_{\Lambda',\xi}(Y \otimes_\beta \mathbb{T}(T(\chi))) = Y \otimes_\beta (d_\Lambda(\phi_1)d_\Lambda(\phi_2))_{\widetilde{\Lambda}} \circ \varepsilon_{\Lambda',\xi}(Y \otimes_\beta Tw_\chi^*(\mathbb{T}(T)))$$

which is just an instance of the multiplicativity of $\varepsilon_{\Lambda',\xi}(-)$. Then $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ satisfies the base change property with respect to Λ', Y and β.

Proof. We will proceed similarly to the specialisation property. Let Λ', Y and β be as in the proposition. Since the ring homomorphism $Tw : \Lambda \to \Lambda$ is continuous, so is the right-action of Λ on Y via $\beta \circ Tw$ and we can do the following calculations:

$$\begin{aligned}
Y \otimes_\beta \varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))) &= Y \otimes_\beta \Big((d_\Lambda(\phi_1)d_\Lambda(\phi_2))_{\widetilde{\Lambda}} \circ Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))) \circ d_\Lambda(u_\chi)_{\widetilde{\Lambda}}\Big) \\
&= Y \otimes_\beta \big((d_\Lambda(\phi_1)d_\Lambda(\phi_2))_{\widetilde{\Lambda}}\big) \circ Y \otimes_\beta Tw_\chi^* \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) \circ Y \otimes_\beta d_\Lambda(u_\chi)_{\widetilde{\Lambda}} \\
&= Y \otimes_\beta \big((d_\Lambda(\phi_1)d_\Lambda(\phi_2))_{\widetilde{\Lambda}}\big) \circ Y \otimes_{\beta\circ Tw} \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) \\
&= Y \otimes_\beta \big((d_\Lambda(\phi_1)d_\Lambda(\phi_2))_{\widetilde{\Lambda}}\big) \circ \varepsilon_{\Lambda',\xi}(Y \otimes_{\beta\circ Tw} \mathbb{T}(T)) \\
&= Y \otimes_\beta \big((d_\Lambda(\phi_1)d_\Lambda(\phi_2))_{\widetilde{\Lambda}}\big) \circ \varepsilon_{\Lambda',\xi}(Y \otimes_\beta Tw_\chi^*(\mathbb{T}(T))) \\
&= \varepsilon_{\Lambda',\xi}(Y \otimes_\beta \mathbb{T}(T(\chi)))
\end{aligned}$$

The first two lines are just unpacking definitions and rearranging. From the second to the third line, we pulled the twist functor down into the Λ-action on Y. Then we used the base change property of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ along $\beta\circ Tw$. Finally, we use the assumed multiplicativity of $\varepsilon_{\Lambda',\xi}(-)$. $\square$

Proposition 3.3.6 (Change of ξ)**.** *Let $\sigma \in I(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \subset G_{\mathbb{Q}_p}^{ab}$. If there are isomorphisms $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ and $\varepsilon_{\Lambda,\chi_{cycl}(\sigma)\xi}(\mathbb{T}(T))$ compatible with a change of ξ as in conjecture 2.3.5, then $\varepsilon_{\Lambda,\chi_{cycl}(\sigma)\xi}(\mathbb{T}(T(\chi))) = [\mathbb{T}(T(\chi)), \sigma]\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$.*

Proof. Since $\phi_2(T)$ is an isomorphism of $G_{\mathbb{Q}_p}$-modules, the diagram

$$\begin{tikzcd}
Tw_\chi^*(\mathbb{T}(T)) \arrow[r, "Tw_\chi^*(\sigma)=\mathrm{id}_\Lambda \otimes_{Tw_\chi}\sigma"] \arrow[d, "\phi_2(T)"'] & Tw_\chi^*(\mathbb{T}(T)) \arrow[d, "\phi_2(T)"] \\
\mathbb{T}(T(\chi)) \arrow[r, "\sigma"] & \mathbb{T}(T(\chi))
\end{tikzcd}$$

commutes, which shows $[Tw_\chi^*(\mathbb{T}(T)), Tw_\chi^*(\sigma)] = [\mathbb{T}(T(\chi)), \sigma]$. This implies the last equality in the following calculation

$$\begin{aligned}
\varepsilon_{\Lambda,\chi_{cycl}(\sigma)\xi}(\mathbb{T}(T(\chi)) &= \big(d_\Lambda(\phi_1(T))d_\Lambda(\phi_2(T))\big)_{\widetilde{\Lambda}} \circ Tw_\chi^*\big(\varepsilon_{\Lambda,\chi_{cycl}(\sigma)\xi}(\mathbb{T}(T))\big) \circ d_\Lambda(u_\chi) \\
&= \big(d_\Lambda(\phi_1(T))d_\Lambda(\phi_2(T))\big)_{\widetilde{\Lambda}} \circ Tw_\chi^*\big([\mathbb{T}(T), \sigma])\,\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))\big) \circ d_\Lambda(u_\chi) \\
&= \big(d_\Lambda(\phi_1(T))d_\Lambda(\phi_2(T))\big)_{\widetilde{\Lambda}} \circ [Tw_\chi^*(\mathbb{T}(T)), Tw_\chi^*(\sigma)]\; Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))) \circ d_\Lambda(u_\chi) \\
&= [Tw_\chi^*(\mathbb{T}(T)), Tw_\chi^*(\sigma)](d_\Lambda(\phi_1(T))d_\Lambda(\phi_2(T))_{\widetilde{\Lambda}} \circ Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))) \circ d_\Lambda(u_\chi) \\
&= [Tw_\chi^*(\mathbb{T}(T)), Tw_\chi^*(\sigma)]\,\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))) \\
&= [\mathbb{T}(T(\chi)), \sigma]\,\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))).
\end{aligned}$$

At the second equation sign we used that the untwisted ε-isomorphisms are assumed to be compatible with the change of ξ as in conjecture 2.3.5. The third equation holds because the twist functor commutes with the $K_1(\Lambda)$-action by lemma 3.2.3. At the first and fifth equation sign, we used the definition of the ε-isomorphism for a twisted representation (3.3.1). Finally, we used lemma 1.1.24 for the fourth equality. $\square$

Proposition 3.3.7 (Frobenius invariance)**.** *Let $\phi \in G_{\mathbb{Q}_p}$ be a Frobeniuslift with $\chi_{cycl}(\phi) = 1$. Denote by $\varphi_p := \phi|_{\widehat{\mathbb{Z}_p^{nr}}} : \widehat{\mathbb{Z}_p^{nr}} \to \widehat{\mathbb{Z}_p^{nr}}$ the restriction of $\phi : B_{\mathrm{dR}} \to B_{\mathrm{dR}}$. The ring homomorphism $\varphi_p \otimes \mathrm{id}_\Lambda : \widetilde{\Lambda} \to \widetilde{\Lambda}$ induces a base change homomorphism $(\varphi_p \otimes \mathrm{id}_\Lambda)^* : K_1(\widetilde{\Lambda}) \to K_1(\widetilde{\Lambda})$ (see lemma 1.1.10). We put $K_1(\widetilde{\Lambda})_{\mathbb{T}(T)} = \{x \in K_1(\widetilde{\Lambda}) | (\varphi_p \otimes \mathrm{id})^*(x) = [\mathbb{T}(T), \phi]^{-1} x\}$ and $K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))} = \{x \in K_1(\widetilde{\Lambda}) | (\varphi_p \otimes \mathrm{id})^*(x) = [\mathbb{T}(T(\chi)), \phi]^{-1} x\}$. If the ε-isomorphism $\varepsilon_{\Lambda,\xi}(\mathbb{T})$ belongs to $V(\Lambda)\big(\mathbb{1}, d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)) d_\Lambda(\mathbb{T}(T))\big) \overset{K_1(\Lambda)}{\times} K_1(\widetilde{\Lambda})_{\mathbb{T}(T)}$, then $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ belongs to $V(\Lambda)\big(\mathbb{1}, d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))) d_\Lambda(\mathbb{T}(T(\chi)))\big) \overset{K_1(\Lambda)}{\times} K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))}$.*

Proof. Let $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) \in V(\Lambda)\big(\mathbb{1}, d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)) d_\Lambda(\mathbb{T}(T))\big) \overset{K_1(\Lambda)}{\times} K_1(\widetilde{\Lambda})_{\mathbb{T}(T)}$ be given as (f, α). Since the twist functor commutes with the action of $K_1(\widetilde{\Lambda})$ as described in lemma 3.2.3, we have that

$$\begin{aligned} Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) &= Tw_\chi^*((f,\alpha)) \\ &= \Big(Tw_\chi^*(f), Tw_\chi^*(\alpha)\Big) \in V(\Lambda)\big(\, Tw_\chi^*(\mathbb{1}), Tw_\chi^*\left(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T})) d_\Lambda(\mathbb{T})\right)\big) \overset{K_1(\Lambda)}{\times} K_1(\widetilde{\Lambda}). \end{aligned}$$

The isomorphisms with which we pre- and post-compose $Tw_\chi^*(\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ to get $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ are defined over Λ, so that

$$\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)) \in V(\Lambda)\big(\mathbb{1}, d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))) d_\Lambda(\mathbb{T}(T(\chi)))\big) \overset{K_1(\Lambda)}{\times} K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))}$$

holds if and only if $Tw_\chi^*(\alpha)$ lies in $K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))}$. We show more generally, that the map $Tw_\chi^* : K_1(\widetilde{\Lambda}) \to K_1(\widetilde{\Lambda})$ restricts to an isomorphism $K_1(\widetilde{\Lambda})_{\mathbb{T}(T)} \to K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))}$. The ring homomorphism $Tw_\chi : \Lambda \to \Lambda$ has the inverse $Tw_{\chi^{-1}}$. It induces an inverse to the homomorphism $Tw_\chi^* : K_1(\widetilde{\Lambda}) \to K_1(\widetilde{\Lambda})$, which is hence automatically injective. By symmetry, we only need to prove that $Tw_\chi^*(K_1(\widetilde{\Lambda})_{\mathbb{T}(T)}) \subset K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))}$. Let $\alpha \in K_1(\widetilde{\Lambda})_{\mathbb{T}(T)}$. Both homomorphisms Tw_χ^* and $(\varphi_p \otimes \mathrm{id}_\Lambda)^*$ come from base changes. These base changes commute since the first one is only on the first, the second only on the second factor of $\widetilde{\Lambda} = \widehat{\mathbb{Z}_p^{nr}} \otimes \Lambda$. Hence, so do Tw_χ^* and $(\varphi_p \otimes \mathrm{id}_\Lambda)^*$. We get that

$$\begin{aligned} (\varphi_p \otimes \mathrm{id}_\Lambda)^* \left(Tw_\chi^*(\alpha)\right) &= Tw_\chi^*\left((\varphi_p \otimes \mathrm{id}_\Lambda)^*(\alpha)\right) \\ &= Tw_\chi^*\left([\mathbb{T}(T), \phi]^{-1}\alpha\right) \\ &= Tw_\chi^*\left([\mathbb{T}(T), \phi]\right)^{-1} Tw_\chi^*(\alpha), \end{aligned}$$

where at the second equality sign we used that α came from $K_1(\widetilde{\Lambda})_{\mathbb{T}(T))}$. We saw above in the proof of proposition 3.3.6 that $Tw_\chi^*\left([\mathbb{T}(T), \phi]\right) = [\mathbb{T}(T(\chi)), \phi]$. So $Tw_\chi^*(\alpha)$ lies in $K_1(\widetilde{\Lambda})_{\mathbb{T}(T(\chi))}$. $\square$

Proposition 3.3.8 (Duality). *If $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ satisfies the duality statement in conjecture 2.3.5, then so does $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$, i.e. up to a sign as in remark 2.3.7, we have*

$$\varepsilon_{\Lambda,\xi}\big(\mathbb{T}(T(\chi))\big)\varepsilon_{\Lambda^\circ,-\xi}\big(\mathbb{T}(T(\chi))^*(1)\big)^*\overline{d_\Lambda\big(\psi(\mathbb{Q}_p,\mathbb{T}(T(\chi)))\big)}^{-1} = d_\Lambda\left(\mathbb{T}(T(\chi))(-1)\xrightarrow{\cdot\xi}\mathbb{T}(T(\chi))\right)_{\widetilde{\Lambda}}.$$

Proof. We begin by noting that the map $\cdot\xi$ is a natural transformation between the identity on $\mathrm{PMod}(\Lambda)$ and the functor $(-)(-1)$. So, the diagram

$$\begin{tikzcd}
Tw_\chi^*(\mathbb{T}(T)(-1) \arrow[r, "\cdot\xi"] \arrow[d, "\phi_2(T)(-1)"'] & Tw_\chi^*(\mathbb{T}(T)) \arrow[d, "\phi_2(T)"] \\
\mathbb{T}(T(\chi))(-1) \arrow[r, "\cdot\xi"] & \mathbb{T}(T(\chi))
\end{tikzcd}$$

commutes. Clearly, the twist functor Tw_χ^*, being a base change, commutes with $(-)(-1)$. Thus, we get that the left side of the following diagram is just multiplication by ξ:

$$\begin{tikzcd}
d_\Lambda(\mathbb{T}(T(\chi))(-1)) \arrow[r, "\mu"] \arrow[d, "\overline{\phi_2(-1)}"] & \begin{array}{c} d_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}(T(\chi))^*(1))^*\cdot d_\Lambda(\mathbb{T}(T(\chi))^*(1))^* \\ \cdot(d_\Lambda(R\Gamma((\mathbb{Q}_p,\mathbb{T}(T(\chi))^*(1))^*)^{-1}\end{array} \arrow[d, "(\phi_1'\cdot\phi_2'\cdot\phi_1'^{-1})^*"] \\
Tw_\chi^*(d_\Lambda(\mathbb{T}(T)(-1))) \arrow[r, "\mu"] \arrow[d, "\cdot\xi"'] & \begin{array}{c} Tw_\chi^*\big(d_\Lambda(R(\Gamma(\mathbb{Q}_p,\mathbb{T}(T)^*(1))^*\cdot d_\Lambda(\mathbb{T}(T)^*(1)))^* \\ \cdot\, d(R\Gamma(\mathbb{Q}_p,\mathbb{T}(T)^*(1))^*)^{-1}\big)\end{array} \arrow[d] \\
Tw_\chi^*(d_\Lambda(\mathbb{T}(T)) \arrow[r, "\mu"] \arrow[d, "\phi_2"] & Tw_\chi^*\big(d_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}(T)))\cdot d_\Lambda(\mathbb{T}(T))\cdot d_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}(T)))^{-1} \arrow[d, "\phi_1\cdot\phi_2\cdot\phi_1^{-1}"] \\
d_\Lambda(\mathbb{T}(T(\chi)) \arrow[r, "\mu"] & d_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}(T(\chi)))\cdot d_\Lambda(\mathbb{T}(T(\chi)))\cdot d_\Lambda(R\Gamma(\mathbb{Q}_p,\mathbb{T}(T)))^{-1}.
\end{tikzcd}$$

The diagram itself commutes up to the necessary sign from remark 2.3.7. Each horizontal arrow is induced by de-trivialising objects. The upper two also use the isomorphism $(-)^*(1)^* = (-)(-1)$ (1.2.8) and the middle two are post-composed with Tw_χ^*. The twist functor commutes with d_Λ by construction, with duals by remark 3.3.2 and with inverses by lemma 1.1.30 since it is monoidal. Therefore, we can pull it out as done in the middle of the right column. The upper and lower square commute by the definition of inverses of morphisms (1.1.20 (3)) and using the isomorphism $(-)^*(1)^* = (-)(-1)$ in the upper square. The middle square is nothing but the duality for $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ and $\varepsilon_{\Lambda^\circ,-\xi}(\mathbb{T}(T)^*(1))^*$ post-composed with Tw_χ^*. So this square commutes up to Tw_χ^* applied to the sign from remark 2.3.7 for $\mathbb{T}(T)$. We have the Λ-isomorphism

$$\phi_1^0 : Tw_\chi^*(H^0(\mathbb{Q}_p,\mathbb{T}(T))) \cong H^0(\mathbb{Q}_p, Tw_\chi^*(\mathbb{T}(T))) \cong H^0(\mathbb{Q}_p,\mathbb{T}(T(\chi))).$$

Now, if we have a finite resolution $P^\bullet \to H^0(\mathbb{Q}_p,\mathbb{T}(T))$, consisting of finitely generated, projective Λ-modules, we obtain a resolution $\Lambda\otimes_{Tw} P^\bullet \to H^0(\mathbb{Q}_p,\mathbb{T}(T(\chi)))$ with the same properties. This shows that applying Tw_χ^* to the sign associated to $\mathbb{T}(T)$ yields the sign associated to $\mathbb{T}(T(\chi))$.

Therefore, it only remains to prove that the right hand side of the diagram is the product $\varepsilon_{\Lambda,\xi}\big(\mathbb{T}(T(\chi))\big)\cdot\varepsilon_{\Lambda^\circ,-\xi}\big(\mathbb{T}(T(\chi))\big)^*\cdot\overline{d_\Lambda\big(\psi(\mathbb{Q}_p,\mathbb{T}(T(\chi)))\big)}^{-1}$. By definition of $\varepsilon_{\Lambda,\xi}\big(\mathbb{T}(T(\chi))\big)$ and $\varepsilon_{\Lambda^\circ,-\xi}\big(\mathbb{T}(T(\chi))^*(1)\big)$ and because duals commute with d_Λ, this reduces to showing the commutativity of the diagram

$$\begin{tikzcd}
Tw_\chi^*\left(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))\right) \arrow[r, "Tw_\chi^*\left(\psi(\mathbb{Q}_p,\mathbb{T}(T))\right)"] \arrow[dd, "\phi_1"'] & Tw_\chi^*\left(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)^*(1))^*[-2]\right) \arrow[d, "\nu"] \\
& \left(Tw_\chi^*(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)^*(1)))\right)^*[-2] \arrow[d, "\phi_1'^{*-1}[-2]"] \\
R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))) \arrow[r, "\psi(\mathbb{Q}_p,\mathbb{T}(T(\chi))"'] & R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))^*(1))^*[-2].
\end{tikzcd}$$

This, however, is just an instance of the compatibility of the local Tate-duality with base change (1.3.18) and its naturality.

□

Remark 3.3.9. The heart of our construction of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ was a base change along $Tw : \Lambda \to \Lambda$. In the proofs of the properties of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ we established in particular the compatibility of those properties with this base change. However, the techniques are not limited to the base change along $Tw : \Lambda \to \Lambda$. In fact, for any triple $(\Lambda, \mathbb{T}, \xi)$ as in conjecture 2.3.5 and another adic ring Λ' with Y in $\mathrm{PMod}(\Lambda')$ and Λ acting continuously from the right on Y in a way compatible with the Λ'-action, our arguments show that if $\varepsilon_{\Lambda,\xi}(\mathbb{T})$ has some of the properties in 2.3.5, then so does $Y \otimes_\Lambda \varepsilon_{\Lambda,\xi}(\mathbb{T})$.

Remark 3.3.10. The finiteness of the Galois extension F/K is not essential to the proofs in this chapter. The assumption is a remnant of the setting in chapter 4. In fact, the proofs in this chapter go through for certain infinite extensions F/K, too. We only have to make sure that Λ remains an adic ring. In [FK06] 1.4.2 Fukaya and Kato show that if a profinite group G contains a topologically finitely generated pro-p open normal subgroup, then the completed group ring $\mathcal{O}_L[\![G]\!] := \lim_{N\in\mathcal{N}} \mathcal{O}_L[G/N]$, where $\mathcal{N}$ is the set of open normal subgroups of G, is an adic ring. For our purpose, we want to take Λ as $\mathcal{O}_L[\![G]\!]$, where G is the Galois group of F/K. Again by [FK06] 1.4.2, Λ will be an adic ring, for instance, if F/K is a p-adic Lie extension.
The key argument in this chapter was the twist functor. Before we define it in the new situation, we briefly consider the topology on Λ. By definition, Λ carries a natural profinite topology given by the system of fundamental neighbourhoods of 0 consisting of $\pi_L^n\Lambda + I(U) = \ker\left(\Lambda \to \mathcal{O}_L/\pi_L^n[G/U]\right)$, where π_L is a uniformiser of $\mathcal{O}_L$ and the ideal $I(U) = \ker\left(\Lambda \to \mathcal{O}_L[G/U]\right)$ is the augmentation ideal of an open normal subgroup U. On the one hand, this topology is clearly finer than the topology in [FK06] 1.4.2 that makes Λ an adic ring. On the other hand, by lemma 1.3.11 (2) the latter topology is the same as the J-adic topology, where J is the Jacobson radical of Λ. Since Λ is semi-local as an adic ring (1.3.11 (3)), the J-adic topology is finer than the natural profinite topology by [NSW13] 5.2.16 and hence all these topologies are the same.
Now, we can extend the main ingredients of the definition of $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ from the finite to the infinite case:

Lemma 3.3.11. *There is a continuous $\mathcal{O}_L$-algebra homomorphism $\widehat{Tw} : \Lambda \to \Lambda$, which extends the twist morphism $Tw : \mathcal{O}_L[G] \to \mathcal{O}_L[G]$, defined as at the beginning of section 3.2, along the dense embedding*

$$\mathcal{O}_L[G] \hookrightarrow \Lambda, g \mapsto (gN)_{N\in\mathcal{N}}.$$

Proof. We want to use the universal property of the completed group ring (see [Wil98] proposition 7.2.1). Therefore, we check that the map $f : G \to \Lambda^\times, g \mapsto \chi(g)^{-1}(gN)_N$ is continuous if we equip $\Lambda^\times \subset \Lambda$ with the subspace topology. Let $n \geq 1$ and $U \in \mathcal{N}$. Then by the continuity of χ there is an open normal subgroup U' of G inside U such that $\chi(U')^{-1} \subset 1 + \pi^n \mathcal{O}_L$. Since $(gN)_{N \in \mathcal{N}} - (gu'N)_{N \in \mathcal{N}}$ lies in $I(U)$ for all g in G and u' in U', we have

$$\begin{aligned} f(gU') = \chi(g)^{-1}\chi(U')^{-1}(gU'N)_{N\in\mathcal{N}} &\subset \chi(g)^{-1}\chi(U')^{-1}(gN)_{N\in\mathcal{N}} + I(U) = f(g)\chi(U')^{-1} + I(U) \\ &\subset f(g)(1 + \pi^n\Lambda) + I(U) \subset f(g) + (\pi^n\Lambda + I(U)). \end{aligned}$$

By the universal property of the completed group ring, we get $\widehat{Tw}$. It extends Tw since Tw is given by the universal property of the group ring $\mathcal{O}_L[G]$ applied to f and because the injection $G \subset \Lambda$ factors over $\mathcal{O}_L[G]$. $\square$

As a result of this lemma, we get a twist functor as before. The next lemma shows that this twist functor also relates to twisting T with χ as in the finite case.

Lemma 3.3.12. *There is a (Λ, G_K)-isomorphism $\widehat{\phi_2} : \Lambda \otimes_{\widehat{Tw}} (\Lambda^\natural \otimes_{\mathcal{O}_L} T) \to \Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi)$ that is also a homeomorphism.*

Proof. Similar to the definition of ϕ_2 in corollary 3.2.2, we have a $(\mathcal{O}_L[G], G_K)$-isomorphism $\mathcal{O}_L[G] \otimes_{Tw} (\mathcal{O}_L[G]^\natural \otimes_{\mathcal{O}_L} T) \to \mathcal{O}_L[G]^\natural \otimes_{\mathcal{O}_L} T(\chi)$. We tensor this isomorphism with Λ over $\mathcal{O}_L[G]$ and observe that $\Lambda \otimes_{Tw} \mathcal{O}_L[G]^\natural$ and $\Lambda \otimes_{\widehat{Tw}} \Lambda^\natural$ are isomorphic as $(\Lambda, \mathcal{O}_L[G_K])$-bimodules to obtain the desired isomorphism. The continuity is clear since the topologies are induced by the Λ-structure. $\square$

The remaining parts of the finite situation, except the connection to Shapiro's lemma, now carry over to the infinite situation.

Chapter 4

Specialisation and twists

In this chapter we will give an outlook on a stronger compatibility of specialisation and twists than the one in the previous chapter. In addition to the problem setting in chapter 3, we assume that the finite, continuous character $\chi : G_K \to G \to \mathcal{O}_L$ is unramified. Suppose there is an isomorphism $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) : \mathbb{1}_{\widetilde{\Lambda}} \to \big(d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))) \cdot d_\Lambda(\mathbb{T}(T))\big)_{\widetilde{\Lambda}}$ and a continuous ring homomorphism $\alpha : \Lambda \to M_n(L')$ for some finite extension $L'/\mathbb{Q}_p$, such that $L'^n \otimes_\alpha \mathbb{T}(T)$ is an L'-linear de Rham representation of $G_{\mathbb{Q}_p}$, for which $L'^n \otimes_\alpha \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) = \varepsilon_{L',\xi}(L'^n \otimes_\alpha \mathbb{T}(T))$. The question we will address in this chapter is if the specialisation property along α holds for $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ as well, i.e. if in the case of $L'^n \otimes_\alpha \mathbb{T}(T(\chi))$ being de Rham (which we will see to be always the case) we also have $L'^n \otimes_\alpha \varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi))) = \varepsilon_{L',\xi}(L'^n \otimes_\alpha \mathbb{T}(T(\chi)))$.

Remark 4.0.1. The specialisation property in this chapter is different from the one in the previous chapter. In proposition 3.3.3, we assumed that $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ specialised correctly along all α's that yield de Rham representations and concluded that $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ also specialised correctly along all α's which yield de Rham representations. In fact, by the proof of 3.3.3 we actually showed that if $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T))$ specialises correctly along $\alpha \circ Tw$, then $\varepsilon_{\Lambda,\xi}(\mathbb{T}(T(\chi)))$ specialised correctly along α. So the ring homomorphism for which we assumed the desired specialisation behaviour differed from the one for which we wanted to establish the correct specialisation. In this chapter we will work with the same ring homomorphism α in both cases. Alternatively, we could ask whether the two equations

$$L'^n \otimes_\alpha \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) = \varepsilon_{L',\xi}(L'^n \otimes_\alpha \mathbb{T}(T))$$
$$L'^n \otimes_{\alpha\circ Tw} \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) = \varepsilon_{L',\xi}(L'^n \otimes_{\alpha\circ Tw} \mathbb{T}(T))$$

are equivalent for a single α.

The question of this chapter is not solved. We will present some of our thoughts on the matter. From now on, we fix a continuous ring homomorphism $\alpha : \Lambda \to M_n(L')$ such that $L'^n \otimes_\alpha \mathbb{T}(T))$ is an L'-linear de Rham representation and for which $L'^n \otimes_\alpha \varepsilon_{\Lambda,\xi}(\mathbb{T}(T)) = \varepsilon_{L',\xi}(L'^n \otimes \alpha\, \mathbb{T}(T))$ holds.
To simplify the notation, we write $\mathbb{V} := L'^n \otimes_\alpha \mathbb{T}(T)$ and $\mathbb{V}_\chi := L'^n \otimes_\alpha \mathbb{T}(T(\chi))$. We will analyse how the ε-isomorphisms of $\mathbb{V}$ and $\mathbb{V}_\chi$ are related. Recall that $\varepsilon_{L',\xi}(\mathbb{V}) = \Gamma(\mathbb{V}) \cdot \varepsilon^{\mathrm{dR}}_{L'\xi}(\mathbb{V}) \cdot \theta(\mathbb{V})$ with $\varepsilon^{\mathrm{dR}}_{L'\xi}(\mathbb{V}) = t^{-t_H(\mathbb{V})} \cdot \varepsilon_L(D_{\mathrm{pst}}(\mathbb{V}), \xi) \cdot \mathrm{can}_{\mathbb{V}}$.

4.1 Relating the Hodge-Tate structure of $\mathbb{V}$ and $\mathbb{V}_\chi$

The next lemma shows that $\mathbb{V}_\chi$ is automatically de Rham and that the parts of the ε-isomorphisms related to the Hodge-Tate structure of $\mathbb{V}$ and $\mathbb{V}_\chi$ agree. Let F' be the Galois closure over $\mathbb{Q}_p$ of the maximal unramified subextension of F/K.

Lemma 4.1.1. *There is an isomorphism $\mu : F' \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}) \to F' \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}_\chi)$ of filtered $F' \otimes_{\mathbb{Q}_p} L'$-modules.*
In particular, $\mathbb{V}_\chi$ is de Rham, we have $h(r)_{\mathbb{V}} = h(r)_{\mathbb{V}_\chi}$ for all r in $\mathbb{Z}$ and thus $t_H(\mathbb{V}) = t_H(\mathbb{V}_\chi)$ and $\Gamma(\mathbb{V}) = \Gamma(\mathbb{V}_\chi)$.

Proof. The proof rests on Galois descent for D_{dR}. Since χ is unramified and trivial on G_F, it is also trivial on $G_{F'}$. The morphism $\nu : \mathbb{V} \to \mathbb{V}_\chi$ which is induced by the simple map $T \to T(\chi), t \mapsto t \otimes e_\chi$ is an isomorphism of $G_{F'}$-representations, since $G_{F'}$ is normal in $G_{\mathbb{Q}_p}$, and induces an isomorphism of filtered $F' \otimes_{\mathbb{Q}_p} L'$-modules $D_{\mathrm{dR},F'}(\nu) : D_{\mathrm{dR},F'}(\mathbb{V}) \to D_{\mathrm{dR},F'}(\mathbb{V}_\chi)$. Moreover, by Galois descent (proposition 1.2.30,) we have filtered $F' \otimes_{\mathbb{Q}_p} L'$-isomorphisms $F' \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}) \cong D_{\mathrm{dR},F'}(\mathbb{V})$ and $F' \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}_\chi) \cong D_{\mathrm{dR},F'}(\mathbb{V}_\chi)$. Now, one can easily read off that the $\mathbb{Q}_p$-dimensions of $D_{\mathrm{dR}}(\mathbb{V})$ and $D_{\mathrm{dR}}(\mathbb{V}_\chi)$ agree, so that $\mathbb{V}_\chi$ is de Rham since $\mathbb{V}$ is. As μ is an isomorphism of filtered modules, we immediately get the remaining results. $\square$

Corollary 4.1.2. *We have*

$$d_{L'}(\nu)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'} \cdot d_{F' \otimes_{\mathbb{Q}_p} L'}(\mu)^{-1}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'} \circ \mathrm{can}_{\mathbb{V}} = \mathrm{can}_{\mathbb{V}_\chi}$$

as maps

$$\mathbb{1} \to d_{L'}(\mathbb{V}_\chi)_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'} \cdot d_{L'}(D_{\mathrm{dR}}(\mathbb{V}_\chi))^{-1}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'}.$$

Proof. The diagram

$$\begin{tikzcd}
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}) \arrow[r, equal] \arrow[rrr, bend left, "\mathrm{can}_{\mathbb{V}}"] & B_{\mathrm{dR}} \otimes_{F'} F' \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}) \arrow[r] \arrow[d, "\mathrm{id}_{B_{\mathrm{dR}}} \otimes \mu"'] & B_{\mathrm{dR}} \otimes_{F'} D_{\mathrm{dR},F'}(\mathbb{V}) \arrow[r] \arrow[d, "\mathrm{id}_{B_{\mathrm{dR}}} \otimes D_{\mathrm{dR},F'}(\nu)"'] & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{V} \arrow[d, "\mathrm{id}_{B_{\mathrm{dR}}} \otimes \nu"'] \\
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}_\chi) \arrow[r, equal] \arrow[rrr, bend right, "\mathrm{can}_{\mathbb{V}_\chi}"'] & B_{\mathrm{dR}} \otimes_{F'} F' \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{V}_\chi) \arrow[r] & B_{\mathrm{dR}} \otimes_{F'} D_{\mathrm{dR},F'}(\mathbb{V}_\chi) \arrow[r] & B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{V}_\chi
\end{tikzcd}$$

commutes. Here, the first horizontal arrow in each row is the Galois descent and the second one is the comparison isomorphism when the representations are viewed as $G_{F'}$-representations. We multiply the outer two ways from the top left to the bottom right corner with $d_{F' \otimes_{\mathbb{Q}_p} L'}(\mu)^{-1}_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'}$, rearrange and de-trivialise appropriately to obtain the statement. $\square$

Remark 4.1.3. Note while being F'-linear, the map μ need not be the base change of some isomorphism $D_{\mathrm{dR}}(\mathbb{V}) \to D_{\mathrm{dR}}(\mathbb{V}_\chi)$ to F'. If K is unramified over $\mathbb{Q}_p$, then F' is unramified over $\mathbb{Q}_p$ and hence the morphism $d_{F' \otimes_{\mathbb{Q}_p} L'}(\mu)$ lives over $\widetilde{L}'$.

4.2 Relating the ε-factors of $\mathbb{V}$ and $\mathbb{V}_\chi$

We collect some results on the behaviour of $D_?$ with respect to induction (see for instance [BB05a] 2.13.).

Lemma 4.2.1.

(1) Let ? be any of dR, st or cris and let W be an object of $\mathrm{Rep}_{L'}(G_K)$. Then the map

$$D_{?,K}(W) \to D_?(\mathrm{Ind}_K^{\mathbb{Q}_p} W), \sum_i b_i \otimes w_i \mapsto \sum_{\bar{g} \in G_{\mathbb{Q}_p}/G_K} \sum_i g(b_i) \otimes g \otimes w_i,$$

where g is a lift of $\bar{g}$ to $G_{\mathbb{Q}_p}$, is an isomorphism of filtered L'-vector spaces (of (φ, N)-modules, of φ-modules).

(2) The map

$$\psi : \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}} D_{\mathrm{pst},K}(W) \to D_{\mathrm{pst}}(\mathrm{Ind}_K^{\mathbb{Q}_p}(W)), g \otimes \sum_i b_i \otimes w_i \mapsto \sum_i g\varphi^{-v(g)}(b_i) \otimes g \otimes w_i$$

is an isomorphism of $\mathbb{Q}_p^{nr}$-linear $W_{\mathbb{Q}_p}$-representations.

Proof.

(1) Note that the map is independent of the choice of representatives g of $\bar{g}$. Let h be in G_K. We have $\sum_i gh(b_i) \otimes gh \otimes w_i = \sum_i g(h(b_i)) \otimes g \otimes h(w_i)$. But $\sum_i b_i \otimes w_i$ is invariant under G_K. Hence, we get $\sum_i g(h(b_i)) \otimes g \otimes h(w_i) = \sum_i g(b_i) \otimes g \otimes w_i$.
Up to the $G_{\mathbb{Q}_p}$-equivariant L'-isomorphism

$$B_? \otimes_{\mathbb{Q}_p} \mathrm{Ind}_K^{\mathbb{Q}_p} W \to \mathrm{Ind}_K^{\mathbb{Q}_p}(B_? \otimes_{\mathbb{Q}_p} W), b \otimes g \otimes w \mapsto g \otimes g^{-1}(b) \otimes w,$$

where $G_{\mathbb{Q}_p}$ acts diagonally on the left side, the statement is just Shapiro's lemma 1.3.8 for the zero-th cohomology group of the module $\mathrm{Ind}_K^{\mathbb{Q}_p}(B_? \otimes_{\mathbb{Q}_p} W)$. The filtration, φ and N are all compatible with the $G_{\mathbb{Q}_p}$-action and thus with the isomorphism in the statement. Alternatively, one can prove the statement more explicitly like (2).

(2) We start by checking that ψ is well-defined. Let $\sum_i b_i \otimes w_i \in D_{\mathrm{pst},K}(W)$, $h \in W_K$ and $g \in W_{\mathbb{Q}_p}$. We have

$$\begin{aligned}\psi\left(gh \otimes \sum_i b_i \otimes w_i\right) &= \sum_i gh(\varphi^{-v(gh)}(b_i)) \otimes gh \otimes w_i \\ &= \sum_i g\varphi^{-v(g)}(h\varphi^{-v(h)}(b_i)) \otimes g \otimes h(w_i) = \psi\left(g \otimes h\left(\sum_i b_i \otimes w_i\right)\right).\end{aligned}$$

Moreover, ψ actually maps to $D_{\mathrm{pst}}(\mathrm{Ind}_K^{\mathbb{Q}_p}(W))$. Let $g \otimes \sum_i b_i \otimes w_i$ be in $\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(D_{\mathrm{pst},K}(W))$. Without loss of generality, we can assume that $\sum_i b_i \otimes w_i$ lies in $(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} W)^{G_M}$ for M being a finite Galois extension of $\mathbb{Q}_p$ containing K. For $h \in G_M$, such that $hg = gh'$ with $h' \in G_M$, we have

$$\begin{aligned}h(\psi(g \otimes \sum_i b_i \otimes w_i)) &= \sum_i hg(\varphi^{-v(g)}(b_i)) \otimes hg \otimes w_i = \sum_i gh'(\varphi^{-v(g)}(b_i)) \otimes g \otimes h'(w_i) \\ &= \sum_i g\varphi^{-v(g)}(h'(b_i)) \otimes g \otimes h'(w_i) = \sum_i g\varphi^{-v(g)}(b_i) \otimes g \otimes w_i \\ &= \psi(g \otimes \sum_i b_i \otimes w_i).\end{aligned}$$

We used $\sum_i h'(b_i) \otimes h'(w_i) = \sum_i b_i \otimes w_i$ at the fourth equality sign. Another such computation shows that ψ is $W_{\mathbb{Q}_p}$-equivariant with respect to the linearised $W_{\mathbb{Q}_p}$-action on the right and left multiplication on the left. ψ is $\mathbb{Q}_p^{nr}$-linear since $g \in W_{\mathbb{Q}_p}$ acts on $\mathbb{Q}_p^{nr}$ as the arithmetic Frobenius to the power $v(g)$ and φ acts on $\mathbb{Q}_p^{nr}$ as the arithmetic Frobenius as well 1.2.14. It remains to prove bijectivity. The map is clearly injective. For surjectivity pick an element x in $D_{\mathrm{pst}}(\mathrm{Ind}_K^{\mathbb{Q}_p}(W))$. We choose a left-transversal τ_g of W_K in $W_{\mathbb{Q}_p}$. So, for all elements g of $W_{\mathbb{Q}_p}$, we have $gW_K = \tau_g W_K$ inside $W_{\mathbb{Q}_p}$. This transversal is also a G_K-left-transversal in $G_{\mathbb{Q}_p}$. So, without loss of generality, we can assume x to have the form $\sum_{\bar{g} \in W_{\mathbb{Q}_p}/W_K} \sum_i b_{i,\bar{g}} \otimes \tau_g \otimes w_{i,\bar{g}}$ and to be G_M-invariant for some finite Galois extension M of $\mathbb{Q}_p$ containing K. Then we have for all $h \in G_M$ and all $\bar{g} \in W_{\mathbb{Q}_p}/W_K$ some $h_{\bar{g}} \in G_M$ such that $h_{\bar{g}} \tau_g = \tau_g h$. In particular, left multiplication of $h_{\bar{g}}$ does not change the G_K-coset in $G_{\mathbb{Q}_p}$, since $G_M \subset G_K$. By assumption, $h_{\bar{g}}(x) = x$ and so we have for each $h \in G_M$ on the $\bar{g}$-component of x:

$$\sum_i b_{i,\bar{g}} \otimes \tau_g \otimes w_{i,\bar{g}} = \sum_i h_{\bar{g}}(b_{i,\bar{g}}) \otimes h_{\bar{g}} \tau_g \otimes w_{i,\bar{g}} = \sum_i \tau_g(h\tau_g^{-1}(b_{i,\bar{g}})) \otimes \tau_g \otimes h(w_{i,\bar{g}}).$$

This implies that for all $\bar{g} \in W_{\mathbb{Q}_p}/W_K$ we have

$$\forall h \in G_M : \sum_i \tau_g(\tau_g^{-1} b_{i,\bar{g}}) \otimes w_{i,\bar{g}} = \sum_i \tau_g(h\tau_g^{-1}(b_{i,\bar{g}})) \otimes h(w_{i,\bar{g}}),$$

so that $y_{\bar{g}} := \sum_i \tau_g^{-1}(b_{i,\bar{g}}) \otimes w_{i,\bar{g}}$ is an element of $(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} W)^{G_M}$. A preimage of x under ψ is given as $\sum_{\bar{g} \in W_{\mathbb{Q}_p}/W_K} \tau_g \otimes y_{\bar{g}} \in \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}}(D_{\mathrm{pst},K}(W))$.

□

Consider the assumption that $\mathrm{im}(\mathcal{O}_L \hookrightarrow \Lambda \xrightarrow{\alpha} M_n(L'))$ consists of diagonal matrices with a single value of L' on the diagonal. We label this assumption $(\star)$. The prime example we have in mind in which $(\star)$ holds is the case where L' is a finite extension of L and α comes from a representation $G \to \mathrm{GL}_n(L')$, so that $\mathcal{O}_L$ is unchanged by α. This is the situation which we need for the compatibility of the ETNC with the functional equation (see [Ven05b] 5.11). In these cases $\mathcal{O}_L \subset \Lambda$ is a subring of the "coefficients" of the motive.

Lemma 4.2.2.

(1) Let R be a ring, k a field, G a group with subgroup H. Let M be an R-module with R-linear H-action and W a k-vector space with a compatible right action of R. Then the map

$$W \otimes_R \mathrm{Ind}_H^G M \to \mathrm{Ind}_H^G(W \otimes_R M), w \otimes g \otimes m \mapsto g \otimes w \otimes m$$

is a well-defined isomorphism of k-linear G-representations.

(2) Assuming $(\star)$, there is an $(L', G_{\mathbb{Q}_p})$-isomorphism

$$\mathbb{V}_\chi \cong \mathrm{Ind}_K^{\mathbb{Q}_p}\left(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T) \otimes_{L'} L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)\right)$$

where G_K acts on $L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)$ only via χ.

Proof. (1) This can easily be checked.

(2) We can pull the induction out of $\mathbb{V}_\chi$ as in the first part of this lemma. With the assumption $(\star)$, the expression $L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)$ makes sense and the map

$$\begin{aligned} L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi)) &\to L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T) \otimes_{L'} L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)) \\ l \otimes \bar{g} \otimes t \otimes e_\chi &\mapsto (l \otimes \bar{g} \otimes t) \otimes (1 \otimes e_\chi \end{aligned}$$

is visibly a G_K-equivariant L'-isomorphism.

□

The next lemma shows how the local ε-factors differ for $\mathbb{V}$ and $\mathbb{V}_\chi$.

Lemma 4.2.3. *Let f be the residue degree of the extension $K/\mathbb{Q}_p$. If we assume $(\star)$, then*

$$\varepsilon(D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma, \xi) = \varepsilon(D_{\mathrm{pst}}(\mathbb{V})_\sigma, \xi) \cdot \sigma(\alpha(\chi(\mathrm{Fr}^f)))^{a(Y_\sigma)+nd\cdot v(\mathfrak{d})},$$

where $\mathfrak{d}$ is the different of $K/\mathbb{Q}_p$. We put $Y_\sigma := \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T))$, write $a(Y_\sigma)$ for the Artin-conductor of the representation Y_σ and denote by d the rank of T as $\mathcal{O}_L$-module.

Proof. Let $\sigma : L' \hookrightarrow \overline{\mathbb{Q}_p}$ be a $\mathbb{Q}_p$-linear embedding. By the lemmata 4.2.1 (2) and 4.2.2 (1), we have

$$D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma = \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst}}(\mathbb{V}_\chi) \cong \mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}} \left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi)))\right)$$

as $\overline{\mathbb{Q}_p}$-linear $W_{\mathbb{Q}_p}$-representations. Let ψ be the character $\mathbb{Q}_p \to \overline{\mathbb{Q}_p}^\times$ with kernel $\mathbb{Z}_p$ corresponding to ξ (see remark 2.1.2). Using theorem 2.1.1 (4), we get

$$\begin{aligned} \varepsilon(D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma, \xi) &= \varepsilon\left(\mathrm{Ind}_{W_K}^{W_{\mathbb{Q}_p}} \left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi)))\right), \xi\right) \\ &= \varepsilon\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi))), \psi \circ Tr_{K/\mathbb{Q}_p}\right) \cdot \lambda(K/\mathbb{Q}_p, \psi)^{\dim_{\overline{\mathbb{Q}_p}}(Y_\sigma)}, \end{aligned}$$

where $\lambda(K/\mathbb{Q}_p, \psi)$ is a factor independent of the representation. The G_K-representations $L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T)$ and $L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)$ are potentially semi-stable. For the former one sees this by comparing dimensions in lemma 4.2.1 (2). The latter is unramified and hence even crystalline by completed unramified descent 1.2.30. As a result, we can use the compatibility of $D_{\mathrm{pst},K}$ with tensor products from lemma 1.2.39 (3) and obtain

$$\begin{aligned} &\varepsilon\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi))), \psi \circ Tr_{K/\mathbb{Q}_p}\right) \\ =&\varepsilon\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T) \otimes_{\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)), \psi \circ Tr_{K/\mathbb{Q}_p}\right). \end{aligned}$$

The one-dimensional $\overline{\mathbb{Q}_p}$-linear W_K-representation $Z_\sigma := \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi))$ is unramified since χ is. So by part (5) of theorem 2.1.1, we get

$$\begin{aligned} &\varepsilon\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T) \otimes_{\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi)), \psi \circ Tr_{K/\mathbb{Q}_p}\right) \\ =&\varepsilon\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T), \psi \circ Tr_{K/\mathbb{Q}_p}\right) \cdot \det_{\overline{\mathbb{Q}_p}}(\mathrm{Fr}^f | Z_\sigma)^{a(Y_\sigma)+\dim_{\overline{\mathbb{Q}_p}}(Y_\sigma)\cdot n(\psi \circ Tr_{K/\mathbb{Q}_p})}, \end{aligned}$$

where $a(Y_\sigma)$ is the Artin-conductor of Y_σ and $n(\psi \circ Tr_{K/\mathbb{Q}_p})$ the greatest integer such that $\pi_K^{-n} \in \ker(\psi \circ Tr_{K/\mathbb{Q}_p})$. The kernel of ψ is $\mathbb{Z}_p$, so $\pi_K^{-n} \in \ker(\psi \circ Tr_{K/\mathbb{Q}_p})$ is equivalent to $Tr_{K/\mathbb{Q}_p}(\pi_K^{-n}) \in \mathbb{Z}_p$. So $n(\psi \circ Tr_{K/\mathbb{Q}_p}) = v(\mathfrak{d})$, with $\mathfrak{d}$ the different of the extension $K/\mathbb{Q}_p$. By lemma 1.2.36, we have $\dim_{\overline{\mathbb{Q}_p}}(Y_\sigma) = \dim_{L'}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T)) = nd$. Using theorem 2.1.1 part (4) once again to pull the induction back inside D_{pst}, we get

$$\begin{aligned}
&\varepsilon\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T)), \psi \circ Tr_{K/\mathbb{Q}_p}\right)\\
&\cdot \lambda(K, \mathbb{Q}_p, \psi)^{\dim_{\overline{\mathbb{Q}_p}}(Y_\sigma)} \det_{\overline{\mathbb{Q}_p}}(\mathrm{Fr}^f | Z_\sigma)^{a(Y_\sigma)+\dim_{\overline{\mathbb{Q}_p}}(Y_\sigma)\cdot n(\psi\circ Tr_{K/\mathbb{Q}_p})}\\
=&\varepsilon(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst}}(L'^n \otimes_\alpha \mathbb{T}(T)), \xi) \cdot \det_{\overline{\mathbb{Q}_p}}(\mathrm{Fr}^f | Z_\sigma)^{a(Y_\sigma)+nd+v(\mathfrak{d})}\\
=&\varepsilon(D_{\mathrm{pst}}(\mathbb{V})_\sigma, \xi) \cdot \det_{\overline{\mathbb{Q}_p}}(\mathrm{Fr}^f | Z_\sigma)^{a(Y_\sigma)+nd\cdot v(\mathfrak{d})}.
\end{aligned}$$

By lemma 4.2.4 below, we know that Fr^f acts on $D_{\mathrm{pst}}(L' \otimes_{\alpha,\mathcal{O}_L} \mathcal{O}_L(\chi))$ via multiplication with $\alpha(\chi(\mathrm{Fr}^f))$. So, on Z_σ the element Fr^f acts via multiplication with $\sigma(\alpha(\chi(\mathrm{Fr}^f)))$. □

The following lemma is inspired by an exercise from the tutorial of the lecture "L-Funktionen und ε-Konstanten II" held in the summer term of 2017 at the University of Heidelberg.

Lemma 4.2.4. *Let $\eta : G_K \to L'^\times$ be an unramified, continuous character and W the induced one-dimensional L'-linear G_K-representation. Then $\tau \in W_K$ acts on $D_{\mathrm{pst},K}(W)$ via η.*

Proof. Since η is unramified, W is a crystalline (thus semi-stable) G_K-representation. Let d be a non-trivial element of $D_{\mathrm{st},K}(W)$. The ring $K_0 \otimes_{\mathbb{Q}_p} L'$ is a product of fields by lemma 1.2.4. So d is not a torsion-element and thus $(K_0 \otimes_{\mathbb{Q}_p} L)d$ is a free $K_0 \otimes_{\mathbb{Q}_p} L'$-submodule of $D_{\mathrm{st},K}(W)$ of rank one. By complete unramified descent, we have

$$\widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0} D_{\mathrm{st},K}(W) \cong D_{\mathrm{st},\widehat{K^{nr}}}(W) = \widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} W = \widehat{\mathbb{Q}_p^{nr}} \otimes_{K_0} K_0 \otimes_{\mathbb{Q}_p} W.$$

The first equality sign is due to $G_{\widehat{K^{nr}}} = I_K$ acting trivially on the unramified representation W. As a result, we get $\dim_{K_0}(D_{\mathrm{st},K}(W)) = \dim_{K_0}(K_0 \otimes_{\mathbb{Q}_p} L')$. But this means that the K_0-linear injection $(K_0 \otimes_{\mathbb{Q}_p} L)d \hookrightarrow D_{st,K}(W)$ is an isomorphism and $D_{\mathrm{st},K}(W)$ is a free $K_0 \otimes_{\mathbb{Q}_p} L'$-module of rank one. Let w be a L'-basis of W. The element $1 \otimes w$ is a $K_0 \otimes_{\mathbb{Q}_p} L'$-basis of $D_{\mathrm{st},K}(W)$. An element τ of W_K sends $1 \otimes w$ to $\tau(\varphi^{-v(\tau)}(1)) \otimes \tau(w) = 1 \otimes \eta(\tau)w$. Since W is K-semi-stable, we have $D_{\mathrm{pst},K}(W) = \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\mathrm{st},K}(W)$ and so the action of τ on $D_{\mathrm{pst},K}(W)$ is by multiplication with $\eta(\tau)$. □

We recall a lemma on the invariants of an induced representation.

Lemma 4.2.5. *Let G be a group with a subgroup H and a normal subgroup N. Let M be an H-module. Then the morphism*

$$\mathrm{Ind}_{H/H\cap N}^{G/N}\left(M^{H\cap N}\right) \to \left(\mathrm{Ind}_H^G(M)\right)^N, gN \otimes m \mapsto \sum_{\bar{n}\in N/N\cap H} gn \otimes m$$

is an isomorphism of G/N-representations.

Since we could not find a reference for this standard result, we give a proof.

Proof. We omit the simple calculations establishing that the map is well-defined. Let $n_1, \ldots, n_r$ be a left-transversal of $N \cap H$ in N and $g_1N, \ldots, g_sN$ a left-transversal of $H/N \cap H$ in G/N. Then it is easy to check that g_in_j for $i = 1, \ldots, s$ and $j = 1, \ldots r$ is a left-transversal for H in G. This shows the injectivity of the morphism. For the subjectivity, consider an element $x = \sum_{i,j} g_in_j \otimes m_{ij}$ of $\left(\mathrm{Ind}_H^G(M)\right)^N$. Let h be in $H \cap N$. Then $n_{ij} := g_in_jh(g_in_j)^{-1}$ lies in the

normal subgroup N. We have $n_{ij}(x) = \sum_{ij} g_i n_j \otimes h(m_{ij})$. But also $n_{ij}(x) = x$ by the choice of x. Comparing the (i,j)-th component, we get $h(m_{ij}) = m_{ij}$. This holds for any h and thus we get $m_{ij} \in M^{N \cap H}$ for all i and j.
We need to show that m_{ij} is independent of j. To see this, we choose some h in $N \cap H$ and define elements $n_{jj'} \in N$ so that $n_{jj'} n_{j'} = n_j h$. Put $n_{ijj'} := g_i n_{jj'} g_i^{-1}$. Again, by normality of N, we know that $n_{ijj'}$ lies in N. The element $n_{ijj'}$ sends $g_i n_{j'} \otimes m_{ij'}$ to $g_i n_j \otimes m_{ij'}$ since we saw that $m_{ij'}$ is $H \cap N$-invariant. So the (i,j)-th component of the equation $n_{ijj'}(x) = x$ yields $m_{ij} = m_{ij'}$, which we now call m_i. A preimage of x under the map is given by $\sum_i g_i N \otimes m_i$. The G/N-equivariance of the map is clear. $\square$

We can now compute how the correction factor for $\mathbb{V}_\chi$ is related to the one for $\mathbb{V}$.

Lemma 4.2.6. *Let f be the residue degree of $K/\mathbb{Q}_p$. Assuming $(\star)$, we have*

$$\det_{L'}\big(-\varphi \big| D_{\text{st}}(\mathbb{V}_\chi)/D_{\text{cris}}(\mathbb{V}_\chi)\big) = \det_{L'}\big(-\varphi \big| D_{\text{st}}(\mathbb{V})/D_{\text{cris}}(\mathbb{V})\big) \cdot \alpha(\chi(\mathrm{Fr}^f))^{\dim_{L'}(D_{\text{st}}(\mathbb{V})/D_{\text{cris}}(\mathbb{V}))/f}.$$

Proof. By lemma 2.2.7, it suffices to consider $\det_{\overline{\mathbb{Q}_p}}\big(-\mathrm{Fr}\big| D_{\text{pst}}(\mathbb{V}_\chi)^I_\sigma/(D_{\text{pst}}(\mathbb{V}_\chi)^I_\sigma)^{N=0}\big)$ for some $\mathbb{Q}_p$-linear embedding $\sigma : L' \hookrightarrow \overline{\mathbb{Q}_p}$. We have

$$\det_{\overline{\mathbb{Q}_p}}\big(-\mathrm{Fr}\big| D_{\text{pst}}(\mathbb{V}_\chi)^I_\sigma/(D_{\text{pst}}(\mathbb{V}_\chi)^I_\sigma)^{N=0}\big) = \sigma\det_A\big(-\mathrm{Fr}\big| D_{\text{pst}}(\mathbb{V}_\chi)^I/((D_{\text{pst}}(\mathbb{V}_\chi)^I)^{N=0}\big),$$

where $\sigma : A \to \overline{\mathbb{Q}_p}$ sends $q \otimes l$ to $q\sigma(l)$. Using the lemmata 4.2.1 (2) and 4.2.5, we pull the induction out of $D_{\text{pst}}(\mathbb{V}_\chi)^I$:

$$D_{\text{pst}}(\mathbb{V}_\chi)^I = \mathrm{Ind}_{W_K/I_K}^{W_{\mathbb{Q}_p}/I}\left(D_{\text{pst},K}\big(L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi))\big)^{I_K}\right).$$

By lemma 4.2.2 (2) we can further write

$$L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T(\chi)) = L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T) \otimes_{L'} L' \otimes_\alpha \mathcal{O}_L(\chi)$$

Now, by lemma 4.2.1 (1) we know that $V := L'^n \otimes_\alpha (\Lambda^\natural \otimes_{\mathcal{O}_L} T)$ is de Rham as G_K-representation and hence potentially semi-stable. $W := L' \otimes_\alpha \mathcal{O}_L(\chi)$ is unramified and hence crystalline and (potentially) semi-stable. The compatibility of the tensor products with $D_{\text{pst},K}$ (1.2.39 (3)) yields

$$D_{\text{pst},K}(V \otimes_{L'} W)^{I_K} \cong \big(D_{\text{pst},K}(V) \otimes_A D_{\text{pst},K}(W)\big)^{I_K} = D_{\text{pst},K}(V)^{I_K} \otimes_A D_{\text{pst},K}(W).$$

For the equality, we used that since W is unramified, it is crystalline and semi-stable, so that $D_{\text{pst},K}(W) = \mathbb{Q}_p^{nr} \otimes_{K_0} D_{\text{cris},K}(W)$, which is clearly fixed element-wise by I_K. In total, we get

$$D_{\text{pst}}(\mathbb{V}_\chi)^I \cong \mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}\big(D_{\text{pst},K}(V)^{I_K} \otimes_A D_{\text{pst},K}(W)\big)$$

as $W_{\mathbb{Q}_p}/I \cong \mathbb{Z}$-representations. As the monodromy operator commutes with everything we have done and we have $D_{\text{pst},K}(W)^{N=0} = D_{\text{pst},K}(W)$ since W crystalline, we also get

$$\big(D_{\text{pst}}(\mathbb{V}_\chi)^I\big)^{N=0} \cong \mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}\big((D_{\text{pst},K}(V)^{I_K})^{N=0} \otimes_A D_{\text{pst},K}(W)\big)$$

as $W_{\mathbb{Q}_p}/I \cong \mathbb{Z}$-representations and so

$$D_{\text{pst}}(\mathbb{V}_\chi)^I/(D_{\text{pst}}(\mathbb{V}_\chi)^I)^{N=0} \cong \mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}\big(D_{\text{pst},K}(V)^{I_K}/(D_{\text{pst},K}(V)^{I_K})^{N=0} \otimes_A D_{\text{pst},K}(W)\big).$$

Let X be a W_K-representation over $\overline{\mathbb{Q}_p}$. We want to relate the determinant of Fr on $\mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}(X)$ to that of Fr^f on X. As transversal of W_K in $W_{\mathbb{Q}_p}$, we choose $\mathrm{Fr}^0, \ldots, \mathrm{Fr}^{f-1}$. Then $\mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}(X)$ is isomorphic to $\bigoplus_{i=0}^{f-1} \mathrm{Fr}^i \otimes X$ as $\overline{\mathbb{Q}_p}$-vector space. Fr maps $\mathrm{Fr}^i \otimes x$ to $\mathrm{Fr}^{i+1} \otimes x$ for $i = 0, \ldots f-2$. On the determinant, this introduces a sign, which only depends on f and $\dim_{\overline{\mathbb{Q}_p}}(X)$. Now, $\mathrm{Fr}^{f-1} \otimes x$ maps to $\mathrm{Fr}^f \otimes x = \mathrm{Fr}^0 \otimes \mathrm{Fr}^f(x)$ under Fr. Therefore we have up to a sign, which only depends on f and $\dim_{\overline{\mathbb{Q}_p}}(X)$,

$$\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}\left|\mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}(X)\right.\right) = \pm\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}^f \left|X\right.\right).$$

We apply this to $X = \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0} \otimes_{\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\sigma,\alpha} D_{\mathrm{pst},K}(W)$ and, using the formula for determinants of a Kronecker-product of two matrices, we obtain

$$\begin{aligned}
&\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}\left|\mathrm{Ind}_{W_K/I_K}^{W_{\mathbb{Q}_p}/I}\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0} \otimes_{\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(W)\right)\right.\right)\\
&= \pm\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}^f\left|\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0} \otimes_{\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(W)\right.\right)\\
&= \pm\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}^f\left|\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0}\right.\right)^{\dim_{\overline{\mathbb{Q}_p}}\left(\overline{\mathbb{Q}_p}\otimes_{\sigma,A} D_{\mathrm{pst},K}(W)\right)}\\
&\quad\cdot\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}^f\left|\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(W)\right.\right)^{\dim_{\overline{\mathbb{Q}_p}}\left(\overline{\mathbb{Q}_p}\otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0}\right)}.
\end{aligned}$$

As W is potentially semi-stable, $\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(W)$ is one-dimensional. Moreover, since

$$\begin{aligned}
&\dim_{\overline{\mathbb{Q}_p}}\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0}\right)\\
&=\dim_{\overline{\mathbb{Q}_p}}\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0} \otimes_{\overline{\mathbb{Q}_p}} \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(W)\right),
\end{aligned}$$

we know that the sign is the same as in

$$\begin{aligned}
&\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}\left|\mathrm{Ind}_{W_K/I_K}^{W_{\mathbb{Q}_p}/I}\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0}\right)\right.\right)\\
&= \pm\det_{\overline{\mathbb{Q}_p}}\left(\mathrm{Fr}^f\left|\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0}\right.\right).
\end{aligned}$$

By lemma 4.2.4 we have $\det_{\overline{\mathbb{Q}_p}}(\mathrm{Fr}^f|\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst}}(W)) = \sigma(\alpha(\chi(\mathrm{Fr}^f)))$. The isomorphisms

$$\begin{aligned}
\overline{\mathbb{Q}_p} \otimes_{\sigma,A} \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} (D_{\mathrm{st}}(\mathbb{V})/D_{\mathrm{cris}}(\mathbb{V})) &\cong \overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst}}(\mathbb{V})^I/(D_{\mathrm{pst}}(\mathbb{V})^I)^{N=0}\\
&\cong \mathrm{Ind}_{f\mathbb{Z}}^{\mathbb{Z}}\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I_K})^{N=0}\right)
\end{aligned}$$

show $f\cdot\dim_{\overline{\mathbb{Q}_p}}\left(\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)^{I_K}/(D_{\mathrm{pst},K}(V)^{I})^{N=0}\right) = \dim_{L'}\left(D_{\mathrm{st}}(\mathbb{V})/D_{\mathrm{cris}}(\mathbb{V})\right)$. Lastly, the dimensions of $D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma^I/(D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma^I)^{N=0}$ and $D_{\mathrm{pst}}(\mathbb{V})_\sigma^I/(D_{\mathrm{pst}}(\mathbb{V})_\sigma^I)^{N=0}$ are the same, so that we get

$$\begin{aligned}
\det_{\overline{\mathbb{Q}_p}}\left(-\mathrm{Fr}\left|D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma^I/(D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma^I)^{N=0}\right.\right) =&\det_{\overline{\mathbb{Q}_p}}\left(-\mathrm{Fr}\left|D_{\mathrm{pst}}(\mathbb{V})_\sigma^I/(D_{\mathrm{pst}}(\mathbb{V})_\sigma^I)^{N=0}\right.\right)\\
&\cdot\sigma(\alpha(\chi(\mathrm{Fr}^f)))^{\dim_{L'}(D_{\mathrm{st}}(\mathbb{V})/D_{\mathrm{cris}}(\mathbb{V}))/f}.
\end{aligned}$$

Combining these equalities for all σ, we get the statement of the lemma. □

Remark 4.2.7. When we put lemma 4.2.3 and lemma 4.2.6 together, we obtain

$$\begin{aligned}
\varepsilon_{L'}(D_{\mathrm{pst}}(\mathbb{V}_\chi)_\sigma, \xi) =&\varepsilon_{L'}(D_{\mathrm{pst}}(\mathbb{V})_\sigma, \xi)\\
&\cdot\sigma(\alpha(\chi(\mathrm{Fr}^f)))^{a(\overline{\mathbb{Q}_p}\otimes_{\sigma,A} D_{\mathrm{pst},K}(V))+\dim_{L'}(D_{\mathrm{st}}(\mathbb{V})/D_{\mathrm{cris}}(\mathbb{V}))/f+nd\cdot v(\mathfrak{d})}.
\end{aligned}$$

If $K/\mathbb{Q}_p$ is unramified, such that $v(\mathfrak{d}) = 0$, then the exponent of the factor $\sigma(\alpha(\chi(\mathrm{Fr}^f)))$ is precisely the conductor that Tate defines in [Tat79] 4.1.6 for the Weil-Deligne representation $\overline{\mathbb{Q}_p} \otimes_{\sigma,A} D_{\mathrm{pst},K}(V)$ of W_K over $\overline{\mathbb{Q}_p}$.

4.3 Relating $\theta(\mathbb{V})$ and $\theta(\mathbb{V}_\chi)$

We now turn our attention to $\theta(\mathbb{V}_\chi)$. As we noted in remarks 4.1.3 and 4.2.7 it might be useful to assume that K is unramified over $\mathbb{Q}_p$. We will make this assumption in the following. This implies that $F' = F_0$ is the maximal unramified subextension of $F/\mathbb{Q}_p$ and $K = K_0$.
The morphisms $\theta(\mathbb{V}_\chi)$ and $\theta(\mathbb{V})$ can be defined via exact sequences $\Sigma_{l,\mathbb{V}}$ and $\Sigma_{l,\mathbb{V}_\chi}$ as in remark 2.2.4. We can relate them via the diagrams 4.1 and 4.2 on the next page.
Let us elaborate the first diagram 4.1. Its first line of isomorphisms is given by Shapiro's lemma using the isomorphism

$$B_? \otimes \operatorname{Ind}_K^{\mathbb{Q}_p}(V) \cong \operatorname{Ind}_K^{\mathbb{Q}_p}(B_? \otimes V), b \otimes g \otimes v \mapsto g \otimes g^{-1}(b) \otimes v$$

and the compatibility of induction with Kummer duals. The isomorphism from Shapiro's lemma is natural and compatible with the connecting homomorphisms in long exact sequences, so that all the unlabelled squares in that row are commutative. Square (1) is commutative if Shapiro's lemma is compatible with the local Tate duality. The (duals of) restriction morphisms from the second to the fourth line are injective (surjective) by [NSW13] corollary 1.5.7 and page 138. The injections in square (2) map x to $1 \otimes x$. The square (3) is dual to square (2). Both clearly commute. They have $(1 - \varphi, \bar{1})$ and $(1 - \phi \otimes \varphi, \bar{1})$ or their duals as top and bottom arrows, respectively. The isomorphisms from the third to the fourth line are (duals of) the unramified Galois descent (see 1.2.30) $D_{\mathrm{cris},F_0}(V) \cong F_0 \otimes_{K_0} D_{\mathrm{cris},K_0}(V)$ under which the action of φ on the left hand side corresponds to $\phi \otimes \varphi$ on the right hand side, where ϕ is an arithmetic Frobenius in $G_{\mathbb{Q}_p}$, and the Galois descent $D_{\mathrm{dR},F_0}(V) \cong F_0 \otimes_K D_{\mathrm{dR},K}(V)$ of filtered F_0-vector spaces (1.2.30). The squares including two Galois descents also commute. The injections in (2) followed by the Galois descents are just the restriction maps $res : H^0(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V) \to H^0(F_0, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)$ or $res : H^0(K, B_{\mathrm{dR}}/B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \to H^0(F_0, B_{\mathrm{dR}}/B^0_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. Since the restriction maps are natural and commute with the connecting homomorphism in long exact cohomology sequences ([NSW13] 1.5.2), we get the commutativity of all squares between the second and fourth row except square (4). It commutes as well due to the compatibility of restriction maps with cup-products ([NSW13] 1.5.3) and thus with the local Tate duality. The equalities in the last row are due to the fact that W is trivial on G_{F_0}. From there on, one can extend the diagram in the same way to get to the defining sequence for $\theta(\mathbb{V}_\chi)$. We assumed $(\star)$ in the last row of the diagram for notational convenience.
By similar arguments the second diagram 4.2 commutes. We could tensor $\Sigma_{l,\mathbb{V}}$ and $\Sigma_{l,\mathbb{V}_\chi}$ over $\mathbb{Q}_p$ with F_0. Using Galois descent (1.2.30), this turns the injections and projections involving D_{cris} and the tangent spaces into $F_0 \otimes_{\mathbb{Q}_p} L$-isomorphisms. One could be inclined to deduce that the maps $F_0 \otimes_{\mathbb{Q}_p} res$ also become $F_0 \otimes_{\mathbb{Q}_p} L'$-isomorphisms. However, this does not follow. The squares labelled (5) do not commute after the tensoring. We saw above that under Galois descent $1 - \varphi$ on $D_{\mathrm{cris},F_0}(\mathbb{V})$ corresponds to $1 - \phi \otimes \varphi$ on $F_0 \otimes_{\mathbb{Q}_p} D_{\mathrm{cris}}(\mathbb{V})$. But in $F_0 \otimes_{\mathbb{Q}_p} \Sigma_{l,\mathbb{V}}$ we have $1 - \mathrm{id}_{F_0} \otimes \varphi$, which corresponds under Galois descent to $1 - \phi^{-1}\varphi$, where ϕ is an arithmetic Frobenius acting nonlinearly on $D_{\mathrm{cris},F_0}(\mathbb{V})$. The action of ϕ on $D_{\mathrm{cris},F_0}(\mathbb{V})$ and on $D_{\mathrm{cris},F_0}(\mathbb{V}_\chi)$ differs. Our hope is that one can relate them in a way that we get an isomorphism in $V(F_0 \otimes_{\mathbb{Q}_p} L')$ between $d_{F_0 \otimes_{\mathbb{Q}_p} L'}(F_0 \otimes_{\mathbb{Q}_p} \Sigma_{l,\mathbb{V}})$ and $d_{F_0 \otimes_{\mathbb{Q}_p} L'}(F_0 \otimes_{\mathbb{Q}_p} \Sigma_{l,\mathbb{V}_\chi})$ induced by the Galois descents up to some factor, which ideally cancels the factor in remark 4.2.7.

$$
\begin{tikzcd}[column sep=small]
H^0(\mathbb{Q}_p,\mathbb{V}) \arrow[r, hook] \arrow[d, phantom, sloped, "\cong"] & D_{\mathrm{cris}}(\mathbb{V}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & D_{\mathrm{cris}}(\mathbb{V})\oplus t(\mathbb{V}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H^1(\mathbb{Q}_p,\mathbb{V}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] \arrow[dr, phantom, "(1)"] & D_{\mathrm{cris}}(\mathbb{V}^*(1))^*\oplus t(\mathbb{V}^*(1))^* \arrow[r] \arrow[d, phantom, sloped, "\cong"] & D_{\mathrm{cris}}(\mathbb{V}^*(1))^* \arrow[r, two heads] \arrow[d, phantom, sloped, "\cong"] & H^0(\mathbb{Q}_p,\mathbb{V}^*(1))^* \arrow[d, phantom, sloped, "\cong"] \\
H^0(K,V) \arrow[r, hook] \arrow[dd, hook, "res"] & D_{\mathrm{cris},K}(V) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(2)"] & D_{\mathrm{cris},K}(V)\oplus t_K(V) \arrow[r] \arrow[d, hook] & H^1(K,V) \arrow[r] \arrow[dd, hook, "res"] & D_{\mathrm{cris},K}(V^*(1))^*\oplus t_K(V^*(1))^* \arrow[r] \arrow[dr, phantom, "(3)"] & D_{\mathrm{cris},K}(V^*(1))^* \arrow[r, two heads] & H^0(K,V^*(1))^* \\
& F_0\otimes_{K_0} D_{\mathrm{cris},K}(V) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \begin{array}{c} F_0\otimes_{K_0} D_{\mathrm{cris},K}(V) \\ \oplus F_0\otimes_K t_K(V)\end{array} \arrow[d, phantom, sloped, "\cong"] & \arrow[r, phantom, "(4)"] & \begin{array}{c}\big(F_0\otimes_{K_0} D_{\mathrm{cris},K}(V^*(1))\big)^* \\ \oplus\big(F_0\otimes_K t_K(V^*(1))\big)^*\end{array} \arrow[r] \arrow[u, two heads] \arrow[d, phantom, sloped, "\cong"] & \big(F_0\otimes_{K_0} D_{\mathrm{cris},K}(V^*(1))\big)^* \arrow[u, two heads] \arrow[d, phantom, sloped, "\cong"] & \\
H^0(F_0,V) \arrow[r, hook] \arrow[d, equal] & D_{\mathrm{cris},F_0}(V) \arrow[r] \arrow[d, equal] & D_{\mathrm{cris},F_0}(V)\oplus t_{F_0}(V) \arrow[r] \arrow[d, equal] & H^1(F_0,V) \arrow[r] \arrow[d, equal] & D_{\mathrm{cris},F_0}(V^*(1))^*\oplus t_{F_0}(V^*(1))^* \arrow[r] \arrow[d, equal] & D_{\mathrm{cris},F_0}(V^*(1))^* \arrow[r, two heads] \arrow[d, equal] & H^0(F_0,V^*(1))^* \arrow[uu, two heads, "res^*"'] \arrow[d, equal] \\
H^0(F_0,V\otimes W) \arrow[r, hook] \arrow[d, dotted, no head] & D_{\mathrm{cris},F_0}(V\otimes W) \arrow[r] \arrow[d, dotted, no head] & \begin{array}{c} D_{\mathrm{cris},F_0}(V\otimes W) \\ \oplus t_{F_0}(V\otimes W)\end{array} \arrow[r] \arrow[d, dotted, no head] & H^1(F_0,V\otimes W) \arrow[r] \arrow[d, dotted, no head] & \begin{array}{c} D_{\mathrm{cris},F_0}((V\otimes W)^*(1))^* \\ \oplus t_{F_0}((V\otimes W)^*(1))^*\end{array} \arrow[r] \arrow[d, dotted, no head] & D_{\mathrm{cris},F_0}((V\otimes W)^*(1))^* \arrow[r, two heads] \arrow[d, dotted, no head] & H^0(F_0,(V\otimes W)^*(1))^* \arrow[d, dotted, no head] \\
{} & {} & {} & {} & {} & {} & {}
\end{tikzcd}
$$

Figure 4.1: Diagram relating $\Sigma_{l,\mathbb{V}}$ and $\Sigma_{l,\mathbb{V}_\chi}$ via Shapiro's lemma and Galois descent.

$$
\begin{tikzcd}[column sep=small]
\Sigma_{l,\mathbb{V}}: & 0 \arrow[r] & H^0(\mathbb{Q}_p,\mathbb{V}) \arrow[r] \arrow[d, hook, "res"] & D_{\mathrm{cris}}(\mathbb{V}) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(5)"] & D_{\mathrm{cris}}(\mathbb{V})\oplus t(\mathbb{V}) \arrow[r] \arrow[d, hook] & H^1(\mathbb{Q}_p,\mathbb{V}) \arrow[r] \arrow[d, hook, "res"] & D_{\mathrm{cris}}(\mathbb{V}^*(1))^*\oplus t(\mathbb{V}^*(1))^* \arrow[r] \arrow[dr, phantom, "(5)"] & D_{\mathrm{cris}}(\mathbb{V}^*(1))^* \arrow[r] & H^0(\mathbb{Q}_p,\mathbb{V}^*(1))^* \arrow[r] & 0 \\
\Sigma_{l,F_0,\mathbb{V}} & 0 \arrow[r] & H^0(F_0,\mathbb{V}) \arrow[r] \arrow[d, equal] & D_{\mathrm{cris},F_0}(\mathbb{V}) \arrow[r] \arrow[d, equal] & D_{\mathrm{cris},F_0}(\mathbb{V})\oplus t_{F_0}(\mathbb{V}) \arrow[r] \arrow[d, equal] & H^1(F_0,\mathbb{V}) \arrow[r] \arrow[d, equal] & D_{\mathrm{cris},F_0}(\mathbb{V}^*(1))^*\oplus t_{F_0}(\mathbb{V}^*(1))^* \arrow[r] \arrow[u, two heads] \arrow[d, equal] & D_{\mathrm{cris},F_0}(\mathbb{V}^*(1))^* \arrow[r] \arrow[u, two heads] \arrow[d, equal] & H^0(F_0,\mathbb{V}^*(1))^* \arrow[r] \arrow[u, two heads, "res^*"'] \arrow[d, equal] & 0 \\
\Sigma_{l,F_0,\mathbb{V}_\chi} & 0 \arrow[r] & H^0(F_0,\mathbb{V}_\chi) \arrow[r] & D_{\mathrm{cris},F_0}(\mathbb{V}_\chi) \arrow[r] \arrow[dr, phantom, "(5)"] & D_{\mathrm{cris},F_0}(\mathbb{V}_\chi)\oplus t_{F_0}(\mathbb{V}_\chi) \arrow[r] & H^1(F_0,\mathbb{V}_\chi) \arrow[r] & D_{\mathrm{cris},F_0}(\mathbb{V}_\chi^*(1))^*\oplus t_{F_0}(\mathbb{V}_\chi^*(1))^* \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(5)"] & D_{\mathrm{cris},F_0}(\mathbb{V}_\chi^*(1))^* \arrow[r] \arrow[d, two heads] & H^0(F_0,\mathbb{V}_\chi^*(1))^* \arrow[r] \arrow[d, two heads, "res^*"] & 0 \\
\Sigma_{l,\mathbb{V}_\chi} & 0 \arrow[r] & H^0(\mathbb{Q}_p,\mathbb{V}_\chi) \arrow[r] \arrow[u, hook, "res"] & D_{\mathrm{cris}}(\mathbb{V}_\chi) \arrow[r] \arrow[u, hook] & D_{\mathrm{cris}}(\mathbb{V}_\chi)\oplus t(\mathbb{V}_\chi) \arrow[r] \arrow[u, hook] & H^1(\mathbb{Q}_p,\mathbb{V}_\chi) \arrow[r] \arrow[u, hook, "res"] & D_{\mathrm{cris}}(\mathbb{V}_\chi^*(1))^*\oplus t(\mathbb{V}_\chi^*(1))^* \arrow[r] & D_{\mathrm{cris}}(\mathbb{V}_\chi^*(1))^* \arrow[r] & H^0(\mathbb{Q}_p,\mathbb{V}_\chi^*(1))^* \arrow[r] & 0
\end{tikzcd}
$$

Figure 4.2: Diagram relating $\Sigma_{l,\mathbb{V}}$ and $\Sigma_{l,\mathbb{V}_\chi}$ via restrictions.

Assuming such an isomorphism, the Galois descents on $t(\mathbb{V})$ and on $t(\mathbb{V}^*(1))^*$ add up to the map μ from corollary 4.1.2 and the ones on D_{cris} cancel. This yields an isomorphism

$$f_\alpha : d_{F_0 \otimes_{\mathbb{Q}_p} L'}(F_0 \otimes_{\mathbb{Q}_p} R\Gamma(\mathbb{Q}_p, \mathbb{V})) \to d_{F_0 \otimes_{\mathbb{Q}_p} L'}(F_0 \otimes_{\mathbb{Q}_p} R\Gamma(\mathbb{Q}_p, \mathbb{V}_\chi)).$$

We would get $f_\alpha \cdot d_{F_0 \otimes_{\mathbb{Q}_p} L'}(\mu) \circ \theta(\mathbb{V}) = \theta(\mathbb{V}_\chi)$. Multipliying this identity after base change to $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'$ with the identity in corollary 4.1.2 for can, the $d_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} L'}(\mu)$'s cancel.
The morphism ν in lemma 4.1.1 comes from the Λ-isomorphism

$$\begin{aligned} \nu' : \mathbb{T}(T) &\longrightarrow Tw_\chi^*(\mathbb{T}(T)) \xrightarrow{\phi_2} \mathbb{T}(T(\chi)) \\ \sigma \otimes \bar{g} \otimes t &\longmapsto \bar{g} \otimes \sigma \otimes \bar{1} \otimes t \longmapsto \sigma \otimes \bar{g} \otimes t \otimes e_\chi. \end{aligned}$$

Hence, the problem of this chapter would be solved if, in addition to the above assumptions, f_α came from some isomorphism

$$f : d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))) \to d_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi))))$$

via base change along $L'^n \otimes_{\alpha,\Lambda} -$. However, ν' is not $G_{\mathbb{Q}_p}$-equivariant, so that we cannot set $f = d_\Lambda(R\Gamma(\mathbb{Q}_p, \nu'))$. In fact, if $\mathbb{T}(T)$ and $\mathbb{T}(T(\chi))$ were $(\Lambda, G_{\mathbb{Q}_p})$-isomorphic, then T and $T(\chi)$ need to be isomorphic as $\mathcal{O}_L[G_K]$-modules. In addition, in the section on Shapiro's lemma and in corollary 3.2.2 (3), we saw that $R\Gamma(\mathbb{Q}_p, \mathbb{T}(T(\chi)))$ is the same as $R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))(\chi)$ as Λ-modules. We do not know if there is a suitable f and if so, how to define it. If it existed, it would also provide a new angle to look at the problem in chapter 3.

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